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The remarkable nature of radially symmetric equilibrium states of aeolotropic nonlinearly elastic bodies

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Abstract. This paper treats the radially symmetric equilibrium states of aeolotropic nonlinearly elastic solid cylinders and balls under constant normal forces on their boundaries. It is shown that the aeolotropy gives rise to solutions describing both intact and cavitating states, which exhibit an array of remarkable new phenomena, not suggested by the solutions for isotropic bodies. E.g., it is shown that there are materials having a critical pressure such that for applied pressures on the boundary below the critical value, the normal pressures at the center of the body are zero and for applied pressures above the critical value, the normal pressures at the center are infinite. There are also materials for which there is no equilibrium state with center intact when the boundary is subjected to uniform tension. It is also shown that the equilibrium states treated here are the only radially symmetric equilibrium states. Thus the strange phenomena discovered here must be present in such stable equilibrium states.

Contents

1. Introduction

By 1960 a comprehensive theory of aeolotropic nonlinearly elastic bodies had been established and a variety of specific problems for them had been treated (cf. Green and Adkins [10]). But for the most part, these studies of the statics of compressible bodies left open important questions on the existence, multiplicity, and qualitative behavior of solutions.

In this paper we examine what are conceptually the simplest of such problems: the deformation under constant normal pressure or tension on the boundary of transversely isotropic solid cylinders and balls. Using the most elementary of mathematical techniques we are able to show that the solutions exhibit the most striking departures from the behavior of solutions for the equations for isotropic bodies. Not only do we obtain new nonexistence and multiplicity results, but we show that certain materials admit a family of remarkable bifurcations of so catastrophic a nature that the snap bucklings associated with the applications of catastrophe theory to structures seem innocuous by comparison. E.g., in Theorem 7.15 we show that there are materials having a critical pressure such that for applied pressures on the boundary below the critical value, the normal pressures at the center are zero, and for applied pressures above the critical value, the normal pressures at the center are infinite. In Theorem 7.24, we show that there are materials for which there is no equilibrium state with center intact when the boundary is subjected to a uniform tension. We also show that a jumping phenomenon can occur in certain cavitation problems.

The most difficult part of our elementary analysis is to establish a precise count of the number of possible equilibrium states. Since the solutions we study can exhibit pathological behavior, such results are necessary to show that these solutions are unavoidable. We also show how the cavitating solutions for aeolotropic bodies differ markedly from those for isotropic bodies, the theory of which was magisterially established by Ball [4]. An underlying theme of our work is that the properties of solutions to our problems do not depend stably upon the divergence of material symmetry from that for isotropic bodies.

As part of our analysis we show that the stress at the center of a compressed azimuthally reinforced body is zero and that that for a radially reinforced body is infinite. Results like these have been observed for various problems of linear aeolotropic elasticity [1, 15, 19]. It is reasonable to extrapolate the case of zero stress to nonlinear problems, but unreasonable to do so for the case of infinite stress. Here the very physical validity of the linear model is in question. In particular, an infinite compressive stress, if taken seriously, corresponds through linear stress–strain laws to an infinite deformation in which the orientation has been changed. There are numerous singularities for problems of linear elasticity that are absent in the corresponding problems of nonlinear elasticity [3]. Indeed, one of the main responsibilities of nonlinear elasticity is to clarify the nature of such singularities. Our results for the most general models of nonlinearly elastic response that respect the requirement that all deformations preserve orientation confirm that our infinite stresses are intrinsic for radially reinforced bodies, and not an artifact of a linearization of dubious validity.

We remark that aeolotropic bodies with the symmetries we study arise naturally. In the casting of metals, the temperature gradient in the freezing process causes molecules to line up in a way that creates a radially symmetric kind of transverse isotropy. (See the figures of [11] and [22].) The cross-section of a tree trunk clearly enjoys similar properties. Bodies with this kind of aeolotropy can also be manufactured by the suitable disposition of fibers in a matrix with different material properties. Such composites can be of great technological importance (cf. [16]).

2. Formulation of the governing equations

Let the unstressed reference configuration of a nonlinearly elastic body be either a solid cylinder of radius 1 or a ball of radius 1. We consider only axisymmetric deformations of the cylinder and spherically symmetric deformations of the ball in which material points with reference radius s are constrained to move along their rays a distance depending only on s. Let the radial distance of such a material point in a deformed configuration be denoted $\varrho(s)$. Then $\varrho'(s)$ is the radial stretch and $\varrho(s)/s$ is the azimuthal stretch. The shear strains with respect to polar coordinates are all zero.

We assume that the materials of these bodies have enough symmetry that the stresses enjoy the same symmetries as the deformations. E.g., for the ball undergoing the deformation just described there should be zero shear on any spherical surface centered at the center of the ball and on any plane passing through the center. Moreover, all normal stresses in any azimuthal direction for a given radius s should be equal in magnitude. Let $\hat{N}(s)$ and $\hat{T}(s)$ be the normal Piola-Kirchhoff stresses of the first kind at the radius s in the radial and azimuthal directions, respectively. If the boundary of the body is subjected to a uniform normal force of intensity $\lambda g(\varrho(1))$ per unit reference area, then the equilibrium equations expressing the balance of forces on an arbitrary sector of the body (lying between cylinders or spheres of radii s and 1 and bounded by rays) are

$$s^{\alpha}\hat{N}(s) = \lambda g(\varrho(1)) - \int_s^1 \alpha t^{\alpha-1}\hat{T}(t)\,\mathrm{d}t, \tag{2.1}$$

where $\alpha = 1$ for cylinders and $\alpha = 2$ for spheres. We assume that

$$g(\varrho) > 0, \quad g'(\varrho) \geqslant 0 \quad \text{for} \quad \varrho > 0. \tag{2.2}$$

We shall be particularly concerned with dead loads, for which $g(\varrho) = 1$, with hydrostatic loads for cylinders, for which $g(\varrho) = \varrho$, and with hydrostatic loads for spheres, for which $g(\varrho) = \varrho^2$. λ is a constant accounting for the magnitude of the loads. $\lambda < 0$ for compression.

We assume that the material of these bodies is homogeneous and nonlinearly elastic, with the symmetry properties mentioned above. E.g., the ball could be transversely isotropic with respect to polar coordinates. We accordingly assume that there are functions

$$(0, \infty) \times (0, \infty) \ni (\tau, v) \mapsto T(\tau, v), N(\tau, v) \in (-\infty, \infty) \tag{2.3}$$

such that

$$\hat{T}(s) = T(\varrho(s)/s, \varrho'(s)), \hat{N}(s) = N(\varrho(s)/s, \varrho'(s)). \tag{2.4}$$

We assume that T and N are twice continuously differentiable.

We require that (T, N) satisfy the (specialization to our class of deformations of the) *strong ellipticity condition*:

$$T_\tau > 0, \quad N_v > 0 \tag{2.5a, b}$$

and a corresponding set of growth conditions:

$$T(\tau, v) \to \begin{Bmatrix} \infty \\ -\infty \end{Bmatrix} \text{ as } \tau \to \begin{Bmatrix} \infty \\ 0 \end{Bmatrix} \text{ if } v \begin{Bmatrix} \text{has a positive lower bound} \\ \text{is bounded from above} \end{Bmatrix}, \tag{2.6a}$$

$$N(\tau, v) \to \begin{Bmatrix} \infty \\ -\infty \end{Bmatrix} \text{ as } v \to \begin{Bmatrix} \infty \\ 0 \end{Bmatrix} \text{ if } \tau \begin{Bmatrix} \text{has a positive lower bound} \\ \text{is bounded from above} \end{Bmatrix}. \tag{2.6b}$$

We describe refinements of these restrictions in the next section.

Our governing equations are obtained by substituting (2.4) into (2.1). We supplement these by requiring that either the axis of the cylinder or the center of the ball remain intact, in which case

$$\varrho(0) = 0, \tag{2.7}$$

or else cavitation occurs, in which case

$$\varrho(0) > 0, \lim_{s\to 0} N(\varrho(s)/s, \varrho'(s)) = 0. \tag{2.8}$$

Note that N is a Piola–Kirchhoff stress, measuring force per unit reference area. The corresponding Cauchy stress, measuring force per actual area is $(s/\varrho)^\alpha N$. Cavitation is characterized by the vanishing of N at $s = 0$. (A bounded Piola–Kirchhoff stress can produce a zero Cauchy stress at $s = 0$).

If ϱ is restricted to a suitable function space for which ϱ' is defined almost everywhere and for which $s \mapsto s^{\alpha-1}T(\varrho(s)/s, \varrho'(s))$ is integrable on any closed subset of $(0, 1]$, then a standard bootstrap procedure (like that used in the direct methods of the calculus of variations) based upon (2.5b) and (2.6b) shows that any solution ϱ of (2.1), (2.4) in the function space is in fact in $C^2((0, 1])$ and satisfies the classical form of the equilibrium equations:

$$\frac{\mathrm{d}}{\mathrm{d}s}[s^\alpha N(\varrho(s)/s, \varrho'(s))] = \alpha s^{\alpha-1}T(\varrho(s)/s, \varrho'(s)) \tag{2.9}$$

on $(0, 1]$, and the boundary condition

$$N(\varrho(1), \varrho'(1)) = \lambda g(\varrho(1)). \tag{2.10}$$

Near $s = 0$ the effects of aeolotropy become focused. We accordingly refrain from imposing a priori restrictions on the regularity of ϱ here.

Note that the problem for the cylinder is one of plane strain. It is mathematically equivalent to the plane-stress problem for a disk.

3. Constitutive restrictions

We assume that the reference state is stress-free so that

$$T(1, 1) = 0 = N(1, 1). \tag{3.1}$$

Let $T_\tau^0 \equiv T_\tau(1, 1)$, etc. We assume that

$$\begin{pmatrix} T_\tau^0 & T_\nu^0 \\ N_\tau^0 & N_\nu^0 \end{pmatrix} \text{ is positive-definite.} \tag{3.2}$$

For the restricted class of deformations we consider, the material is *isotropic* if and only if

$$T(\tau, \nu) = N(\nu, \tau). \tag{3.3}$$

The material is said to be *azimuthally reinforced at* τ if

$$T_\tau(\tau, \tau) + T_\nu(\tau, \tau) > N_\tau(\tau, \tau) + N_\nu(\tau, \tau) \tag{3.4}$$

and *radially reinforced* at τ if the contrary inequality holds:

$$N_\tau(\tau, \tau) + N_\nu(\tau, \tau) > T_\tau(\tau, \tau) + N_\nu(\tau, \tau). \tag{3.5}$$

The material is *azimuthally reinforced* or *radially reinforced* if (3.4) or (3.5) hold for all $\tau \in (0, \infty)$.

In much of our analysis we employ the assumption that (T, N) is *monotone*:

$$\begin{pmatrix} T_\tau & T_\nu \\ N_\tau & N_\nu \end{pmatrix} \text{ is positive-definite.} \tag{3.6}$$

(This condition, obviously more stringent than (2.5), is a degenerate version of the Coleman-Noll inequality, cf. Truesdell and Noll [21].)

The following result is standard:

3.7. THEOREM. *If* (3.6) *holds and if* $\lambda \leqslant 0$, *then there is at most one solution* ϱ *in* $C^2((0, 1])$ *to the boundary value problem* (2.9), (2.10), (2.7) *for which* $s \mapsto s^\alpha N(\varrho(s)/s, \varrho'(s))$ *is bounded.*

A further set of restrictions corresponding to the usual effects associated with the Poisson ratio of linear elasticity is

$$T_\nu > 0, \quad N_\tau > 0. \tag{3.8a, b}$$

Further growth conditions compatible with (3.6) and (3.8) are

$$T(\tau, \nu) \to \left\{ \begin{matrix} \infty \\ -\infty \end{matrix} \right\} \text{ as } \nu \to \left\{ \begin{matrix} \infty \\ 0 \end{matrix} \right\} \text{ if } \tau \left\{ \begin{matrix} \text{has a positive lower bound} \\ \text{is bounded from above} \end{matrix} \right\}, \tag{3.9a}$$

$$N(\tau, \nu) \to \begin{Bmatrix} \infty \\ -\infty \end{Bmatrix} \text{ as } \tau \to \begin{Bmatrix} \infty \\ 0 \end{Bmatrix} \text{ if } \nu \begin{Bmatrix} \text{has a positive lower bound} \\ \text{is bounded from above} \end{Bmatrix}. \tag{3.9b}$$

The material is hyperelastic if there is a real valued *stored-energy density function* $(\tau, \nu) \mapsto \Phi(\tau, \nu)$ such that

$$T = \Phi_\tau, \quad N = \Phi_\nu. \tag{3.10}$$

Since the eminently reasonable assumption of hyperelasticity is of scant mathematical advantage in our analysis, we shall not adopt it in general. Note that (3.10) reduces (3.4) to

$$T_\tau > N_\nu. \tag{3.11}$$

reduces (3.6) to the requirement that Φ be convex, and reduces (3.8) to a single inequality.

Both to show that the various conditions we have imposed are not inconsistent and to have a class of materials for which we can make specific computations, we consider

$$\Phi(\tau, \nu) = \frac{A_1 \tau^{1-a_1}}{a_1 - 1} + \frac{A_2 \nu^{1-a_2}}{a_2 - 1} + \frac{B_1 \tau^{1+b_1}}{1 + b_1} + \frac{B_2 \nu^{1+b_2}}{1 + b_2} + C\tau^{-c_1}\nu^{-c_2}$$
$$+ D\tau^{d_1}\nu^{d_2} + E_1\tau^{-e_1}\nu^{f_1} + E_2\tau^{f_2}\nu^{-e_2}. \tag{3.12}$$

Here $A_1, A_2, B_1, B_2, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2 > 0$ and $C, D, E_1, E_2 \geqslant 0$. The corresponding functions T, N of (3.10) clearly satisfy the strong ellipticity condition (2.5) and the growth conditions (2.6), (3.9). The first five terms on the right-hand side of (3.12) define convex functions; the last three do not. Thus we can adjust parameters so that (3.10), (3.11) violate (3.6). Conversely, by studying (3.12) separately on each of the four rectangles of (τ, ν)-space bounded by the lines $\tau = 1$, $\nu = 1$, by grouping terms of the right-hand side of (3.12) appropriately on each of these quadrants (e.g., by grouping the third, fourth, and sixth term on the first, second, and fourth quadrants), and by applying the inequality $2xy \leqslant x^2 + y^2$ (or, more generally, Young's inequality), we can prove for suitable ranges of parameters that these groupings and consequently Φ are convex. Condition (3.8) is satisfied if

$$E_1 = 0 = E_2.$$

A reasonable further requirement to impose on solutions of our boundary value problems is that $[0, 1] \ni s \mapsto \Phi(\varrho(s)/s, \varrho'(s))$ is integrable, or equivalently, that the total stored energy of the body is finite.

4. A conjugate formulation

Conditions (2.5b), (2.6b) ensure that $N(\tau, \cdot)$ has an inverse, which we denote by $v^{\#}(\tau, \cdot)$. We set

$$T^{\#}(\tau, n) = T(\tau, v^{\#}(\tau, n)). \tag{4.1}$$

Then we can write the quasilinear equation (2.9) as the equivalent semilinear system

$$\varrho' = v^{\#}(\varrho/s, \hat{N}), \tag{4.2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}[s^{\alpha}\hat{N}] = \alpha s^{\alpha-1}T^{\#}(\varrho/s, \hat{N}). \tag{4.2b}$$

The strong ellipticity condition (2.5) becomes

$$T^{\#}_{\tau} v^{\#}_{n} - T^{\#}_{n} v^{\#}_{\tau} > 0, \quad v^{\#}_{n} > 0, \tag{4.3a, b}$$

and the monotonicity condition (3.6) becomes

$$\begin{pmatrix} T^{\#}_{\tau} v^{\#}_{n} - T^{\#}_{n} v^{\#}_{\tau} & T^{\#}_{n} \\ -v^{\#}_{\tau} & 1 \end{pmatrix} \text{ is positive-definite.} \tag{4.4a}$$

Condition (4.4a) says that the symmetric part of the matrix of (4.4a) is positive-definite. A simple argument then shows that the determinant of the matrix of (4.4a) is positive, i.e., that

$$T^{\#}_{\tau} > 0. \tag{4.4b}$$

Condition (3.8) becomes

$$T^{\#}_{n} > 0, \quad v^{\#}_{\tau} < 0. \tag{4.5a, b}$$

The analogs of the growth conditions (2.6) and (3.9) are

$$T^{\#}(\tau, n) \to \begin{Bmatrix} \infty \\ -\infty \end{Bmatrix} \text{ as } \tau \to \begin{Bmatrix} \infty \\ 0 \end{Bmatrix} \text{ if } n \text{ is bounded from } \begin{Bmatrix} \text{below} \\ \text{above} \end{Bmatrix}, \tag{4.6a}$$

$$v^{\#}(\tau, n) \to \begin{Bmatrix} \infty \\ 0 \end{Bmatrix} \text{ as } n \to \begin{Bmatrix} \infty \\ -\infty \end{Bmatrix} \text{ if } \begin{Bmatrix} \text{is bounded from above} \\ \text{has a positive lower bound} \end{Bmatrix}, \tag{4.6b}$$

$$T^{\#}(\tau, n) \to \pm\infty \text{ as } n \to \pm\infty \text{ if } \tau \begin{Bmatrix} \text{has a positive lower bound} \\ \text{is bounded from above} \end{Bmatrix}, \tag{4.6c}$$

$$v^{\#}(\tau, n) \to \begin{Bmatrix} \infty \\ 0 \end{Bmatrix} \text{ as } \tau \to \begin{Bmatrix} \infty \\ 0 \end{Bmatrix} \text{ if } n \text{ is bounded from } \begin{Bmatrix} \text{below} \\ \text{above} \end{Bmatrix}. \tag{4.6d}$$

Condition (3.1) is equivalent to

$$T^{\#}(1, 0) = 0, \quad v^{\#}(1, 0) = 1. \tag{4.7}$$

If the material is hyperelastic, then we can introduce a conjugate energy Ψ as the Legendre transform of Φ given by

$$\Psi(\tau, n) = nv^{\#}(\tau, n) - \Phi(\tau, v^{\#}(\tau, n)). \tag{4.8}$$

Then

$$T^{\#}(\tau, n) = -\Psi_{\tau}(\tau, n), \quad v^{\#}(\tau, n) = \Psi_{n}(\tau, n). \tag{4.9}$$

Since it is generally difficult to get an explicit representation for the inverse of $N(\tau, \cdot)$ (cf. (3.12)), it is no easy task to construct an explicit form for Ψ that satisfies the analogs of the various subsidiary conditions we have imposed on Φ.

5. Autonomous equations

Let

$$s = e^{\xi}, \quad \varrho(s) = s\tau(\ln s), \quad \hat{N}(s) = n(\ln s). \tag{5.1}$$

Let a superposed dot denote the derivative with respect to ξ. Then (2.9) and (4.2) are respectively converted into the autonomous equations

$$\frac{d}{d\xi} N(\tau(\xi), \tau(\xi) + \dot{\tau}(\xi)) = \alpha\{T(\tau(\xi), \tau(\xi) + \dot{\tau}(\xi)) - N(\tau(\xi), \tau(\xi) + \dot{\tau}(\xi)\}, \tag{5.2}$$

$$\dot{\tau} = v^{\#}(\tau, n) - \tau, \tag{5.3a}$$

$$\dot{n} \doteq \alpha[T^{\#}(\tau, n) - n] \tag{5.3b}$$

for $-\infty < \xi < 0$. The boundary conditions (2.10), (2.7), (2.8) become

$$n(0) = \lambda g(\tau(0)), \tag{5.4}$$

$$\lim_{\xi \to -\infty} e^{\xi}\tau(\xi) = 0, \tag{5.5}$$

$$\lim_{\xi \to -\infty} e^{\xi}\tau(\xi) > 0, \quad \lim_{\xi \to -\infty} n(\xi) = 0. \tag{5.6a, b}$$

(Since ϱ' is required to be positive, it follows that $\lim_{s\to 0} \varrho(s)$ exists, so that the limits in (5.5) and (5.6a) exist.)

Note that the system obtained by substituting (4.9) into (5.3) is *not* Hamiltonian and accordingly cannot be expected to yield an integral or to have the typical kinds of critical points of such systems. Nevertheless the level curves of Ψ and of related functions can be of help in determining the phase portrait of (5.3).

6. Phase portraits

We now determine the phase portraits of (5.3). Let

$$\mathscr{U} \equiv \{(\tau, n): 0 < \tau, -\infty < n < \infty\}. \tag{6.1}$$

The *phase space* $\bar{\mathscr{U}}$ for (5.3) is the union of the closure of $\mathscr{U}$ and the point at infinity:

$$\bar{\mathscr{U}} \equiv \{(\tau, n): 0 \leqslant \tau \leqslant \infty, -\infty \leqslant n \leqslant \infty\}. \tag{6.2}$$

($\bar{\mathscr{U}}$ corresponds to the closure of the stereographic image of $\mathscr{U}$ on a sphere tangent to the (τ, n)-plane.) We introduce the open quadrants

$$\mathscr{Q}_1 \equiv \{(\tau, n): 1 < \tau, 0 < n\}, \mathscr{Q}_2 \equiv \{(\tau, n): 0 < \tau < 1, 0 < n\}, \tag{6.3}$$

$$\mathscr{Q}_3 \equiv \{(\tau, n): 0 < \tau < 1, n < 0\}, \mathscr{Q}_4 \equiv \{(\tau, n): 1 < \tau, n < 0\}.$$

The *vertical isocline* $\mathscr{v}$ is the curve formed of the set of points (τ, n) for which

$$v^{\#}(\tau, n) = \tau, \text{ or equivalently, } n = N(\tau, \tau). \tag{6.4}$$

Note that (2.6b) ensures that

$$N(\tau, \tau) \to \begin{Bmatrix} \infty \\ -\infty \end{Bmatrix} \text{ as } \tau \to \begin{Bmatrix} \infty \\ 0 \end{Bmatrix}. \tag{6.5}$$

We assume that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} N(\tau, \tau) \equiv N_\tau(\tau, \tau) + N_v(\tau, \tau) > 0 \quad \text{for all } \tau. \tag{6.6}$$

This restriction is ensured by (2.5b) and (3.8b). It is also ensured by (3.6) for isotropic materials. The *horizontal isocline* $\mathscr{h}$ is the set of points for which

$$T^{\#}(\tau, n) = n, \text{ or equivalently, } T(\tau, v^{\#}(\tau, n)) = N(\tau, v^{\#}(\tau, n)). \tag{6.7a, b}$$

We shall assume that $\mathscr{h}$ is a single curve. We shall shortly provide conditions ensuring that this is so.

Singular points of (5.3) in $\mathscr{U}$ are points at which $\mathscr{v}$ and $\mathscr{h}$ intersect and thus are points (τ, n) with

$$T(\tau, \tau) = N(\tau, \tau) = n. \tag{6.8a, b}$$

(Other candidates for singular points in $\bar{\mathscr{U}}$ are points on the boundary $\tau = 0$ and points at infinity, at which the direction fields are not immediately apparent from (5.3). These points require separate study.) If the material of the body is isotropic, then (6.8a) is an identity, by (3.3). Thus $\mathscr{v} = \mathscr{k}$ is an entire curve of singular points. Condition (3.1), or its equivalent (4.7), ensures that (1, 0) is always a singular point. If the material is either azimuthally reinforced or radially reinforced, then (3.4), (3.5) show that this is the only singular point in $\mathscr{U}$. Let $\mathscr{S}$ be the open region lying between $\mathscr{k}$ and $\mathscr{v}$. Let $\mathscr{L}$ be the open region lying to the left of both $\mathscr{k}$ and $\mathscr{v}$ and let $\mathscr{R}$ be the open region lying to the right of both $\mathscr{k}$ and $\mathscr{v}$. We illustrate these regions in Fig. 6.9. We define $\bar{\mathscr{L}}$, etc., just as we defined $\bar{\mathscr{U}}$ in (6.2): $\bar{\mathscr{L}}$ is the union of $\mathscr{L}$, the extended n-axis, namely $\{(0, n)\colon -\infty \leqslant n \leqslant \infty\}$, the parts of $\mathscr{k}$ and $\mathscr{v}$ forming the boundary of $\mathscr{L}$, and the point(s) at ∞ of the form (τ, ∞). The collection of all trajectories lying entirely within $\mathscr{S}$ is denoted $\mathscr{s}$. The collection of all trajectories terminating at (1, 0) that do not touch $\mathscr{S}$ is denoted $\mathscr{e}$. (We shall show that $\mathscr{e}$ consists of a pair of trajectories approaching (1, 0).)

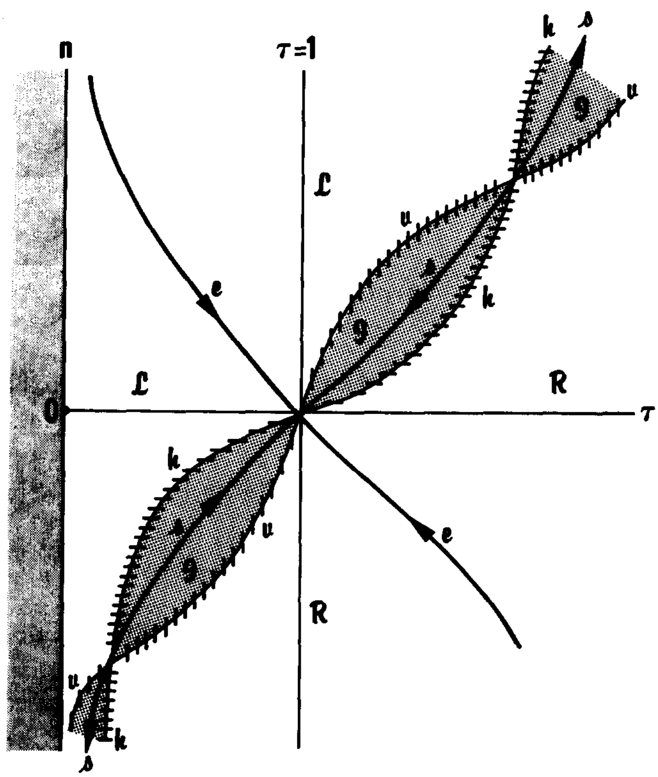

Fig. 6.9.

We now linearize (5.3) about the singular point (1, 0) obtaining

$$N_\nu^0 \delta\dot{\tau} = -(N_\tau^0 + N_\nu^0)\delta\tau + \delta n, \tag{6.10a}$$

$$N_\nu^0 \delta\dot{n} = \alpha(T_\tau^0 N_\nu^0 - T_\nu^0 N_\tau^0)\delta\tau + \alpha(T_\nu^0 - N_\nu^0)\delta n. \tag{6.10b}$$

The eigenvalues $\lambda^\pm N_\nu^0$ of the coefficient matrix of the right-hand side of (6.10) are given by

$$2\lambda^\pm N_\nu^0 = -b \pm [b^2 + 4\alpha N_\nu^0(T_\tau^0 + T_\nu^0 - N_\tau^0 - N_\nu^0)]^{1/2} \tag{6.11a}$$

$$= -b \pm [c^2 + 4\alpha(T_\tau^0 N_\nu^0 - T_\nu^0 N_\tau^0)]^{1/2}, \tag{6.11b}$$

$$b \equiv (\alpha + 1)N_\nu^0 + N_\tau^0 - \alpha T_\nu^0, \quad c \equiv N_\tau^0 + \alpha T_\nu^0 + (1 - \alpha)N_\nu^0. \tag{6.11c}$$

If the material is azimuthally reinforced at $\tau = 1$, then either (6.11a) or (6.11b) supported by (3.2) shows that (1, 0) is a saddle point of (6.10) with separatrices along the lines

$$2n = \{c \pm [c^2 + 4\alpha(T_\tau^0 N_\nu^0 - T_\nu^0 N_\tau^0)]^{1/2}\}(\tau - 1), \tag{6.12$\pm$}$$

the plus sign corresponding to the unstable manifold. Since (3.2) holds, (6.12+) has positive slope and (6.12−) has negative slope. If the material is radially reinforced at $\tau = 1$, then (1, 0) is an attractive improper node for (6.10) with all trajectories except two approaching it along the line (6.12+), the exceptional two approaching along (6.12−).

Since our constitutive functions have been assumed to be twice continuously differentiable, most of these properties are conserved by the phase portrait of (5.3): If the material is azimuthally reinforced at $\tau = 1$, then a pair of separatrices, forming the intersection of $\mathscr{s}$ with the connected components of $\mathscr{I}$ closest to (1, 0), issue from (1, 0) into $\mathscr{Q}_1$ and $\mathscr{Q}_3$ along (6.12+) and another pair forming $\mathscr{e}$ enter (1, 0) from $\mathscr{Q}_2$ and $\mathscr{Q}_4$ along (6.12−). No other trajectories touch (1, 0). If the material is radially reinforced at $\tau = 1$, then a pair of trajectories, forming $\mathscr{e}$, enter (1, 0) from $\mathscr{Q}_2$ and $\mathscr{Q}_4$ along the exceptional direction (6.12−). These are the only trajectories approaching (1, 0) along (6.12−). All other trajectories passing near (1, 0) enter it via $\mathscr{Q}_1$ and $\mathscr{Q}_3$ along (6.12+). Among these are the trajectories belonging to $\mathscr{s}$, which lies entirely in $\mathscr{I}$. Since it will soon be evident that each connected component of $\mathscr{I}$ is positively invariant (so that no trajectory ever entering such a component ever leaves it), the intersection of $\mathscr{a}$ with each such component is nonempty. But in contrast to our results about $\mathscr{a}$ for bodies azimuthally reinforced at $\tau = 1$ and our results for $\mathscr{e}$, we do not know whether the intersections of $\mathscr{a}$ with the two connected components of $\mathscr{I}$ closest to (1, 0) consist of single trajectories. These same remarks apply to the saddles and nodes resulting from transversal crossings of $\mathscr{v}$ and $\mathscr{h}$. (For a discussion of results used in this paragraph cf. Friedrichs [8, Section III.4], e.g.)

Let v be the inverse of $\tau \mapsto N(\tau, \tau)$, which exists by virtue of (6.5) and (6.6). Then $\mathscr{v}$ is set of points (τ, n) for which

$$\tau = v(n). \tag{6.13}$$

Now assume that (4.4b) holds. Then (4.6a) ensures that $T^{\#}(\cdot, n)$ is invertible so that (6.7) is equivalent to an equation of the form

$$\tau = h(n), \quad -\infty < n < \infty. \tag{6.14}$$

(Hence $\mathscr{h}$ is a curve.) If the material is $\left\{\begin{matrix}\text{azimuthally}\\ \text{radially}\end{matrix}\right\}$ reinforced (every where), then

$$[h(n) - v(n)]\, n \left\{\begin{matrix} < \\ > \end{matrix}\right\} 0 \text{ for } n \neq 0. \tag{6.15}$$

In this case the global arrangement of the isoclines is dictated by their behavior near (1, 0).

Now conditions (4.3b) and (4.7) ensure that

$$[v^{\#}(1, n) - 1]n > 0 \quad \text{for} \quad n \neq 0. \tag{6.16}$$

Thus (5.3a) implies that trajectories touching the line $\{(1, n): n > 0\}$ cross it transversally from $\mathscr{Q}_2$ to $\mathscr{Q}_1$ and that trajectories touching the line $\{(1, n): n < 0\}$ cross it transversally from $\mathscr{Q}_4$ to $\mathscr{Q}_3$. If (4.4b) holds, then we likewise obtain

$$(\tau - 1)T^{\#}(\tau, 0) > 0 \quad \text{for} \quad \tau \neq 1. \tag{6.17}$$

Thus trajectories touching the segment $\{(\tau, 0): 0 < \tau < 1\}$ cross it transversally from $\mathscr{Q}_2$ to $\mathscr{Q}_3$ and trajectories touching the line $\{(\tau, 0): \tau > 1\}$ cross it transversally from $\mathscr{Q}_4$ to $\mathscr{Q}_1$. Hence, (6.16), (6.17) imply that $\mathscr{Q}_1$ and $\mathscr{Q}_3$ are

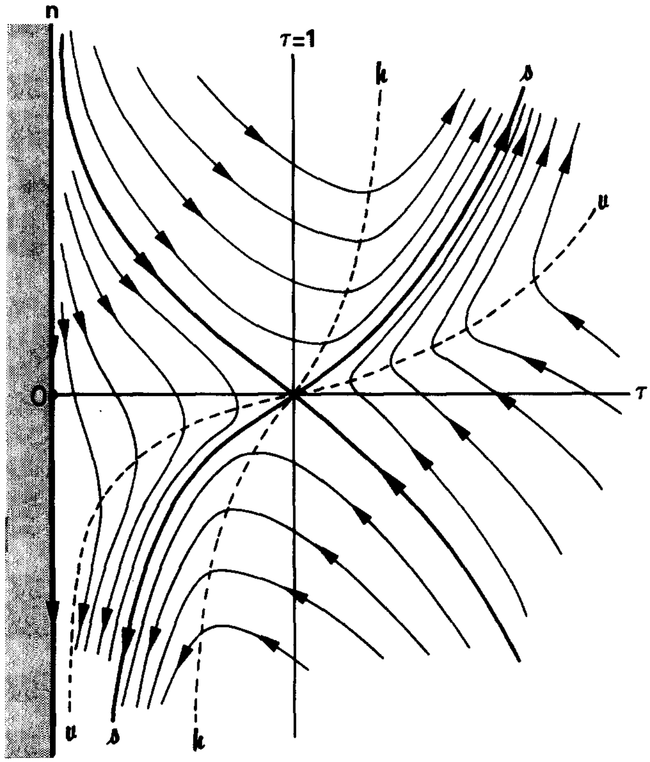

Fig. 6.18. Phase portrait when $\tau = 1$ is azimuthally reinforced and (1, 0) is the only singular point in $\mathscr{U}$. A sufficient condition for this to hold is that the material be azimuthally reinforced. The behavior of trajectories for τ near 0 is discussed in Section 9.

positively invariant regions. The orientation of the vector fields in $\mathscr{L}$ and $\mathscr{R}$ show that they are negatively invariant regions. Thus the phase portraits have the character indicated in Figs 6.18–6.21. The actual portraits shown are based on the further reasonable assumptions that

$$[T^{\#}(1, n) - n]n < 0 \quad \text{for} \quad n \neq 0, \tag{6.22}$$

$$[v^{\#}(\tau, 0) - \tau](\tau - 1) < 0 \quad \text{for} \quad \tau \neq 0. \tag{6.23}$$

An alternative to (6.22), (6.23), having similar effect, is

$$h'(n) \equiv \frac{1 - T_n^{\#}}{T_{\tau\#}} \equiv \frac{N_v - T_v}{T_\tau N_v - T_v N_\tau} > 0. \tag{6.24}$$

The portraits shown satisfy (6.24). Note that (4.6a, c) imply that $h(n) \to 0$ as $n \to -\infty$. Figures 6.20 and 6.21 are merely representative of what can happen if the kind of reinforcement changes with strain.

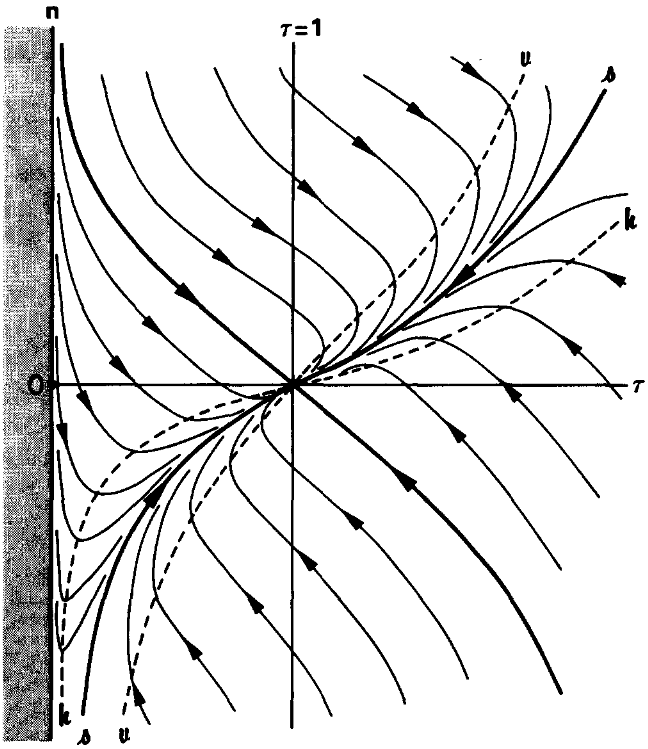

Fig. 6.19. Phase portrait when $\tau = 1$ is radially reinforced and is the only singular point in $\mathscr{U}$. A sufficient condition for this to hold is that the material be radially reinforced.

7. Solutions with the center intact

We now seek solutions of (5.3)–(5.5). Let ℓ denote the curve whose equation is $n = \lambda g(\tau)$. Since ξ ranges from $-\infty$ to 0, such a solution corresponds to a trajectory in $\mathscr{U}$ terminating on ℓ and originating from a singular point in $\mathscr{U}$ or possibly from a point on the line $\tau = 0$ or from a point at ∞. Such a solution must satisfy (5.5), which is automatic if τ is bounded.

Since (5.3) is autonomous, it is invariant under shifts of ξ. Therefore, if a trajectory originates at a point, such as a singular point, for which $\xi = -\infty$, then any point on the trajectory with ξ finite may be regarded as corresponding to $\xi = 0$. For this reason problems for solid cylinders and balls are more easily handled in the present context than those for tubes and shells. In these latter cases, solutions correspond to trajectories for which ξ must range over an internal of fixed finite length. (Cf. Sivaloganathan [18]).

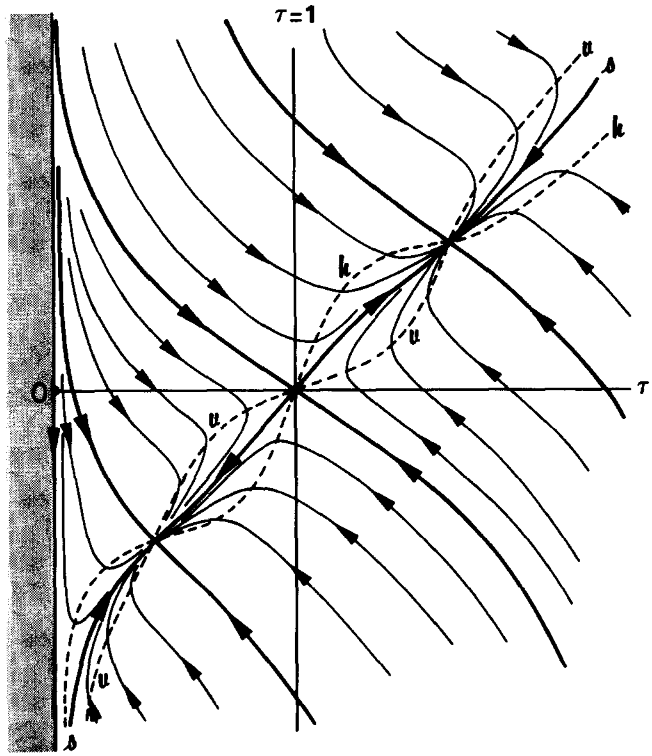

Fig. 6.20. Phase portrait when $\tau = 1$ is azimuthally reinforced and $\mathscr{v}$ and $\mathscr{k}$ have exactly two other intersections, which are transversal, with one lying in $\mathscr{D}_1$ and the other in $\mathscr{D}_3$.

Before attacking (5.3)–(5.5) for aeolotropic bodies, let us pause to study the problem for isotropic bodies so that the results for each class of problems can be readily contrasted. As we noted after (6.8), for an isotropic body $\mathscr{v} = \mathscr{s} = \mathscr{k} = \mathscr{S}$ is an entire curve of singular points. If $\lambda \leqslant 0$, then $\mathscr{s}$ intersects ℓ at a single point, which corresponds to a deformation of the form $\varrho(s) = c(\lambda)s$. Theorem 3.7 gives conditions for the uniqueness of this solution. If $\lambda > 0$ and if λ is sufficiently small, then $\mathscr{s}$ still intersects ℓ uniquely. If $g(\tau) = 1$, then $\mathscr{s}$ intersects ℓ uniquely for all λ. But if $g(\tau) = \tau$ or $g(\tau) = \tau^2$ (see the remarks following (2.2)), then for different materials there may be none, one, or many such intersections. For $\lambda > 0$ there can also be cavitating solutions. These have been beautifully treated in [4, 12, 14, 18, 20] for isotropic bodies. We study cavitation for aeolotropic bodies in Section 8. The behavior of isotropic bodies can be readily ascertained by taking the limiting case of our results.

In this section we show that problem (5.3)–(5.5) admits a solution if a trajectory in $\mathscr{s}$ satisfying (5.5) terminates at ℓ. Except in a couple of cases in

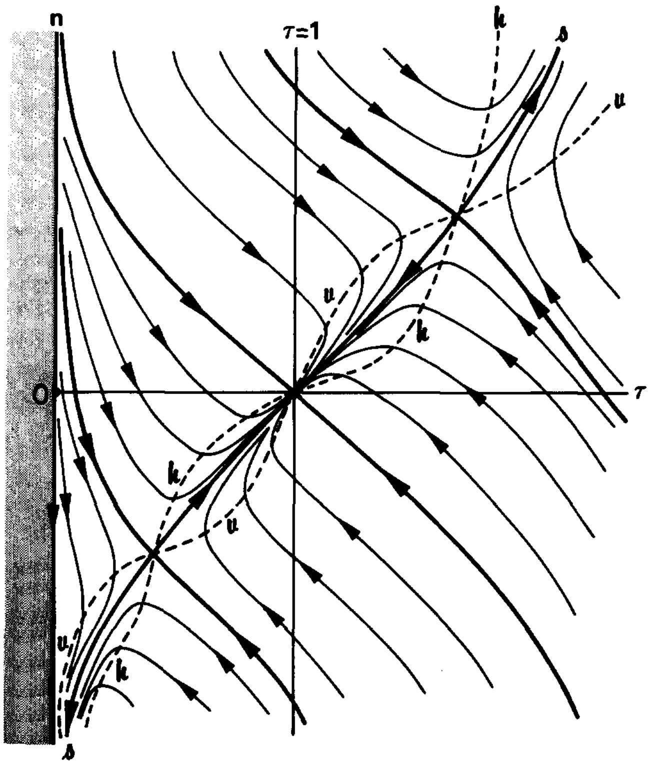

Fig. 6.21. Phase portrait when $\tau = 1$ is radially reinforced and $\mathscr{u}$ and $\mathscr{k}$ have exactly two other intersections, which are transversal, with one lying in $\mathscr{D}_1$ and the other in $\mathscr{D}_3$.

which the verification of (5.5) is delicate, this demonstration follows immediately from an examination of the phase portrait for the material under study. The phase portrait also yields the properties of such solutions. We find that there are instances in which these solutions (i) are relatively innocuous, (ii) exhibit surprising features, or (iii) are nonexistent. To show that there are no other solutions of a more interesting kind (say, with strange states of stress locked in) when (i) holds, that there are no other solutions of a less pathological nature when (ii) holds, and that there are no other solutions at all when (iii) holds, it behooves us to show that there can be no other kinds of solutions.

Other candidates for solutions have other trajectories terminating at ℓ. Because $\bar{\mathscr{I}}$ is positively invariant, these other trajectories must come from $\mathscr{L}$ or $\mathscr{R}$. We begin our analysis in this section by demonstrating that trajectories coming from $\mathscr{L}$ or $\mathscr{R}$ cannot generate solutions to (5.3)–(5.5).

In $\mathscr{L}$, $\dot{\tau} > 0$, $\dot{n} < 0$. If (4.3b) and (4.5b) hold, then

$$\frac{d}{d\xi} v^{\#}(\tau(\xi), n(\xi)) < 0 \tag{7.1}$$

for trajectories lying in $\mathscr{L}$. Since every point in $\mathscr{L}$ lies above ν, we have

$$n > N(\tau, \tau), \text{ or equivalently, } v^{\#}(\tau, n) > \tau \quad \text{for} \quad (\tau, n) \in \mathscr{L}. \tag{7.2}$$

Let $-\infty < \theta < 0$ and let $(\tau(\theta), n(\theta)) \in \mathscr{L}$. Since $\mathscr{L}$ is negatively invariant, the trajectory terminating at $(\tau(\theta), n(\theta))$ lies entirely in $\mathscr{L}$. Inequalities (7.1) and (7.2) imply that on this trajectory

$$v^{\#}(\tau(\xi), n(\xi)) \geqslant v^{\#}(\tau(\theta), n(\theta)) > \tau(\theta) \quad \text{for} \quad \xi \leqslant \theta. \tag{7.3a, b}$$

Substituting (7.3a) into (5.3a) we obtain

$$\dot{\tau} \geqslant v^{\#}(\tau(\theta), n(\theta)) - \tau, \text{ or equivalently, } \frac{d}{d\xi}(e^{\xi}\tau) \geqslant v^{\#}(\tau(\theta), n(\theta))e^{\xi}. \tag{7.4a, b}$$

Integrating (7.4b) from η to θ, we obtain

$$e^{\eta}\tau(\eta) \leqslant e^{\theta}[\tau(\theta) - v^{\#}(\tau(\theta), n(\theta))] + v^{\#}(\tau(\theta), n(\theta))e^{\eta} \quad \text{for} \quad \eta < \theta \tag{7.5}$$

on this trajectory. Since the first term on the right-hand side of (7.5) is negative by (7.3b), the entire right-high side becomes negative as $\eta \to -\infty$. Thus there is a finite negative value of η, depending on $(\tau(\theta), n(\theta))$, at which this trajectory touches the line $\tau = 0$. Hence we obtain

7.6. LEMMA. *Let* (7.1) *hold for trajectories lying in* $\mathscr{L}$. (*This is a constitutive restriction, which can be obtained by performing the differentiations in* (7.1) *and then substituting* (5.3) *into the resulting inequality. Condition* (4.5b) *is sufficient for this requirement.*) *Then a trajectory having a single point in* $\mathscr{L}$ *cannot correspond to a solution of* (5.3)–(5.5) (*because* ξ *cannot assume values in its full range*).

Now let us examine $\mathscr{R}$, in which $\dot{\tau} < 0, \dot{n} > 0$. Condition (4.5b) implies that

$$\frac{d}{d\xi} v^{\#}(\tau(\xi), n(\xi)) > 0 \text{ on } \mathscr{R}. \tag{7.7}$$

This condition and the reverse of (7.2) then yield the reverse of (7.5), which says that $e^{\eta}\tau(\eta)$ has a positive lower bound, so that (5.5) cannot hold. Hence

7.8. LEMMA. *Let* (7.7) *hold for trajectories lying in* $\mathscr{R}$. *Then a trajectory having a single point in* $\mathscr{R}$ *cannot correspond to a solution of* (5.3)–(5.5).

In Section 9 we shall discuss the nature of trajectories in $\mathscr{L}$ for τ near 0 and trajectories in $\mathscr{R}$ for τ near ∞. We now turn to our basic theorems giving the existence, multiplicity, and qualitative properties of solutions of (5.3)–(5.5). In the rest of this section, it is tacitly assumed that (2.5), (2.6), (3.1), (3.2), (6.22), (6.23), (7.1), (7.7) hold. Note the identifications of Section 4.

(7.9) THEOREM. *Let* $\tau = 1$ *be azimuthally reinforced and let* $\mathscr{Q}_3$ *be free of singular points.* (*A sufficient condition for this hypothesis is that the material be azimuthally reinforced for* $\tau \leqslant 1$.) *The for each* $\lambda < 0$, *there is exactly one solution of* (5.3)–(5.5). *This solution describes an equilibrium state in which the center is stress-free and in which the radial normal stress* $\hat{N}(s)$ *and the azimuthal stretch* $\varrho(s)/s$ *strictly decrease with the radius* s.

Proof. We refer to Fig. 6.18. (In general $\ell \cap \{(\tau, n)\colon n < 0\}$ need not be confined to $\mathscr{Q}_3$.) Since the separatrix σ in $\mathscr{Q}_3$ is confined to the positively invariant region $\mathscr{I}$, τ strictly decreases from 1 and n strictly decreases from 0 along σ as ξ increases from $-\infty$. Condition (2.2) ensures that ℓ intersects σ transversally. Hence a solution of (5.3)–(5.5) corresponds to the segment of σ joining (1, 0) to ℓ. The point (1, 0) is occupied when $\xi = -\infty$ or $s = 0$. Here $\tau = 1$, $n = 0$, which imply by the transformations of Sections 4 and 5 the remarkable result that $\hat{T}(0) = 0 = \hat{N}(0)$. Since σ is a curve (by the comments of Section 6), Lemmas 7.6 and 7.8 imply that this solution is unique. □

Note that the uniqueness part of this theorem is stronger than Theorem 3.7 because it allows into competition solutions of (5.3) that need not be admitted by Theorem 3.7. Similar remarks apply to many of the results that follow.

7.10. THEOREM. *Let $\tau = 1$ be radially reinforced and let $\mathscr{Q}_3$ be free of singular points. (A sufficient condition for this hypothesis is that the material be radially reinforced for $\tau \leqslant 1$.) Let* (4.4b) *and* (4.5a) *hold. Then for each $\lambda < 0$ there is at least one solution of* (5.3)–(5.5). *If $\mathscr{s}$ is a curve in $\mathscr{Q}_3$ (a sufficient condition for which is that* (3.6) *hold), then there is exactly one solution. Each such solution describes a configuration in which the center of the body is in complete compression, whence $\hat{T}(0) = -\infty = \hat{N}(0)$. The radial normal stress $\hat{N}(s)$ and the azimuthal stretch strictly increase with the radius s.*

Proof. We refer to Fig. 6.19. On the trajectories of $\mathscr{s} \cap \mathscr{Q}_3$ (which form a continuous one-parameter family) τ increases from 0 to 1 and n increases from $-\infty$ to 0 as ξ increases up to ∞. We must first show that $\xi = -\infty$ when $(\tau, n) = (0, -\infty)$. Now $\dot{\tau} > 0$, $\dot{n} > 0$ on $\mathscr{J} \cap \mathscr{Q}_3$ and a fortiori on $\mathscr{s} \cap \mathscr{Q}_3$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}\xi} T^{\#}(\tau(\xi), n(\xi)) > 0 \quad \text{on } \mathscr{s} \cap \mathscr{Q}_3. \tag{7.11}$$

Let $\theta < \infty$ and let $(\tau(\theta), n(\theta)) \in \mathscr{s} \cap \mathscr{Q}_3$. Let (τ, n) be the solution of (5.3) that terminates at $(\tau(\theta), n(\theta))$ when $\xi = \theta$. Then $(\tau(\xi), n(\xi))$ for $\xi \leqslant \theta$ belongs to the part of $\mathscr{s}$ joining $(0, -\infty)$ to $(\tau(\theta), n(\theta))$, because $\mathscr{s} \cap \mathscr{Q}_3$ is invariant. Then (5.3b) and (7.11) imply that

$$\dot{n}(\xi) \leqq \alpha[T^{\#}(\tau(\theta), n(\theta)) - n(\xi)],$$

$$\text{or } \frac{\mathrm{d}}{\mathrm{d}\xi}(\mathrm{e}^{\alpha\xi} n) \leqq \alpha T^{\#}(\tau(\theta), n(\theta))\mathrm{e}^{\alpha\xi} \quad \text{for} \quad \xi < \theta, \tag{7.12}$$

whence we obtain

$$[n(\xi) - T^{\#}(\tau(\theta), n(\theta))] \geqslant \mathrm{e}^{\alpha(\theta-\xi)}[n(\theta) - T^{\#}(\tau(\theta), n(\theta))]. \tag{7.13}$$

Note that the bracketed term on the right-hand side is negative since $\mathscr{s}$ lies below $\mathscr{k}$. (See (6.7).) Let $n(\xi) \to -\infty$. Then (7.13) implies that $\xi \to -\infty$. Since $\dot{\tau} > 0$, $\dot{n} > 0$ on each trajectory of $\mathscr{s}$, each such trajectory intersects ℓ transversally. A solution of the boundary value problem corresponds to the segment of any trajectory of $\mathscr{s}$ joining $(0, -\infty)$ to ℓ. A rearrangement of (7.13) shows that $\mathrm{e}^{\alpha\xi} n(\xi)$ is bounded below for $\xi \leqq \theta$. Thus we can apply Theorem 3.7 when (3.6) holds to show that the trajectory of $\mathscr{s}$ joining $(0, -\infty)$ to ℓ is unique. Note that (5.5) is automatically satisfied because τ is bounded. Lemmas 7.6 and 7.8 ensure that these solutions are the only solutions. □

7.14. THEOREM. *Let $\tau = 1$ be azimuthally reinforced and let υ and $\hbar$ have exactly one intersection in $\mathscr{Q}_3$, which is transversal. (This intersection accordingly corresponds to an attractive improper node.) Let* (4.4b), (4.5a), (4.6a) *hold. Then for each $\lambda < 0$ there is at least one solution of* (5.3)–(5.5). *There is a critical value $\lambda^* < 0$ of the load parameter such that for $0 > \lambda > \lambda^*$, the solution is unique, the center is stress-free, and the normal stress and azimuthal stretch strictly decrease with the radius, for $\lambda = \lambda^*$ the solution is unique and the stresses have constant finite values throughout the body, and for $\lambda < \lambda^*$ the center is under complete compression so that $\hat{T}(0) = -\infty = \hat{N}(0)$ and the radial normal stress and the azimuthal stretch strictly increase with the radius. The solution is unique for $\lambda < \lambda^*$ if the intersection of $\mathscr{s}$ with the component of $\mathscr{I}$ extending to* $(0, -\infty)$ *is a curve, which happens when* (3.6) *holds. (The intersections of $\mathscr{s}$ with the components of $\mathscr{I}$ next to* $(1, 0)$ *are curves.)*

Proof. We refer to $\mathscr{Q}_3$ of Fig. 6.20. We do not bother to repeat the arguments developed in the proofs of the preceding theorems. λ^* is the unique value of λ for which the curve ℓ passes through the singular point in $\mathscr{Q}_3$. For $0 > \lambda > \lambda^*$ the solution corresponds to the trajectory originating at $(1, 0)$ and terminating at the curve ℓ. For $\lambda = \lambda^*$, the solution corresponds to the singular point in $\mathscr{Q}_3$. For $\lambda < \lambda^*$, the solution corresponds to the trajectory originating at $(0, -\infty)$ and terminating on the curve ℓ. (Note that for $\lambda = \lambda^*$, the trajectories of $\mathscr{s}$ beginning at $(1, 0)$ or $(0, -\infty)$ and ending at the singular point do not generate solutions because their terminal points correspond to $\xi = \infty$.) □

Note the catastrophic change in the nature of the family of solutions parametrized by λ as λ passes through the critical value λ^*. Since $\varrho(1) = \tau(0)$ Fig. 6.20 shows that there is no concomitant jump in the outer radius of the body.

7.15. THEOREM. *Let $\tau = 1$ be radially reinforced and let υ and $\hbar$ have exactly one intersection in $\mathscr{Q}_3$, which is transversal. (The intersection accordingly corresponds to a saddle point.) Then for each $\lambda < 0$ there is exactly one solution of* (5.3)–(5.5). *There is a critical value $\lambda^* < 0$ of the load parameter such that for $0 > \lambda > \lambda^*$ the normal stress and azimuthal stretch strictly increase with the radius from their values $n_0 \in (-\infty, 0)$ and $\tau_0 \in (0, 1)$ at the center, for $\lambda = \lambda^*$ the stress and stretches have constant values throughout the body with $N(s) = n_0$, $\varrho(s)/s = \tau_0$, and for $\lambda < \lambda^*$ the radial normal stress and the azimuthal stretch strictly decrease with the radius from their values n_0 and τ_0 at the center.*

The proof is based on Fig. 6.21. Since it uses the same ideas as those of Theorems 7.9 and 7.14, we omit the details.

It is clear how to generalize Theorems 7.14 and 7.15 to handle problems in which there are any number of singular points in $\mathcal{Q}_3$.

We now turn to problems in which the boundary is subject to tension, so that $\lambda > 0$. We encounter some new effects.

7.16. THEOREM. *Let $\tau = 1$ be azimuthally reinforced and let there be no singular point in $\mathcal{Q}_1$. Let $n = f(\tau)$, $\tau \geqslant 1$, be the equation of the separatrix $\mathscr{s} \cap \mathcal{Q}_1$. ($f$ is well-defined because $\dot{\tau} > 0$, $\dot{n} > 0$ on $\mathscr{s} \cap \mathcal{Q}_1$.) Then for each $\lambda > 0$ there are as many solutions of* (5.3)–(5.5) *as there are intersections of ℓ and $\mathscr{s}$. In particular, if λ is sufficiently small, there is exactly one such solution for each λ. If the load is dead so that $g(\tau) = 1$, then there is exactly one solution for each $\lambda > 0$. If the loading is a hydrostatic tension on a cylinder, so that $g(\tau) = \tau$, then* (i) *there is a solution for each $\lambda > 0$ iff $f(\tau)/\tau \to \infty$ as $\tau \to \infty$ and* (ii) *there is a $\bar{\lambda} > 0$ such that there are no solutions for $\lambda > \bar{\lambda}$ if $f(\tau)/\tau \to 0$ as $\tau \to \infty$. (Since $v(n) > f^{-1}(n) > h(n)$ for $n > 0$, a sufficient condition for $f(\tau)/\tau \to \infty$ as $\tau \to \infty$ is that $N(\tau, \tau)/\tau \to \infty$ as $\tau \to \infty$, by the definition of v, and a sufficient condition for $f(\tau)/\tau \to 0$ as $\tau \to \infty$ is that $h^{-1}(\tau)/\tau \to 0$ as $\tau \to \infty$.) If the loading is a hydrostatic tension on a sphere, so that $g(\tau) = \tau^2$, then* (i) *and* (ii) *hold with $f(\tau)/\tau$ replaced with $f(\tau)/\tau^2$. Each such solution describes a state in which the center is stress-free and the radial normal stress $\hat{N}(s)$ and azimuthal stretch strictly increase with radius s.*

The proof follows immediately from Figure 6.18 by the arguments already developed. The essential difference between this result and Theorem 7.9 is that here f' and g' have the same sign in $\mathcal{Q}_1$ so that f and λg need not intersect. Thus the question of existence is intimately bound up with the material response in large tension. These issues occur all the time in the study of the inflation of elastic rings and shells. The physical interpretation of the nonexistence is simple: The material is incapable of withstanding the forces applied to it and either responds dynamically or else assumes an unsymmetric equilibrium state.

To study solutions of (5.3)–(5.5) for $\lambda > 0$ when the phase portrait is like that of Figs 6.19 and 6.20 we require detailed information on $\mathscr{s} \cap \mathcal{Q}_1$ for large n to show that $\xi \to -\infty$ as $n \to \infty$ and that (5.5) is satisfied. We begin with

7.17. LEMMA. *Let all the singular points in $\mathcal{Q}_1$ be confined to a bounded set and let the unbounded connected component of $\mathscr{I} \cap \mathcal{Q}_1$ lie to the right of ν and to the left of $\hbar$ (as in Figs* 6.19 *and* 6.20). *Then $\xi \to -\infty$ as $n \to \infty$ on $\mathscr{s} \cap \mathcal{Q}_1$.*

Except for some obvious sign changes, the proof is identical with the first part of the proof of Theorem 7.10.

The treatment of (5.5) when $\lambda > 0$ for Figs 6.19 and 6.20 is a more difficult matter because it depends in a delicate way on the nature of the constitutive functions for large τ and n. Our first result depends on the asymptotic behavior of $\mathscr{k}$. The equation (6.7b) for $\mathscr{k}$ can be written as

$$T(\tau, v) = N(\tau, v) \tag{7.18}$$

where

$$v = v^{\#}(\tau, n), \text{ or equivalently, } n = N(\tau, v). \tag{7.19}$$

7.20. THEOREM. *Let $\tau = 1$ be radially reinforced and let $\mathscr{Q}_1$ be free of singular points. Let there be a number $M > 0$ such that*

$$v^{\#}(\tau, n) \geqslant M\tau \textit{ for } (\tau, n) \in \mathscr{s} \cap \mathscr{Q}_1 \textit{ and } \tau \textit{ sufficiently large.} \tag{7.21}$$

(A natural sufficient condition for (7.21) *is that if (τ, v) satisfies* (7.18) *and if τ is sufficiently large, then $v \geqslant M\tau$.) Then for each $\lambda > 0$ there are exactly as many solutions of* (5.3)–(5.5) *as there are intersections of ℓ with trajectories forming $\mathscr{s}$. ($\mathscr{s}$ is a curve if* (3.6) *holds.) The consequences of this statement are the same as those of Theorem* 7.16. *Each such solution describes a state in which $\hat{T}(0) = \infty = \hat{N}(0)$ and in which the radial normal stress and azimuthal stretch strictly decrease with the radius s.*

Proof. The only novelty is to verify that (5.5) holds. Relations (5.3a) and (7.21) imply that

$$\dot{\tau} \geqslant M\tau - \tau \text{ or } \frac{\mathrm{d}}{\mathrm{d}\xi}(\mathrm{e}^{\xi}\tau) \geqslant M\mathrm{e}^{\xi}\tau(\xi) \tag{7.22}$$

on $\mathscr{s} \cap \mathscr{Q}_1$ for τ sufficiently large. Let $-\infty < \eta < \theta < 0$ and let (τ, n) be a solution to (5.3) corresponding to a trajectory of $\mathscr{s} \cap \mathscr{Q}_1$ that terminates at $(\tau(\theta), n(\theta))$ with $\tau(\theta)$ sufficiently large. Then (7.22) implies that

$$\mathrm{e}^{\eta}\tau(\eta) \leqslant \mathrm{e}^{M(\eta-\theta)}\mathrm{e}^{\theta}\tau(\theta), \tag{7.23}$$

which implies (5.5). □

We now complement this result with a nonexistence theorem under an hypothesis that roughly speaking is the negation of (7.21).

7.24. THEOREM. *Let* $\tau = 1$ *be radially reinforced and let* $\mathcal{Q}_1$ *be free of singular points. Let there be numbers* $m > 0$, $\mu < 1$ *such that*

$$v^{\#}(\tau, n) \leqslant m\tau^{\mu} \text{ for } (\tau, n) \in \mathscr{A} \cap \mathcal{Q}_1 \text{ and } \tau \text{ sufficiently large.} \tag{7.25}$$

(*A natural sufficient condition for* (7.25) *is the inequality*

$$v^{\#}(\tau, N(\tau, \tau)) \leqslant m\tau^{\mu} \text{ for } \tau \text{ sufficiently large,} \tag{7.26}$$

which describes constitutive behavior on ν.) *Then* (5.3)–(5.5) *has no solutions.*

Proof. Relations (5.3a) and (7.25) imply that

$$\frac{d}{d\xi}(e^{\xi}\tau) \leqslant m(e^{\xi}\tau)^{\mu}e^{\xi(1-\mu)} \tag{7.27}$$

on $\mathscr{A} \cap \mathcal{Q}_1$ for τ large. Let η, θ, τ, n play the same roles as in the proof of Theorem 7.20. Then (7.27) yields

$$[e^{\eta}\tau(\eta)]^{1-\mu} \geqslant me^{\eta(1-\mu)} + e^{\theta(1-\mu)}[\tau(\theta)^{1-\mu} - m]. \tag{7.28}$$

Since $\tau(\theta) \to \infty$ as $\theta \to -\infty$ (cf. Figure 6.19), we can choose θ so small that the right-hand side of (7.28) has a positive lower bound. Thus (7.28) ensures that (5.5) cannot hold. □

It is a straightforward exercise to use the techniques developed in this section to formulate analogous theorems for material response leading to Figs 6.20, 6.21, and other even more complicated phase portraits. In Section 9 we comment on how (7.26) and the sufficient condition for (7.21) can be determined from specific constitutive functions, such as those determined from (3.12).

8. Cavitation

We now briefly study solutions for which the center of the body opens into a hole. (When the conditions of Theorem 7.24 hold, such solutions generate the only symmetric equilibrium states!). We accordingly seek solutions of (5.3), (5.4), (5.6) for $\lambda > 0$. To handle (5.6) we must study phase portraits for (τ, n) near $(\infty, 0)$. In particular, the proof of Lemma 7.8 shows that (5.6a) is automatically satisfied and also that $\xi \to -\infty$ as $\tau(\xi) \to \infty$ along any trajectory in $\mathscr{R}$. It follows that a condition necessary for the existence of cavitation is the existence of a special trajectory c asymptotic to the τ-axis as $\tau \to \infty$.

Now for (τ, n) near $(\infty, 0)$ neither the effects of aeolotropy nor those of isotropy are discernible. The nature of the phase portrait in this region is in fact dictated by the growth rates of $T^{\#}(\cdot, 0)$ and $\nu^{\#}(\cdot, 0)$ for τ large. (As Figs 6.18–6.21 show, the predominant effect of aeolotropy is manifested in the form of $\mathscr{I}$. For isotropic materials $\mathscr{I}$ collapses to a single curve.) In view of these observations, we enunciate the heuristic principle that c exists for aeolotropic materials if they satisfy the same growth conditions as those ensuring the existence of cavitation for isotropic materials. These conditions have been exhaustively studied in [4, 12, 14, 18, 20]. We describe a more concrete approach to the existence of c in Section 9. In this section, we simply assume that c exists and is unique.

It is illuminating to see what happens when (5.4) is replaced by the requirement that the deformed outer radius be specified:

$$\varrho(1) = \tau(0) = \omega > 1. \tag{8.1}$$

(We get no cavitating solutions unless $\omega > 1$.) When (5.4) is prescribed, we limit our comments to the case that the load is dead. The treatment of other kinds of loads follows the lines of Theorem 7.16.

The existence, multiplicity, and qualitative properties of solutions follow from an examination of the appropriate phase portraits. The stability of solutions does not (although one can make educated guesses about it). We refer to Ball [4] for a treatment of stability for isotropic bodies, using methods that are possibly applicable to the problems at hand.

In Figs 8.2, 8.3, 8.4 we exhibit phase portraits for an isotropic material and for the materials of Figs 6.18 and 6.19. In extracting information from these figures we tacitly rely on Lemmas 7.6 and 7.8 for our statements about multiplicity.

From Fig. 8.2, for isotropic materials, we find that for $0 < \lambda \leqslant n^*$ there is exactly one equilibrium state with the center intact and exactly one cavitating equilibrium state satisfying (5.4). The intact state has constant stress λ throughout the body. The cavitating state has normal stress increasing strictly with the radius from 0 to λ and has azimuthal stretch decreasing strictly with the radius from ∞ to a value exceeding that for the intact state. If λ exceeds n^*, there is exactly one equilibrium state, which is intact. (This state is presumably unstable: Cavitation may well occur, but in a dynamical process. Note that the outer radius of this intact state is the same as that for a cavitating state, but the normal stress λ on the boundary for the intact state

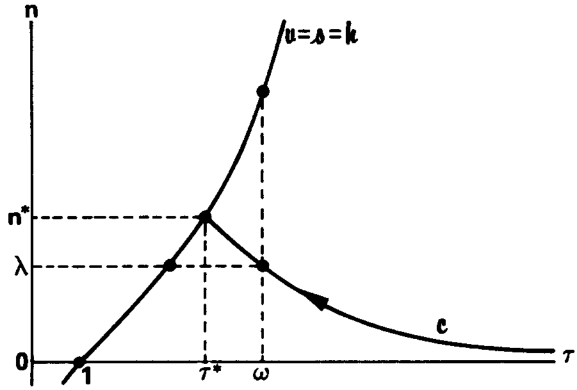

Fig. 8.2. Phase portrait for an isotropic material for which c exists.

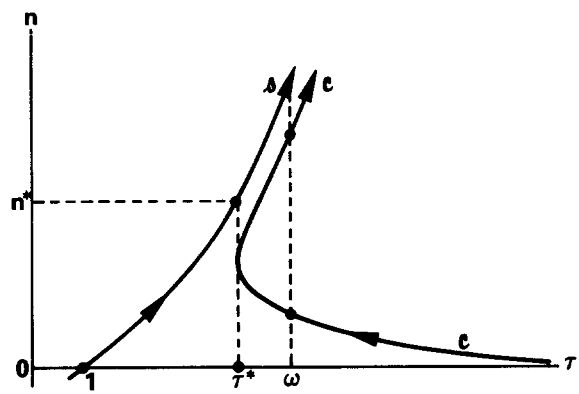

Fig. 8.3. Phase portrait of Fig. 6.18.

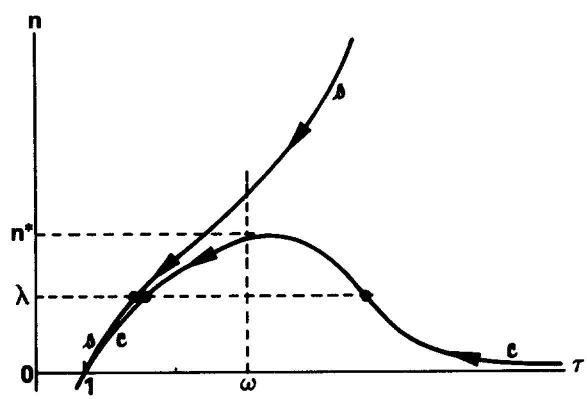

Fig. 8.4. Phase portrait of Fig. 6.19.

exceeds the normal stress for the cavitating state at the same radius. This normal stress for the cavitating state is presumably the most that can be borne in a stable equilibrium state.)

From Fig. 8.2 we also find that for $1 \leqslant \omega < \tau^*$ there is exactly one equilibrium state satisfying (8.1), which is intact. For $\tau^* \leqslant \omega$ there are exactly two equilibrium states satisfying (8.1), one intact, the other not. For hyperelastic materials, the cavitating state is stable and the intact state unstable according to the energy criterion (cf. Ball [4]).

Now we study materials for which $\tau = 1$ is azimuthally reinforced and $\mathscr{D}_1$ is free of singular points. The phase portrait has the form of Fig. 8.3. For each $\lambda > 0$, there is exactly one intact state and one cavitating state satisfying (5.4). For $1 \leqslant \omega < \tau^*$ there is exactly one equilibrium state satisfying (8.1), which is intact. For $\tau^* < \omega$, there is exactly one intact equilibrium state and there are exactly two cavitating equilibrium states satisfying (8.1). Extrapolating from Ball's results about stability for the isotropic body, we surmise that the only one of these three states that is stable is the cavitating state on the lower branch of c. (n is apparently smallest on this branch.) As ω is increased past τ^*, the (presumably stable) equilibrium state jumps (snaps) from one with the center intact to a cavitating state. No such jumping occurs for isotropic materials. For the traction problem, we conjecture that there is no stable equilibrium state for $\lambda \geqslant n^*$.

We finally study materials for which $\tau = 1$ is radially reinforced and $\mathscr{D}_1$ is free of singular points, so that the phase portrait is given by Fig. 8.4. (Similar techniques handle the manifold other possibilities). For $0 < \lambda < n^*$ there are exactly two cavitating equilibrium states satisfying (5.4). If (7.21) holds, there is also exactly one intact state. If (7.25) holds with $\mu < 1$, then these are the only symmetric equilibrium states. For $\omega > 1$, there is exactly one intact state and exactly one cavitating state satisfying (8.1). We surmise that the cavitating state is stable. For $0 < \lambda < n^*$ there are one intact state and two cavitating states and for $\lambda > n^*$, the only symmetric equilibrium state describes an intact state. Since τ seems to be smallest on this branch, we conjecture that it is stable. (Methods now being developed by J. Sivaloganathan may help resolve the question of stability.)

It is interesting to observe that Figs 8.3 and 8.4 are the two simplest unfoldings of Fig. 8.2.

9. The effects of material response on the asymptotic nature of the phase portraits

In this section we examine the behavior of the phase portraits at the extremities of $\mathscr{U}$. For the most part, this study relies on a quantitative analysis of the growth conditions on $T^\#$ and $\nu^\#$. We wish to motivate our results by using the material of (3.12). Since we do not have a closed-form representation for the inverse $\nu^\#(\tau, \cdot)$ of $N(\tau, \cdot)$ in this case, we must employ appropriate asymptotic methods.

We first study (5.3) in $\mathscr{L}$. Since $\dot{\tau} > 0$ and $\dot{n} < 0$ in $\mathscr{L}$ we can describe any trajectory in $\mathscr{L}$ by the function $\tau \mapsto \bar{n}(\tau)$ where $\bar{n}(\tau)$ satisfies

$$\frac{d\bar{n}}{d\tau} = \frac{\alpha[T^\#(\tau, \bar{n}) - \bar{n}]}{\nu^\#(\tau, \bar{n}) - \tau}. \tag{9.1}$$

We are particularly concerned with the behavior of $\bar{n}$ near the n-axis. To determine this, we restrict our attention to the special family of constitutive functions given by (3.10), (3.12) with $E_1 = 0 = E_2$. (Our entire analysis actually goes through for more general Φ's whose behavior at extreme values of τ and ν is captured by our special family of Φ's.)

We now get an asymptotic representation for the right-hand side of (9.1) when τ is small and $\bar{n}$ is bounded. This information suffices to determine $\lim \bar{n}(\tau)$ as $\tau \to 0$. For our special family of materials, the equation $n = N(\tau, \nu)$ has the form

$$n = -A_2\nu^{-a_2} + B_2\nu^{b_2} - Cc_2\tau^{-c_1}\nu^{-c_2-1} + Dd_2\tau^{d_1}\nu^{d_2-1}. \tag{9.2}$$

We seek an asymptotic representation for the solution $\nu^\#(\cdot, n)$ of this equation as $\tau \to 0$ with n held fixed. Since the only functions of τ and ν that appear in (9.1) are powers, we are led to seek the solution of it in the form

$$\nu^\#(\tau, n) = \tau^{-\gamma(n)}\phi(\tau, n) \tag{9.3}$$

where $\lim_{\tau\to 0} \phi(\tau, n)$ exists for each finite n and lies in $(0, \infty)$. $\gamma(n)$ is a positive number to be determined. We substitute (9.3) into (9.2) obtaining an equation that involves essentially a sum of powers of τ. For this equation to be satisfied γ must be chosen so that the two highest powers are equal. The combinatorial problem of determining the right γ is readily treated by the Newton polygon method when the exponents $a_2, \ldots, d_2$ are given specific numerical values. (Cf. Dieudonné [7], e.g. To put (9.2) into standard form for this method, set $\nu = y^{-1}$ and multiply the resulting version of (9.2) by $\tau^{c_1}y^{\max(b_2, d_2-1)}$.) Then $\lim_{\tau\to 0} \phi(\tau, n)$ is determined by the requirement that the sum of terms with highest powers of n vanish in the limit as $\tau \to 0$. In particular, the substitution of (9.3) into (9.2) yields the following asymptotic formula

$$0 \sim B_2\tau^{-\gamma b_2}\phi(0, n)^{b_2} - Cc_2\tau^{-c_1+\gamma(c_2+1)}\phi(0, n)^{-c_2-1}$$
$$+ Dd_2\tau^{d_1-\gamma(d_2-1)}\phi(0, n)^{d_2-1} \quad \text{as} \quad \tau \to 0. \tag{9.4}$$

Here we have retained all terms that cannot be immediately disqualified as negligible. The candidates for γ, obtained by pairwise equating the exponents on τ, are the positive numbers among

$$c_1/(b_2 + c_2 + 1), \quad d_1/(d_2 - 1 - b_2), \quad (c_1 + d_1)/(c_2 + d_2). \tag{9.5}$$

That giving the largest negative exponent on τ in (9.4) is the correct one. Denote it by Γ. (Note that both $\gamma(n)$ and $\phi(0, n)$, found by the prescription given above, are independent of n.) Then (3.10), (3.12), (4.1) yield

$$T^{\#}(\tau,n) \sim -A_1\tau^{-a_1} - Cc_1\Gamma^{-c_2}\tau^{-c_1-1+\gamma c_2} + Dd_1\Gamma^{d_2}\tau^{d_1-1-\gamma d_2} \equiv \alpha^{-1}\Gamma\tau^{-\gamma}f(\tau) \tag{9.6}$$

for n fixed as $\tau \to 0$. Thus for $\bar{n}$ in any bounded interval, (9.1), (9.3), (9.6) yield

$$\frac{\mathrm{d}\bar{n}}{\mathrm{d}\tau} \sim f(\tau). \tag{9.7}$$

If we choose the exponents so that $T^{\#}_{\tau} > 0$ (see (4.4b) and the comments following (3.12)), then we need not worry about the third term in (9.6). (It cannot dominate the other two. At worst we might have to change the constants A and C.) Since the exponents of the C and D-terms of (9.7) are each one less than the corresponding exponents of (9.6), the D cannot be important in (9.4). Thus the only γ is the first from (9.5). It causes the exponent on the C-term in (9.7) to be negative. Hence, for the class of materials described by (3.12), we conclude that

$$\frac{\mathrm{d}\bar{n}}{\mathrm{d}\tau} \to -\infty \quad \text{as} \quad \tau \to 0 \quad \text{for fixed } \bar{n}. \tag{9.8}$$

In view of the remarks on the D-term we obtain from (9.7) that

$$\bar{n}(\tau) \sim \bar{n}(\omega) + \alpha\Gamma^{-1}\int_{\tau}^{\omega} [A_1\sigma^{\gamma-a_1} + Cc_1\Gamma^{-c_2}\sigma^{-(c_1+1)+\gamma(c_2+1)}]\,\mathrm{d}\sigma \tag{9.9}$$

for $0 \leqslant \tau < \omega$ with ω small, provided $\bar{n}$ is bounded on $[\tau, \omega]$. Thus $\bar{n}$ would be bounded on $[0, \omega]$, so that $\bar{n}(0)$ would be a finite number (depending on $\bar{n}(\omega)$) if, and only if

$$\gamma - a_1 > -1, \quad \gamma(c_2 + 1) - (c_1 + 1) > -1. \tag{9.10}$$

where γ is the first entry from (9.5). But the second inequality is false. Thus we conclude that all trajectories in $\mathscr{L}$ originate at $(\tau, n) = (0, \infty)$.

To treat (7.21) we use the sufficient condition for it given in Theorem 7.20. We substitute (3.10), (3.12) into (7.18) and use our informal Newton Polygon Method to solve it for an asymptotic representation for v as a function of τ for τ large. To treat (7.26) we first solve (9.4) for $v^{\#}(\tau, n)$ for τ and n large and then replace n by $N(\tau, \tau)$.

We finally turn to the study of cavitation, the existence of which devolves on the existence of the special trajectory $\mathscr{c}$. $\mathscr{c}$ has the defining property that

$$n(\xi) \to 0, \tau(\xi) \to \infty \quad \text{as} \quad \xi \to -\infty. \tag{9.11}$$

We begin by seeking the solution of $v^{\#}(\tau, n)$ of (9.2) for n fixed and finite and τ large in the form (9.3) with $\lim_{\tau\to\infty} \phi(\tau, n) \in (0, \infty)$ existing for each finite n. As the analog of (9.4) we get

$$\begin{aligned} 0 \sim &- A_2\tau^{\gamma a_2}\phi(\infty, n)^{-a_2} - Cc_2\tau^{-c_2+\gamma(c_2+1)}\phi(\infty, n)^{-c_2-1} \\ &+ Dd_2\tau^{d_2-\gamma(d_2-1)}\phi(\infty, n)^{d_2-1} \quad \text{as} \quad \tau \to \infty. \end{aligned} \tag{9.12}$$

We find γ and $\phi(\infty, n)$ as above, noting that each is independent of n. Then (3.10), (3.12), (4.1) yield

$$T^{\#}(\tau, n) \sim B_1\tau^{b_1} - Cc_1\Gamma^{-c_2}\tau^{-c_1-1+\gamma c_2} + Dd_1\Gamma^{d_2}\tau^{d_1-1-\gamma d_2} \tag{9.13}$$

as $\tau \to \infty$. Here $\Gamma \equiv \phi(\infty, n)$. If $\tau \mapsto \bar{n}(\tau)$ denotes the graph of a typical trajectory in $\mathscr{R}$, then

$$\alpha^{-1}\frac{d\bar{n}}{d\tau} \sim -B_1\tau^{b_1-1} + Cc_1\Gamma^{-c_2}\tau^{-c_1-2+\gamma c_2} - Dd_1\Gamma^{d_2}\tau^{d_1-2-\gamma d_2} \equiv f(\tau) \tag{9.14}$$

as $\tau \to \infty$. If ω is a large number and if $\tau > \omega$, then (9.14) implies that

$$\alpha^{-1}\bar{n}(\tau) \sim \bar{n}(\omega) + \int_\omega^\tau f(\sigma)\,d\sigma \quad \text{as} \quad \tau \to \infty. \tag{9.15}$$

Thus $\bar{n}(\tau)$ approaches a constant depending on $\bar{n}(\omega)$ if f is integrable on (ω, ∞). Since the trajectory described by $\bar{n}$ lies in $\mathscr{R}$, $\bar{n}$ is decreasing here. It follows that the C-term in (9.14) cannot be dominant and may be discarded (provided, in the worst case, that B_1 or D be changed). Thus $\mathscr{c}$ exists if $b_1 < 1$, $d_1 < 2 + \gamma d_2$ (and a fortiori if $d_1 \leqslant 2$).

10. Conclusion

If a material is aeolotropic, but has constitutive equations in some sense close to those for an isotropic material, the various phenomena we have discovered would be qualitatively unaffected by the size of the discrepancy. In short, the properties of solutions, especially those related to infinite stresses at the center, do not depend continuously on the material properties. (The notion of continuous dependence can be made mathematically precise. It corresponds to the continuous dependence of solutions of ordinary differential equations on the equations, a topic discussed in advanced texts on ordinary differential equations.) The source of trouble is the singularity at the center at which the effects of aeolotropy are focused.

A similar phenomenon is manifested in Theorems 7.20 and 7.24. There is a critical rate of growth for the constitutive equations of radially reinforced materials dividing the constitutive equations into two classes. In one class there is no symmetric equilibrium state with center intact when the boundary is subjected to any uniform tension. As frequently happens, this critical rate of growth is exactly that of linear elasticity (cf. Antman [3]).

In fact, some of the phenomena we have studied occur in linear elasticity (cf. [3, 15, 19]). The governing equations for linearly elastic media are exactly those of (6.10). The phase-plane is then either that for a saddle or node of the linear theory. Consequently, $\mathscr{v}$, $\mathscr{k}$, $\mathscr{s}$ are straight lines with positive slope. But we confront the characteristic conundrum of linear elasticity when we attempt to interpret the meaning of unbounded solutions generated by the trajectories of $\mathscr{s}$ that start at $(\tau, n) = (\pm\infty, \pm\infty)$. This issue never arises in nonlinear elasticity. (Indeed, one of the most important justifications for the study of nonlinear elasticity is that it gives clear resolutions of the paradoxical interpretation of unbounded solutions of the equations of linear elasticity.)

The presence of infinite stresses at the center due to suitable kinds of small aeolotropy makes it tempting to suggest that this aeolotropy may contribute to the mechanism of fracture. Indeed, poorly cast cylindrical rods, which may be regarded as radially reinforced (cf. [11, 22]), fracture along their axes. The same phenomenon can be more readily viewed in a piece of ice cream frozen into a cylinder. Of course, fracture occurs because of mechanisms operating on the molecular level, but the use of continuum mechanics has continued to illuminate the subject.

The jumps in stress in certain bodies (cf. Theorem 7.15) suggest applications to switches and transducers. By implanting in a body reinforcing fibers or sheets that respond mechanically to magnetic fields it may be possible to greatly alter the sensitivity of the body and even to switch the aeolotropy on and off (cf. Savage and Adler [17]).

For simplicity, we assumed that our constitutive functions are twice continously differentiable. Virtually our entire analysis goes through, except that with the reduced smoothness, we are never certain if $\mathscr{S}$ is a curve. Consequently, some of our results on uniqueness and multiplicity would lose their sharpness.

The qualitative properties of the phase portraits of (5.3) are unaffected by the dimension of the body: 2 for cylinders and 3 for balls. But as Theorems 7.16 and 7.20 and as the results of Section 8 indicate, the nature of hydrostatic tension greatly depends on dimension and greatly influences the number and type of solutions.

Phase-plane methods have been used to treat radially symmetry problems of nonlinear elasticity by several authors, among whom are Ball [4], Biot [5], Callegari, Reiss and Keller [6], and Sivaloganathan [18]. Such problems have also been treated by direct methods of the calculus of variations by Ball [4] and Sivaloganathan [18], by methods of variational inequalities by Antman [2], by shooting methods by Stuart [20], and by methods based on the Leray–Schauder degree theory by Gauss and Antman [9] and Negrón-Marrero [13]. Methods based upon the calculus of variations give useful insights into the stability of equilibrium states. But for the problems we have treated, methods other than phase-plane methods are not only technically far more demanding, but also fail to deliver the detailed qualitative information about solutions that we have observed. They enjoy, however, two compelling virtues: They can be applied to problems in which the material is not homogeneous, so that the governing equations typically fail to have a formulation as an autonomous system, and they can be applied to systems with more than two dependent variables, for which many of the advantages of two-dimensional phase spaces are lost.

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