The Numerical Solution of the v. Kármán equations using Multigrid Methods

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Abstract

On the first part of this work, we compare the performance of three iterative methods: Gauss-Jacobi, Gauss-Seidel, and Red-black Gauss-Seidel. The comparison is based on their theoretical rate of convergence and their amenability for vectorization and parallelization on a particular problem. We further compare these methods with a multigrid method. On the second part of the paper, we use a multigrid method to solve numerically the v. Kármán equations (semilinear system of partial differential equations) that describe the buckling of a rectangular plate.

1 Introduction

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In Section 2 we describe and compare the iterative methods of Jacobi, Gauss-Seidel, Red-Black Gauss-Seidel, and the Multigrid method. We do so for a model problem that consist of a linear partial differential equation whose solution is known. We consider both sequential and optimized (in terms of vectorization and parallelization) versions of the algorithms and measure their performance on an ALLIANT FX/8.

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In Section 3, after defining the v. Kármán equations, we describe a numerical scheme to approximate solutions of these equations. The existence of solutions of these equations has been established among others by Wolkowisky (1967) (using the Schauder Fixed Point Theorem), Berger (1967) and Berger & Fife (1968) (using the variational methods of Lyusternick and Shnirel'man). In order to solve the v. Kármán system we first apply a substitution to reduce its order from four to two, thus allowing us to work directly with the Laplacian operator instead of the biharmonic. Next we discretize the domain and approximate the derivatives using finite differences at each discretized point. This process renders a large scale system of nonlinear equations consisting of four blocks of equations. To obtain numerical solutions of this system we use a fixed point iteration. To solve the intermediate linear systems of equations in the fixed point iteration, we apply a multigrid method. We satisfactorily compute the first two branches of nontrivial solutions.

2 Description and comparison of the Iterative Methods and the Multigrid Method

In this section we describe and compare the iterative methods of Jacobi, Gauss-Seidel, Red-Black Gauss-Seidel, and the Multigrid method. We compare them on the model problem

$$\begin{aligned} \Delta u = u_{xx}+u_{yy} &= \pi^2 \sin(\pi x) \sin(\pi y) \ \text{in } \Omega, \\ u &= 0 \ \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\Omega = (0,1)^2$ is the unit square, and whose exact solution is

$$u = \frac{1}{2} \sin(\pi x) \sin(\pi y).$$

In order to find numerical solutions to the problem (2.1) we divide the domain Ω into (N+1)x(N+1) subsquares as follows. Define the points

$$(x_i, y_j) = (ih, jh) \ , \ \text{for } 0 \le i,j \le N, \ h = \frac{1}{N}.$$

We approximate the partial derivatives in (2.1) by finite difference of second order at each interior point of the discretized domain. This yields the following system of equations:

$$\frac{(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})}{h^2} + \frac{(u_{i,j+1} - 2u_{i,j} + u_{i,j-1})}{h^2} = f_{i,j} \ ,$$

for $1 \le i,j, \le N-1$ and where

$$f_{i,j} = \pi^2 \sin(\pi x_i) \sin(\pi y_j) \ ,$$

for $1 \le i,j \le N-1$. This can be written as

$$u_{i+1,j} - 4u_{i,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} = h^2 f_{i,j} , \qquad (2.2)$$

for all $1 \le i,j \le N-1$. To solve (2.2) we first employ the following iterative or relaxation methods.

The Jacobi method

The Jacobi method starts iterating with an initial approximation of $\{u_{i,j}\}$, and during each iteration it constructs a new approximation using only values from the previous iteration. That is, we generate the sequence $\{u_{i,j}^{(k)}\}_{k=0}^{\infty}$ according to:

$$u_{i,j}^{(k)} = \frac{1}{4}[u_{i+1,j}^{(k-1)} + u_{i-1,j}^{(k-1)} + u_{i,j+1}^{(k-1)} + u_{i,j-1}^{(k-1)} - h^2 f_{i,j}] , \qquad (2.3)$$

where $1 \le i,j \le N-1$, $k \ge 1$. It can be shown (see Isaacson and Keller (1966), pags. 463-470) that this method converges linearly to a solution of (2.2).

The Gauss-Seidel method

In the Gauss-Seidel method, we start iterating with an initial approximation of $\{u_{i,j}\}$, and in each iteration a new approximation is found using values from the previous and the current iteration. That is, we generate the sequence $\{u_{i,j}^{(k)}\}_{k=0}^{\infty}$ according to:

$$u_{i,j}^{(k)} = \frac{1}{4}[u_{i+1,j}^{(k-1)} + u_{i-1,j}^{(k)} + u_{i,j+1}^{(k-1)} + u_{i,j-1}^{(k)} - h^2 f_{i,j}], \qquad (2.4)$$
$$1 \le i,j \le N-1 , \; k \ge 1.$$

Although this method converges linearly also, it has a better rate of convergence than the Jacobi method.

The Red-Black Gauss-Seidel method

The Red-Black Gauss-Seidel method is a mixture of the Jacobi and Gauss-Seidel methods. At each iteration, this method updates the approximation of $\{u_{i,j}\}$ over the points (ih, jh) first when i,j are either both even or both odd, using a Jacobi type iteration, and then updates the other points with a Gauss-Seidel type iteration. That is, we generate the sequence $\{u_{i,j}^{(k)}\}_{k=0}^{\infty}$ according to:

$$u_{2i,2j}^{(k)} = \frac{1}{4}[u_{2i+1,2j}^{(k-1)} + u_{2i-1,2j}^{(k-1)} + u_{2i,2j+1}^{(k-1)} + u_{2i,2j-1}^{(k-1)} - h^2 f_{2i,2j}],$$

$$u_{2i-1,2j-1}^{(k)} = \frac{1}{4}[u_{2i,2j-1}^{(k-1)} + u_{2i-2,2j-1}^{(k-1)} + u_{2i-1,2j}^{(k-1)} + u_{2i-1,2j-2}^{(k-1)} - h^2 f_{2i-1,2j-1}], \quad (2.5)$$

$$u_{2i-1,2j}^{(k)} = \frac{1}{4}[u_{2i,2j}^{(k)} + u_{2i-2,2j}^{(k)} + u_{2i-1,2j+1}^{(k)} + u_{2i-1,2j-1}^{(k)} - h^2 f_{2i-1,2j}],$$

$$u_{2i,2j-1}^{(k)} = \frac{1}{4}[u_{2i+1,2j-1}^{(k)} + u_{2i-1,2j-1}^{(k)} + u_{2i,2j}^{(k)} + u_{2i,2j-2}^{(k)} - h^2 f_{2i,2j-1}],$$

where $1 \le i,j \le \frac{N}{2}$, $k \ge 1$. The Red-Black Gauss-Seidel method has a rate of convergence somewhere in between those of the Jacobi and Gauss-Seidel.

A comparison of the iterative methods

We used an Alliant FX/8 computer to implement each of these methods. This computer has both vector and parallel capabilities. We used $N = 128$ and as convergence criterion the relative error

$$\frac{\| \mathbf{u} - \mathbf{v} \|_\infty}{\| \mathbf{u} \|_\infty} < 1.0 \times 10^{-4}, \quad (2.6)$$

where $\mathbf{u}$ is the exact solution to the equation (2.1) evaluated at the mesh points, and $\mathbf{v}$ is an approximate solution of (2.2) obtained using (2.3), (2.4) or (2.5).

We used the execution time of the corresponding implementations of (2.3), (2.4), and (2.5) as the comparison criterion for each method. First we do a comparison between the programs without making use of parallelization or vectorization. In Table 2.1 we show the execution times, in seconds, and the number of iterations, for the programs corresponding to each method. Without using any vectorization or parallelization, Gauss-Seidel has the smaller execution time, which is the case because it has the best theoretical rate of (sequential) convergence of the three methods.

In Table 2.2 we show the execution time, in seconds, and the number of iterations, for each program now making use of parallelization and vectorization. We see that the Jacobi method is better than the Gauss-Seidel method with the use of parallelization and vectorization. This is because equation (2.3) has no data dependencies making it perfect for vectorization. Also each update, for given (i,j), can be done independent of the others, thus yielding perfect parallelism also. The Red-Black Gauss-Seidel, however, has the best running time. The reason for this is that this method "borrows" in a sense the good properties of the other two methods. That is, in (2.5) we achieve perfect vectorization and parallelization for the updates in which both indexes are either even or odd (first two equations), and good (sequential) rate of convergence on the last two equations.

Iterative Method	Time	Number of Iterations
Jacobi	11954.14	16800
Gauss-Seidel	6020.47	8450
Red-Black Gauss-Seidel	6326.14	8450

Table 2.1. Execution time in seconds and number of iterations for each program without using vectorization or parallelization.

Iterative Method	Time	Number of Iterations
Jacobi	628.20	16800
Gauss-Seidel	1102.51	8450
Red-Black Gauss-Seidel	315.20	8450

Table 2.2. Execution time in seconds and number of iterations for each program using vectorization and parallelism.

Method	Time
Multigrid	0.66
Red-Black Gauss-Seidel	315.20

Table 2.3. Execution time in seconds for each program using vectorization and parallelism.

The Multigrid method

The Multigrid method is a powerful method to solve large scale linear systems of equations. Consider the equation

$$\boldsymbol{Au} = \boldsymbol{b}, \tag{2.7}$$

where A is an NxN matrix and $\boldsymbol{u},\boldsymbol{b}$ are vectors of N elements. Define the error

$$\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{v}, \tag{2.8}$$

where $\boldsymbol{u}$ correspond to the exact solution of (2.7) and $\boldsymbol{v}$ is an approximation to $\boldsymbol{u}$. The residual is defined by

$$\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{Av}. \tag{2.9}$$

It follows now that $\boldsymbol{e}$ and $\boldsymbol{r}$ satisfy the residual equation

$$\boldsymbol{Ae} = \boldsymbol{r}. \tag{2.10}$$

The simplest cycle of the Multigrid method is as follows;

1. Compute an approximation $\boldsymbol{v}$ of (2.7) on the discretized domain (called the fine grid) using an iterative method.
2. Compute the residual (2.9) and project it, using an interpolation method, to a discrete domain with less point (called the coarse grid).
3. Compute $\boldsymbol{e}$ from (2.10) on the coarse grid using an iterative method.
4. Extrapolate $\boldsymbol{e}$ to the fine grid and correct $\boldsymbol{v}$ as follows

$$\boldsymbol{v} = \boldsymbol{v} + \boldsymbol{e}.$$

5. With the corrected $\boldsymbol{v}$, do a few more iterations with the iterative method again on the fine grid.

In practice, steps (1), (3), and (5) involve at most three iterations of an iterative method which could be either (2.3), (2.4), or (2.5). This process can be done recursively up to the coarsest grid possible. For further details about the multigrid method see Briggs (1987), Hackbusch (1985), McCormick (1987), and Stuben & Trottenberg (1982).

We now use the Multigrid method to solve (2.2) with the Red-Black Gauss-Seidel method (2.5) for the relaxations. We use for the fine grid a domain with $N = 128$ and for the coarsest grid $N = 2$. We can satisfy the convergence criterion (2.6) with just one relaxation in steps (1), (3), and (5) of the above description. In Table 2.3 we show the execution times, in seconds, for the Multigrid method using the Red-Black Gauss-Seidel method for the relaxations, and the Red-Black Gauss-Seidel method alone. Note that for the model problem (2.2), the Multigrid method runs about 478 times faster than the Red-Black Gauss-Seidel method alone which in turn runs about four times faster than the Gauss-Seidel method

optimized. Thus with vectorization and parallelization and the Multigrid Method, in reference to Table 2.1, we are running about 9000 times faster than the best sequential method.

3. Numerical Solutions for the v. Kármán's Equations

In this section we apply the Multigrid method to find numerical solutions to the v. Kármán's equations.

Consider a plate of some material that occupies a unit square region in the plane. We apply a uniform pressure of λ units on the two sides normal to the x axis. The other two sides are free to move. When the applied pressure λ is small the plate remains uniformly compressed in the plane thus forming a smaller rectangle. When the pressure exceeds a critical value the plate deforms going out of the x,y plane (this is called a nontrivial deformation of the plate).

We assume that the nonlinear plate theory of v. Kármán (1910) describes the deformation of the plate. If $w(x,y)$ represents the lateral deviation and $f(x,y)$ represents the excess stress, then the following boundary value problem describes the deformation of the plate (Bauer & Reiss (1965)),

$$\begin{aligned} \Delta^2 f &= -\tfrac{1}{2}[w,w] \ , \\ \Delta^2 w &= [f,w] - \lambda w_{xx} \ , \ in \ \Omega \ , \\ w &= f = \Delta w = \Delta f = 0, \ on \ \partial\Omega. \end{aligned} \tag{3.1}$$

The parameter λ is the applied pressure, $\Omega = (0,1)^2$, and $\partial\Omega$ is the boundary of Ω. The nonlinear operator $[f,w]$ is defined by

$$[f,w] = f_{xx}w_{yy} + f_{yy}w_{xx} - 2f_{xy}w_{xy}. \tag{3.2}$$

We begin by reducing the order of the derivatives in (3.1), from order four to order two. We do this in order to work directly with the Laplacian operator because its numerical approximations have well known stability properties.

If we substitute $u = \Delta f$ and $v = \Delta w$ in (3.1) we obtain

$$\begin{aligned} \Delta u &= -\tfrac{1}{2}[w,w] \ , \\ \Delta f &= u \ , \\ \Delta v &= [f,w] - \lambda w_{xx} \ , \\ \Delta w &= v \ , \ in \ \Omega \ , \\ u &= v = f = w = 0, \ on \ \partial\Omega. \end{aligned} \tag{3.3}$$

Note that the boundary conditions in (3.3) are much simpler than those in (3.1).

In order to solve the problem (3.3) we introduce the following fixed point iteration:

$$
\begin{aligned}
\Delta u^{(k+1)} &= -\tfrac{1}{2}[w^{(k)}, w^{(k)}] \ , \\
\Delta f^{(k+1)} &= u^{(k+1)} \ , \\
\Delta v^{(k+1)} &= [f^{(k+1)}, w^{(k)}] - \lambda w_{xx}^{(k)} \ , \\
\Delta \omega^{(k+1)} &= v^{(k+1)} \ , \quad in \ \Omega \ , \\
w^{(k+1)} &= \theta \omega^{(k+1)} + (1 - \theta) w^{(k)} \ , \\
u^{(k)} &= v^{(k)} = f^{(k)} = \omega^{(k)} = 0, \ on \ \partial\Omega,
\end{aligned}
\tag{3.4}
$$

where θ is an acceleration parameter for the fixed point iteration. If the equations are solved in the order they appear in (3.4), then the right side of each equation is a known function of (x,y) for each k^{th} iteration. This is true except for the case where k=0 that $w^{(0)}$ is unknown. In order to start the iterations we need values for λ and $w^{(0)}$.

We will construct numerical approximations for the nontrivial branches, that bifurcate from the trivial branch $w = f = 0$. These branches are identified as first mode, second mode, etc. depending on the deformation of the plate on this branch. Using the fixed point iteration we constructed numerical approximations for the first and second modes.

Note that for any value of λ, the system (3.1) has the solution $w = f = 0$. The set of triples

$$
\{ (w, f, \lambda) \mid w=0 \ , \ f=0 \ , \ \lambda \in \mathbb{R} \}
$$

is called the trivial branch. The nontrivial solutions of the v. Kármán equations are branches that bifurcate from the trivial branch. The bifurcation points occur at the eigenvalues of the problem linearized about the trivial branch.

The linearization of the equations (3.1) about the trivial solution is formally obtained by setting the nonlinear operators equal to zero. This implies that $f = 0$ and that w is solution of the following eigenvalue problem:

$$
\begin{aligned}
\Delta^2 w + \lambda w_{xx} &= 0, \ in \ \Omega \ , \\
w = \Delta w &= 0 \ , on \ \partial\Omega.
\end{aligned}
\tag{3.5}
$$

This problem has nontrivial solutions (eigenfunctions)

$$
w = w_{m,n} = A_{m,n} \sin (m\pi x) \sin (n\pi y), \qquad m, n = 1, 2, \ldots \tag{3.6}
$$

if and only if

$$\lambda = \lambda_{m,n} = \pi^2 \left[m + \frac{n^2}{m} \right]^2, \qquad m,n=1,2\ldots, \tag{3.7}$$

where $\{A_{m,n}\}$ are arbitrary constants (see Bauer & Reiss (1965)).

Computational method

Consider the system (3.3) where each equation has the form of the model problem (2.1), if we suppose that the right hand side is known. In order to discretize this system we discretize each equation as in Section 2. This process renders a system that consists of four blocks of equations where each block has the form of the equation (2.2). To find numerical approximations for this system we use the fixed point iteration (3.4) together with the Multigrid method to solve each block of equations. We repeat the iteration (3.4) until satisfactory convergence is obtained.

We use the following relative error estimate as a convergence criterion

$$\| w_{k-1} - w_k \|_\infty < 10^{-4} \| w_k \|_\infty \ , \tag{3.8}$$

with a similar condition on f.

In order to compute numerical approximations to the nontrivial branches, we start the iterations with a value of λ in a neighborhood of a given eigenvalue. That is, we use $\lambda_{m,n} + \mu_0$, where μ_0 is some real number, and $\lambda_{m,n}$ is an eigenvalue. We use for $w^{(0)}$ in (3.4), the eigenfunction corresponding to this eigenvalue. We iterate (3.4) until we get satisfactory convergence in the sense of (3.8). Then we move along the nontrivial branch by first increasing the value of λ by a small value μ, and then iterating in (3.4) again using as starting point the values of the functions obtained with the last λ. We move in this way on the nontrivial branch until we arrive to a prescribed maximum value of λ.

The number of iterations necessary to satisfy (3.8) for a fixed λ depends on the acceleration parameter θ in (3.4). The optimal value of θ is defined as the one that minimizes the number of iterations to satisfy (3.8). Since the exact determination of this parameter is in general very difficult, it is usually estimated experimentally.

Numerical results

We computed numerical approximations for the branches corresponding to the first and second modes. We used $N = 32$ and the value of λ was increased until 700.0.

The eigenvalue corresponding to first mode is $\lambda_{1,1} = 4\pi^2 \approx 39.48$, and that for the second mode is $\lambda_{2,1} = \frac{25\pi^2}{4} \approx 61.69$.

For the computation of the first mode we used $\mu_0 = 0.1$, $\mu = 0.1$. Figure 3.1 shows values for $w(x, \frac{1}{2})$ for different values of λ, corresponding to the first mode.

In the approximation of the second mode the fixed point iteration for $61.68 \leq \lambda \leq 76.0$ converged to the first mode. A possible explanation for this is that for these values of λ the fixed point corresponding to first mode is attractive while that corresponding to the second mode is repulsive or unstable.

To start the iterations for the second mode we have to jump until $\lambda = 76.0$. We use $\mu_0 = 14.22$, and $\mu = 0.1$. Figure 3.2 shows values for the function $w(x, \frac{1}{2})$ for different values of λ, corresponding to the second mode.

The eigenvalue corresponding to the third mode is $\lambda_{3,1} = \frac{100\pi^2}{9} \approx 109.66$. The third mode could not be computed using this method. We could not find a neighborhood for $\lambda_{3,1}$ such that the iteration (3.4) converge to this mode. The iteration always converged to the first mode.

4 Conclusions

In this work we described and compared three iterative methods: Jacobi, Gauss-Seidel, and Red-Black Gauss-Seidel, using a model problem that consists of a linear partial differential equation. We found that the Red-Black Gauss-Seidel method, although is not the fastest method when running sequentially, when properly optimized for vectorization and parallelization, performs the best. We further observed that the Multigrid method provides further speedup (in the order of 500 times for the model problem) by solving the associated residual equation (2.10) on several coarse grids.

We constructed some numerical solutions for the first and second modes of the v. Kármán system of equations (3.1). Our approach was to use the fixed point iteration (3.4) together with the Multigrid method, which was used to solve each of the equations in (3.4). We found that the numerical solutions, at some points of the second mode, and at all points of the third mode, converged to the first mode. This occurs because for these values of λ the fixed point corresponding to the first mode is attractive, while those for the second and third modes are repulsive or unstable. To avoid this problem, one could proceed by eliminating the fixed point iteration completely, and include the nonlinear terms in each iteration. This would require the use of the Full Approximation Scheme (see Hackbusch (1985) and K. Stüben & Trottenberg (1982)) which is a generalization for nonlinear problems of the (linear) multigrid method that we used in Sections 2 and 3.

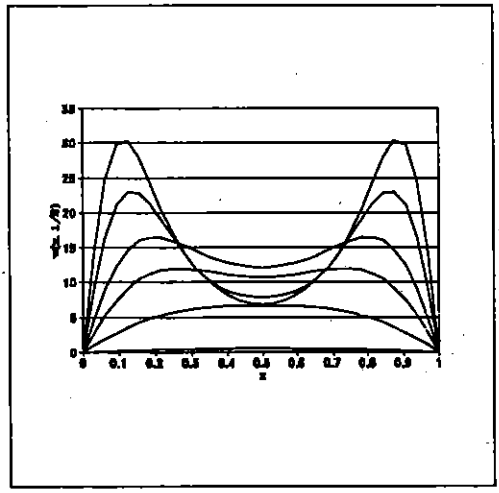

Figure 3.1. Graph of w(x,1/2) for different values of λ corresponding to the first mode.

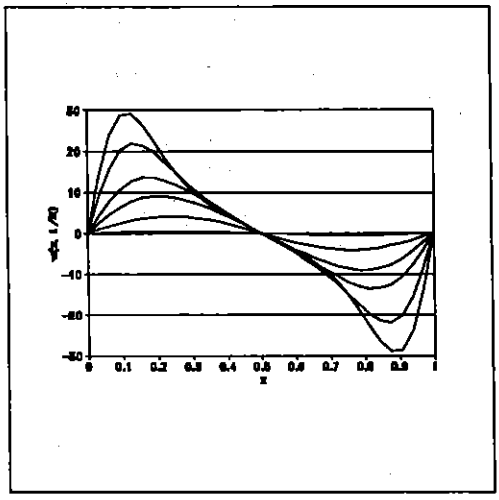

Figure 3.2. Graph of w(x,1/2) for different values of λ corresponding to the second mode.

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