Proceedings of The National Conference
On Undergraduate Research (NCUR) 2007
Dominican University of California
San Rafael, California
April 12 – 14, 2007

The Brachistochrone Problem on a Vertical Plane and Over Surfaces

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Abstract

The classical Brachistochrone Problem consists of the following: given two points A and B on a vertical plane, find the shape of the curve along which a particle, under the influence of gravity, can slide from A to B in minimum time. This problem was first posed by Johann Bernoulli in 1696 and solved that same year by Newton, Leibniz, the Bernoulli brothers, and L'Hopital. They found that the solution to this problem is given by a curve called a "cycloid". For the classical problem we developed a graphical interface in MATLAB where the user can experiment with different types of curves, such as the straight line, parabolic type, exponential type, and a cycloid. The user can also see an actual animation of the particle sliding through the selected curve and the time that it took for it to reach the end point B. We also developed some routines in MATLAB that directly minimize the discretized time integral using the method of steepest descent. These routines compute numerically the curve of minimum descent.

We also studied the Brachistochrone Problem over Surfaces, which consists of finding the curve traced by a particle that slides from point A to point B on a given frictionless surface and under the influence of gravity, in the shortest time. For this problem we developed some routines in MATLAB that directly minimize the discretized time integral. These routines compute numerically the curve of minimum descent over a given surface. As an example, we computed curves of minimum descent when the surface is given by an inclined plane, for different angles of inclination. We then joined together these curves to construct an envelope or surface of minimum curves.
Keywords: Calculus of Variation, Optimization, Visualization.

1. Introduction

The Classical Brachistochrone Problem was originally posed by Johann Bernoulli in 1696, as a challenge for the most famous mathematicians in that time. This problem consists on finding the shape of the curve traced out by a particle that slides from a point A to a point B on a vertical plane, under the influence of gravity, in the shortest time. The solution to this problem is the cycloid, which is the curve traced out by a point on the rim of a rolling circle [2,5,6,7].

We studied the Classical Brachistochrone Problem and developed a GUI (Graphical User Interface) on MATLAB where we were able to experiment with different types of curves and determine the time of descent for each curve by calculating the time integral. In addition we developed some routines on MATLAB to minimize the Discretized Time Integral by using the method of steepest descent. This involves the minimization of a function of n variables, where n is the number of mesh points in the discretization.

The Brachistochrone Problem over a surface S consists on finding the curve over S traced out by a particle that slides from point A to point B in S, under the influence of gravity, with no friction, in minimum time. In this case, we developed as well some routines on MATLAB to study the numerical aspects of this problem by directly minimizing the corresponding time integral. The function to be minimized is now one of $2n$ variables where n is as before. Note that in the brachistochrone problem over a surface, the solution depends on the surface under consideration, contrary to the classical problem in which the solution is given by a cycloid.

2. The Classical Brachistochrone Problem

We consider the problem of finding the curve traced out by a particle that slides from the origin $(0,0)$ to an end point (a, b) on a vertical plane in the shortest time, under the influence of gravity. We assume that the curve is given by $(x, y(x))$ with $0 \le x \le a$. By conservation of energy we have that the speed of the particle as a function of x, is given by $v(x) = \sqrt{v_0^2 - 2gy(x)}$, where v_0 is the initial speed and g is the acceleration due to gravity. Since the arc length along the curve is given by $ds = \sqrt{1 + y'(x)^2}\,dx$, we have that the time that it takes for the particle to slide along the curve and reach the final point (a, b) is given by the *Time Integral*:

$$T[y] = \sqrt{\frac{1 + y'(x)^2}{v_0^2 - 2gy(x)}}dx \,, \qquad (1)$$

The problem then is to find a function $y(x)$ with $y(0) = 0, y(a) = b$ that minimizes T.

2.1 The graphical user interface

We developed a graphical user interface (GUI) on MATLAB to experiment with the Classical Brachistochrone Problem.

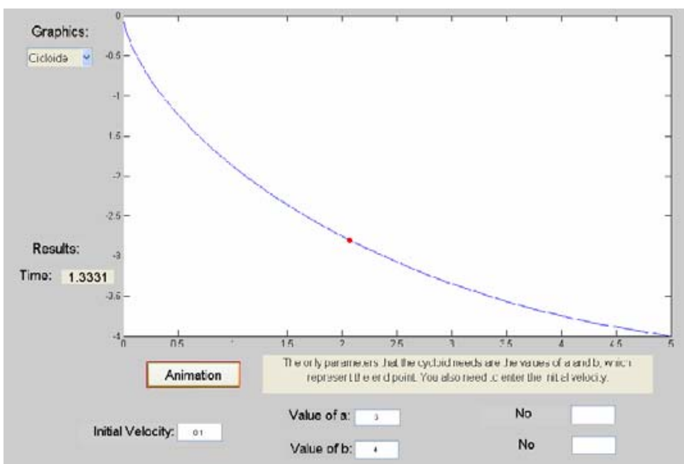

Figure 1. The Graphical User Interface.

With this interface the user can experiment with different types of curves, such as the straight line, the cycloid, $y(x) = \beta_1 x^2 + \beta_2 x$ (parabolic type), and $y(x) = c_1 e^{\beta_1 x} + c_2 e^{\beta_2 x}$ (exponential type), where β_1, β_2 are given by the user. The user can observe and actual animation of a particle sliding through the selected curve. In addition they can also see the time that it takes for the particle to reach the end point. The end point is also given by the user together with the initial speed. This way, the user can give the same final point and initial speed to each curve and see on which curve, the particle takes the least time to reach the final point. For a given function $y(x)$, the time integral (1) is approximated with the mid-point rule for integrals[1] .

Here we present the results we obtained using the graphical user interface for the sets of data: final point and initial speed.

Table 1. results of the GUI, using the end point $(5,4)$ and the initial speed 3

Curve	**End Point**	**Initial Speed**	**Total Time**
Straight Line	(5,4)	3	1.037
Parabola	(5,4)	3	1.006
Exponential	(5,4)	3	0.998
Cycloid	(5,4)	3	0.985

Table2. results of the GUI, using the end point $(4,7)$ and the initial speed 0.1

Curve	**End Point**	**Initial Speed**	**Total Time**
Straight Line	(4,7)	0.1	1.425
Parabola	(4,7)	0.1	1.421
Exponential	(4,7)	0.1	1.402
Cycloid	(4,7)	0.1	1.393

In each case we can see that the cycloid yields the minimum time of descent.

2.2 The discretized problem

In this section we discuss the problem of computing the curve y(x) that minimizes the time integral (1). Let $h = a/n, n \geq 1$ and define

$$x_j = jh,\ 0 \leq j \leq n, \quad x_{j-\frac{1}{2}} = \frac{x_{j-1} + x_j}{2} = \left(j - \frac{1}{2}\right)h, \quad 1 \leq j \leq n.$$

We denote by y_j an approximation of $y(x_j), 0 \leq j \leq n.$ With the following approximations

$$y\left(x_{j-\frac{1}{2}}\right) \approx \frac{y_{j-1} + y_j}{2} = y_{j-\frac{1}{2}}, \quad y'\left(x_{j-\frac{1}{2}}\right) \approx \frac{y_j - y_{j-1}}{h} = \delta y_{j-\frac{1}{2}},$$

and the mid-point rule for approximating integrals[1] we can approximate (1) with

$$T_h\left[y^h\right]= h\sum_{i=1}^{n}\sqrt{\frac{1+\left(\delta y_{j-\frac{1}{2}}\right)^2}{v_0^2-2gy_{j-\frac{1}{2}}}}, \tag{2}$$

where $y_0 = 0,\ y_n = b$. We call (2) the *Discretized Time Integral*. To simplify, we write $f(\bar{y})$ with $\bar{y} = (y_1, y_2, ..., y_{n-1})$ instead of $T_h\left[y^h\right]$. The discretized minimization problem is now:

$$\min_{y\in A} f(y_1, y_2, ..., y_{n-1}), \tag{3}$$

where $A = \left\{ y \in R^{n-1} : v_0^2 - 2gy_{j-\frac{1}{2}} > 0, 1 \le j \le n \right\}$.

In our calculations we used the method of steepest descent (MSD) for approximating a solution of (3). The (MSD) is given by the iteration:

$$\bar{y}_{k+1} = \bar{y}_k - \alpha_k \nabla f(\bar{y}_k),\ k = 0, 1, 2, ... \tag{4}$$

where $\alpha_k \ge 0$ is a line search parameter chosen to approximate the minimum of f along the line containing $\bar{y}_k$ with direction $-\nabla f(\bar{y}_k)$, and ∇f is the gradient (vector) of f.[3] For the function (2) the gradient is given by

$$\frac{\partial f(\bar{y})}{\partial y_j} = \frac{h}{2}\left[A_j (B_j + C_j) - A_{j+1}(B_{j+1} - C_{j+1})\right],\ 1 \le j \le n, \tag{5}$$

where

$$A_j = \left[\frac{1+\left(\delta y_{j-\frac{1}{2}}\right)-}{v_0^2 - 2gy_{j-\frac{1}{2}}}\right]^{-1/2},\ B_j = \frac{2\delta y_{j-\frac{1}{2}}}{h\left(v_0^2 - 2gy_{j-\frac{1}{2}}\right)},\ C_j = \frac{g\left(1+\left(\delta y_{j-\frac{1}{2}}\right)^2\right)}{\left(v_0^2 - 2gy_{j-\frac{1}{2}}\right)^2},\ 1 \le j \le n.$$

We developed some routines on MATLAB that implement the method (4) using (5). For the case $v_0 = 4, a = 6, \mathrm{b} = -1$, we show in Figure 2 the behavior of the errors and functional values in the (MSD), and in Figure 3 we show the approximate minimum computed by the (MSD). Note that the approximate minimum resembles very well a section of a cycloid.

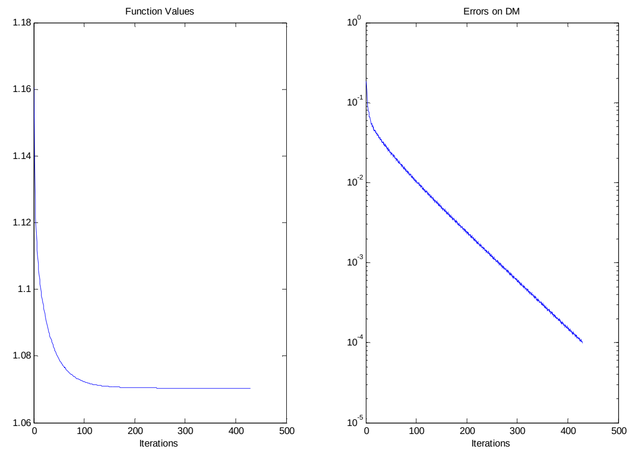

Figure 2. Function values and error iterations in the method of steepest descent.

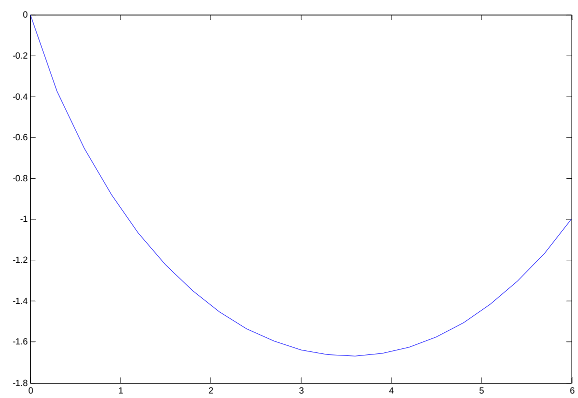

Figure 3. The minimum curve computed by the method of steepest descent.

3. The Brachistochrone Problem Over Surfaces

We study now the problem of the curve of quickest descent but over a surface. We assume that the surface S is given as a function of (x, y), that is

$$z = f(x, y), \qquad (x, y) \in D,$$

where $D \subset R^2$ is open a $f \in C^2$. A curve in S is given by $(\vec{u}(\tau), z(\tau)) = (x(\tau), y(\tau), z(\tau))$, $\tau \in [0,1]$ where

$$z(\tau) = f(\vec{u}(\tau)), \ t \in [0,1].$$

An argument similar to the one leading to equation (1) based on conservation of energy and the formula for arc length, shows that the time it takes for a particle to slide from $(\vec{a}, f(\vec{a}))$ to $(\vec{b}, f(\vec{b}))$, $\vec{a}, \vec{b} \in D$, under the influence of gravity with no friction, is given by the time integral:

$$T[\vec{u}] = \int_0^1 \sqrt{\frac{\|\vec{u}'(\tau)\|^2 + (\nabla f(\vec{u}(\tau)) \cdot \vec{u}'(\tau))^2}{\alpha - 2gf(\vec{u}(\tau))}}\, d\tau, \qquad (6)$$

where $\alpha = v_0^2 + 2gf(a)$, v_0 is the initial speed, and gravity is along the vertical z - direction.

The problem then is to find a function $\vec{u}(\tau)$ with $\vec{u}(0) = \vec{a}$ and $\vec{u}(1) = \vec{b}$ that minimizes the time integral (6).

3.1 The discretized problem

Let $h = 1/n, n \geq 1$ and define

$$\tau_j = jh, 0 \leq j \leq n, \quad \tau_{j-\frac{1}{2}} = \frac{\tau_{j-1} + \tau_j}{2} = \left(j - \frac{1}{2}\right)h, \qquad 1 \leq j \leq n.$$

We let $\vec{u}_j$ be an approximation of $\vec{u}(\tau_j)$, $0 \leq j \leq n.$ With the approximations

$$\vec{u}\left(\tau_{j-\frac{1}{2}}\right) \approx \frac{\vec{u}_{j-1} + \vec{u}_j}{2} = \vec{u}_{j-\frac{1}{2}}, \quad \vec{u}'\left(\tau_{j-\frac{1}{2}}\right) \approx \frac{\vec{u}_j - \vec{u}_{j-1}}{h} = \delta\vec{u}_{j-\frac{1}{2}},$$

and the mid-point rule for approximating integrals, we can approximate (6) with:

$$T_h = h \sum_{j=1}^{n} \sqrt{\frac{\left\|\delta\vec{u}_{j-\frac{1}{2}}\right\|^2 + \left(\nabla f\left(\vec{u}_{j-\frac{1}{2}}\right) \cdot \delta\vec{u}_{j-\frac{1}{2}}\right)^2}{\alpha - 2gf\left(\vec{u}_{j-\frac{1}{2}}\right)}}, \qquad (7)$$

where $\vec{u}_0 = a, \vec{u}_n = b.$ We call (7) the *Discretized Time Integral*. Note that T_h is a function of $\vec{y} = (\vec{u}_1, \vec{u}_2, ..., \vec{u}_{n-1})$, which we denote by $F(\vec{y})$. The minimization problem is given now by:

$$\min_A F(\vec{y}), \qquad (8)$$

where $A = \left\{(\vec{u}_1, \vec{u}_2, ..., \vec{u}_{n-1}) \in R^{2\times(n-1)} : \alpha - 2gf\left(\vec{u}_{j-\frac{1}{2}}\right) > 0,\ 1 \leq \mathrm{j} \leq \mathrm{n}\right\}.$

3.2. Numerical results

We developed some routines on MATLAB to study the numerical aspects of the Brachistochrone Problem Over Surfaces. These routines compute numerically the curve of minimum descent over a given surface. As in section (2.2) we used the method of steepest descent to find approximate solutions of (8). For the function $F(\bar{y})$, given by (7), the gradient vector is given by the following equations:

$$\frac{\partial f(y)}{\partial \bar{u}_j} = \frac{h}{2}\left[A_j\left(B_j^1 + B_j^2 + B_j^3 + C_j\right) - A_{j+1}\left(B_{j+1}^1 + B_{j+1}^2 + B_{j+1}^3 + C_{j+1}\right)\right],\ 1 \le j \le n-1,$$

where

$$A_j = \left[\frac{\left\|\delta\bar{u}_{j-\frac{1}{2}}\right\|^2 + \left(\nabla f\left(\bar{u}_{j-\frac{1}{2}}\right)\cdot\delta\bar{u}_{j-\frac{1}{2}}\right)^2}{\alpha - 2gf\left(\bar{u}_{j-\frac{1}{2}}\right)}\right]^{-1/2}, \quad B_j^1 = \frac{2\delta\bar{u}_{j-\frac{1}{2}}}{h\left(\alpha - 2gf\left(\bar{u}_{j-\frac{1}{2}}\right)\right)},$$

$$B_j^2 = \frac{\left(\nabla f\left(\bar{u}_{j-\frac{1}{2}}\right)\cdot\delta\bar{u}_{j-\frac{1}{2}}\right)H_f\left(\bar{u}_{j-\frac{1}{2}}\right)\delta\bar{u}_{j-\frac{1}{2}}}{\alpha - 2gf\left(\bar{u}_{j-\frac{1}{2}}\right)},\ B_j^3 = \frac{2\left(\nabla f\left(\bar{u}_{j-\frac{1}{2}}\right)\cdot\delta\bar{u}_{j-\frac{1}{2}}\right)\nabla f\left(\bar{u}_{j-\frac{1}{2}}\right)}{h\left(\alpha - 2gf\left(\bar{u}_{j-\frac{1}{2}}\right)\right)},$$

$$C_j = \frac{g\left(\left\|\delta\bar{u}_{j-\frac{1}{2}}\right\|^2 + \left(\nabla f\left(\bar{u}_{j-\frac{1}{2}}\right)\cdot\delta\bar{u}_{j-\frac{1}{2}}\right)^2\right)\nabla f\left(\bar{u}_{j-\frac{1}{2}}\right)}{\left(\alpha - 2gf\left(\bar{u}_{j-\frac{1}{2}}\right)\right)^2},\ 1 \le j \le n.$$

We consider the special case where S is a non-vertical plane, that is when

$$f(x,y) = ax + by,\ (x,y) \in R^2.$$

For the case where the initial and final points are given by (1, 2, 3) and (6, 2, 8), and the initial speed is $v_0 = 6$, we show in Figure 4 (left) the curves of minimum descent computed by the (MSD) for different initial angles of inclination. In Figure 4 (right) the curves are joined together to form a surface or envelope of curves of minimum descent.

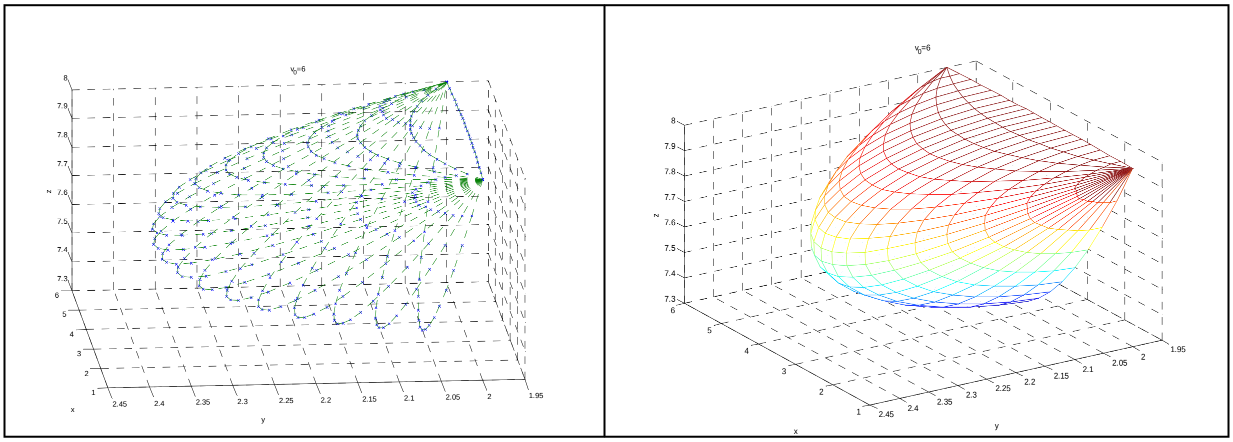

Figure 4. Curves of minimum descent (left). Envelope of the curves of minimum descent (right).

7. Comments and conclusions

Even though the Classical Brachistochrone Problem is a more than three hundred years old problem,. it is still a very interesting problem to study, both from the physical and mathematical point of views. This problem provides a playground to experiment with advanced mathematics like calculus of variations, numerical methods and simulations. For the Classical Problem we developed various numerical routines to solve the problem directly, and developed a GUI to experiment with different candidate curves. For the problem over a surface we developed as well various numerical routines to approximate the curves of minimum descent and constructed an envelope of curves of minimum descent. We do not know of any mathematical properties of this surface or envelope.

8. Acknowledgements

This work has been funded by the National Security Agency, Grant Num. H98230-04-C-0486, and by the National Science Foundation CSEMS program at the UPRH, Grant Num. 0123169.

9. References

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