Proceedings of the Royal Society of Edinburgh, **112A**, 277–291, 1989

Necked states of non-linearly elastic plates

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(MS received 8 June 1988. Revised MS received 4 October 1988)

Synopsis

In this paper we study the equilibrium equations for axisymmetric deformations of isotropic circular plates in tension. We give results on the global multiplicity of solutions and study the stability of the trivial homogeneous solution for large displacements.

1. Introduction

The equilibrium equations for non-linearly elastic circular plates under compression have been studied in **[2]** and **[13]**, in which results on the global multiplicity of solutions are established. In the present paper we study the multiplicity of solutions of isotropic circular plates in tension. We employ the general non-linear plate theory developed in **[2]**, modified to account for transverse deformations.

The corresponding problem for non-linearly elastic bars in tension has been studied among others by Antman and Carbone (see **[1]**, **[4]**) and recently in **[14]**, which includes a complete stability analysis. The geometry of deformation in this problem renders the equilibrium equations as an autonomous 2×2 system of equations which is naturally analysed by phase plane techniques. The detailed qualitative analysis of the solutions obtained in this way allowed Owen in **[14]** to show that non-monotone solutions (representing multiple necks) cannot be minimisers of the total energy in any sense. Crucial in the analysis of the bars is the strong ellipticity condition which implies that the stored energy function is convex in (v, p) (the radial strain and the derivative of the transverse deformation, respectively; cf. (2.11a)) making the stored energy functional weakly lower semicontinuous in the appropriate spaces.

For circular plates in tension, however, the polar singularity at the centre of the plate makes the equilibrium equations form a 2×2 non-autonomous system. Since the leading differential operators in these equations are of the form $(sy')'-y/s$ and $(sy')'-asy$, $a>0$, (cf. (6.3)), there is no apparent transformation of the independent variable s which yields an autonomous system. Moreover, our problem has the additional strain τ, the circumferential elongation, and although strong ellipticity still implies convexity of the stored energy function in (v, p), this is not the case in (τ, v, p). Thus in this respect our problem is similar to that studied in **[7]**, in which non-convex stored energy functions for solids undergoing phase transitions are considered. Despite these difficulties, we are

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able to use the degree theoretic methods in **[6, 10, 12, 15–18]** to show the existence of non-trivial solutions to the equilibrium equations (representing necked states) and to give a global qualitative description of the bifurcating branches.

The paper is organised as follows. In Section 2 we give the boundary value problem for circular plates in tension which can suffer transverse deformations. In Section 3 we present the basic bifurcation theory which will be used to study our boundary value problem. In Section 5 we study the linearised equations and obtain conditions for the existence of eigenvalues and eigenfunctions. Section 6, which is the main section of this paper, will be devoted to transforming our boundary value problem to the form required by the global branching theorem (Theorem 3.1). In Section 7 we study the stability of the trivial homogeneous state and in the Appendix (Section 9) we give a derivation of our equation from the three-dimensional theory of non-linear elasticity.

Notation. $\mathbb{R}$ denotes the set of real numbers. We let

$$C^k[0,1]=\{u\colon[0,1]\to\mathbb{R}\mid u^{(j)}\text{ is continuous in }[0,1],\ 0\leqq j\leqq k\},$$

$$\|u;C^k[0,1]\|=\max_{0\leqq j\leqq k}\ \max_{0\leqq x\leqq 1}|u^{(j)}(x)|.$$

We use the notation

$$f(s)\sim g(s)\quad\text{as}\quad s\to 0^+,$$

which means that $f(s)/g(s)\to 1$ as $s\to 0^+$.

2. Formulation of the governing equations

In this section we give the equations for axisymmetric unbuckled states of non-linearly elastic plates which can suffer radial, circumferential, and transverse deformations. The derivation of these equations from the three-dimensional theory of non-linear elasticity is given in the Appendix (Section 9).

Consider an axisymmetric deformation of a homogeneous non-linearly elastic circular plate of radius one which is a pure compression or expansion, i.e. no bending and no shear. Let $(\rho(s),\ \varepsilon(s))$ be the radius and half of the height, respectively, in the deformed configuration of the cylindrical cross-section of radius s and height $2h$ in the reference configuration. The strains in this problem are

$$\mathbf{w}(s)=(\tau(s),\ v(s),\ p(s),\ \varepsilon(s))=(\rho(s)/s,\ \rho'(s),\ \varepsilon'(s),\ \varepsilon(s)),\tag{2.1}$$

where $\rho'(s)=d\rho(s)/ds$, etc. Here $\tau(s)$ represents the elongation of a circumferential fibre, $v(s)$ accounts for the elongation of a radial fibre, and $p(s)$ accounts for thickness variations of a mid-cross-section of the plate. The requirement that the deformation be orientation preserving leads to the inequalities

$$\rho(s)/s,\ \rho'(s),\ \varepsilon(s)>0,\quad 0\leqq s\leqq 1.\tag{2.2a,b,c}$$

The stresses corresponding to (2.1) are

$$(\hat{T}(s),\ \hat{N}(s),\ \hat{P}(s),\ \hat{U}(s)).\tag{2.3}$$

The classical form of the equilibrium equations (balance of linear and angular momentum) now yields the following system of equations:

$$(s\hat{N}(s))' = \hat{T}(s), \tag{2.4a}$$

$$(s\hat{P}(s))' = s\hat{U}(s), \quad 0 < s < 1. \tag{2.4b}$$

The requirement that the centre of the plate remains intact during the deformation is equivalent to

$$\rho(0) = 0. \tag{2.5}$$

At $s = 1$ we can impose two types of boundary conditions, namely a displacement boundary condition,

$$\rho(1) = \lambda, \quad \lambda > 0, \tag{2.6}$$

or a dead load boundary condition,

$$\hat{N}(1) = \gamma, \quad \gamma \in \mathbb{R}. \tag{2.7}$$

On ε we shall consider the boundary conditions

$$\lim_{s \to 0^+} s\hat{P}(s) = 0 = \hat{P}(1), \tag{2.8a,b}$$

which are the natural boundary conditions corresponding to a suitable variational formulation of (2.4), (2.5), (2.6) or (2.7) (see Appendix). We shall contrast (2.8b) with the corresponding and more realistic Dirichlet boundary condition on ε (see Section 8).

That the material of the plate be homogeneously elastic means that there exist thrice continuously differentiable functions

$$(0, \infty)^2 \times \mathbb{R} \times (0, \infty) \ni \mathbf{w} \mapsto T(\mathbf{w}), N(\mathbf{w}), P(\mathbf{w}), U(\mathbf{w}), \tag{2.9}$$

such that

$$\hat{T}(s) = T(\mathbf{w}(s)) = T(\rho(s)/s, \rho'(s), \varepsilon'(s), \varepsilon(s)), \text{ etc.} \tag{2.10}$$

The strong ellipticity condition of three-dimensional elasticity implies that the constitutive functions $T(\tau, \nu, p, \varepsilon)$, etc., satisfy

$$\begin{pmatrix} N_\nu & N_p \\ P_\nu & P_p \end{pmatrix} \text{ is positive definite,} \tag{2.11a}$$

$$T_\tau, U_\varepsilon > 0, \quad \text{for} \quad \tau, \nu, \varepsilon > 0, \quad p \in \mathbb{R}. \tag{2.11b,c}$$

We impose the further monotonicity condition, which is implied by a plate theoretic version of the Coleman–Noll condition (see e.g. [**20**]), that

$$\begin{pmatrix} N_\nu & N_\tau \\ T_\nu & T_\tau \end{pmatrix} \text{ is positive definite,} \tag{2.11d}$$

where the arguments of the functions are $(\tau, \nu, 0, \varepsilon)$. This condition guarantees that solutions with $\varepsilon' = 0$ of our displacement boundary value problem are unique.

We impose the physically reasonable conditions that

$$N, T, U \text{ are even in } p, \tag{2.12a}$$

$$P \text{ is odd in } p. \tag{2.12b}$$

Conditions (2.10), (2.11a), and (2.12b) imply that for classical solutions of our equations, the boundary conditions (2.8) are equivalent to

$$\lim_{s\to 0^+} s\varepsilon'(s) = 0 = \varepsilon'(1). \tag{2.13a,b}$$

The condition that the material of the plate be isotropic implies that (see Appendix):

$$N(\tau, \nu, 0, \varepsilon) = T(\nu, \tau, 0, \varepsilon), \tag{2.14a}$$

$$U(\tau, \nu, 0, \varepsilon) = U(\nu, \tau, 0, \varepsilon). \tag{2.14b}$$

Finally, we impose the following growth conditions;

$$N \to \infty \quad \text{as} \quad \nu \to \infty, \tag{2.15a}$$

$$N \to -\infty \quad \text{as} \quad \nu \to 0^+, \tag{2.15b}$$

$$T \to \infty \quad \text{as} \quad \tau \to \infty, \tag{2.16a}$$

$$T \to -\infty \quad \text{as} \quad \tau \to 0^+, \tag{2.16b}$$

$$P \to \pm\infty \quad \text{as} \quad p \to \pm\infty, \tag{2.17}$$

$$U \to \infty \quad \text{as} \quad \varepsilon \to \infty, \tag{2.18a}$$

$$U \to -\infty \quad \text{as} \quad \varepsilon \to 0^+. \tag{2.18b}$$

These limits are to hold for fixed values of the remaining variables.

Our (displacement) boundary value problem consists of (2.4), (2.5), (2.6), (2.10), and (2.13). The solutions must satisfy the inequalities (2.2). The (dead load) boundary value problem is as above, but using (2.7) instead of (2.6).

3. The global branching theorem

We now give the basic branching theorem of bifurcation theory which we shall use to study the boundary value problem of Section 2. Most of our effort in this paper will be devoted to the verification of the hypotheses of this theorem.

Let B be a real Banach space with norm $\|\cdot\|$ and let V be the closure of an open subset of B that contains the zero vector $\mathbf{0}$ of B. Consider the equation

$$\mathbf{u} = G(\mathbf{u}, \lambda) \equiv L(\lambda)\mathbf{u} + F(\mathbf{u}, \lambda), \tag{3.1}$$

where $G: V \times \mathbb{R} \to B$ is compact and continuous, $L(\lambda): B \to B$ is linear and compact, $L(\cdot)$ is in $C^1(\mathbb{R}, \operatorname{Lin}(B, B))$, and $F(\mathbf{u}, \lambda) = o(\|\mathbf{u}\|)$ as $\|\mathbf{u}\| \to 0$ uniformly for λ in bounded intervals.

A pair $(\mathbf{u}, \lambda)$ satisfying (3.1) is called a *solution pair*. The *trivial branch* of solutions is the set $\{(\mathbf{0}, \lambda) \mid \lambda \in \mathbb{R}\}$ of trivial solution pairs. Let S denote the closure of non-trivial solution pairs in $V \times \mathbb{R}$. The *linearisation* of (3.1) about the trivial branch is the equation

$$\mathbf{v} = L(\lambda)\mathbf{v}. \tag{3.2}$$

A value of λ for which (3.2) has a non-trivial solution is an *eigenvalue* of (3.2) and a corresponding solution is a corresponding *eigenvector*. An eigenvector λ of (3.2) is *simple* if

$$\text{Null}\,(I - L(\lambda)) = \text{span}\,\{\mathbf{v}\}, \tag{3.3a}$$

$$L'(\lambda)\mathbf{v} \notin \text{Range}\,(I - L(\lambda)), \tag{3.3b}$$

where I is the identity mapping on B. A general definition of the algebraic multiplicity of an eigenvalue of (3.2) is given in [**9, 10, 12**].

The basic theorem of this section is the following, due to Rabinowitz [**16–18**] and Magnus [**12**].

THEOREM 3.1 (global branching theorem). *Let G have the properties listed above. If μ is an eigenvalue of* (3.2) *with odd algebraic multiplicity, then S contains a maximal closed connected subset $C(\mu)$ which contains $(\mathbf{0}, \mu)$ and has at least one of the following three properties*:

(i) *$C(\mu)$ is unbounded in $V \times \mathbb{R}$.*
(ii) *$C(\mu)$ contains a point of $\partial V \times \mathbb{R}$.*
(iii) *$C(\mu)$ contains a point of the form $(\mathbf{0}, \hat{\mu})$, where $\hat{\mu}$ is another eigenvalue of* (3.2).

Suppose, moreover, that μ is simple and that there is a neighbourhood N of $(\mathbf{0}, \mu)$ and a continuous, increasing function Φ: $[0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$\|F(\mathbf{x}, \lambda) - F(\mathbf{y}, \eta)\| \leqq \Phi(\|\mathbf{x}\| + \|\mathbf{y}\|)[\|\mathbf{x} - \mathbf{y}\| + (\|\mathbf{x}\| + \|\mathbf{y}\|)\,|\lambda - \eta|] \tag{3.4}$$

for all $(\mathbf{x}, \lambda)$, $(\mathbf{y}, \eta) \in N$. Let $\mathbf{v}$ be the eigenfunction of (3.2) corresponding to μ and let $P(\mu)$ be the projection of B onto Null $[I - L(\mu)]$. Then there exists a number $\bar{\varepsilon} > 0$ and continuous functions $\mathbf{z}$: $[-\bar{\varepsilon}, \bar{\varepsilon}] \to [I - P(\mu)]B$ and k: $[-\bar{\varepsilon}, \bar{\varepsilon}] \to \mathbb{R}$ with $\mathbf{z}(0) = \mathbf{0}$, $k(0) = 0$ such that $(\varepsilon(\mathbf{v} + \mathbf{z}(\varepsilon)),\ \mu + k(\varepsilon))$ is a solution pair of (3.1) for $|\varepsilon| \leqq \bar{\varepsilon}$. Moreover, there is a neighbourhood N_0 of N such that if $(\mathbf{u}, \lambda)$ is a solution of (3.1) lying in N_0, then either $\mathbf{u} = \mathbf{0}$ or else there is an $\varepsilon \in [-\bar{\varepsilon}, \bar{\varepsilon}]$ such that

$$(\mathbf{u}, \lambda) = (\varepsilon(\mathbf{v} + \mathbf{z}(\varepsilon)), \quad \mu + k(\varepsilon)). \quad \square$$

4. The trivial solution

We look for solutions of the displacement boundary value problem of Section 2 for which $\varepsilon' = 0$, i.e. $\varepsilon =$ constant. We take ρ to be of the form $\rho = \lambda s$ which clearly satisfies (2.5) and (2.6), and also (2.4a) by the isotropy condition (2.14a). The condition (2.12b) then reduces (2.4b) to

$$U(\lambda, \lambda, 0, \varepsilon) = 0. \tag{4.1}$$

Now conditions (2.11c), (2.18), and a global implicit function theorem imply that there exists a C^1 function $\hat{\varepsilon}$: $(0, \infty) \to (0, \infty)$ such that

$$U(\lambda, \lambda, 0, \hat{\varepsilon}(\lambda)) = 0. \tag{4.2}$$

Moreover,

$$d\hat{\varepsilon}(\lambda)/d\lambda = -(U^0_{\nu} + U^0_{\tau})U^0_{\varepsilon}, \tag{4.3a}$$

where

$$U^0_{\nu} = U_{\nu}(\lambda, \lambda, 0, \hat{\varepsilon}(\lambda)), \text{ etc.} \tag{4.3b}$$

Thus $\rho(s)=\lambda s$, $\varepsilon(s)=\hat{\varepsilon}(\lambda)$ is a solution pair for the displacement boundary value problem of Section 2 and it is easy to check that (2.11d) implies that this is the only solution with $\varepsilon'=0$.

For the dead load boundary value problem the situation is not as "simple" as above. Taking $\varepsilon'=0$ and looking for ρs of the form $\rho=ks$, the boundary value problem (2.4), (2.5), (2.7), and (2.8) reduces (by means of (2.12b) and (2.14a) to

$$N(k, k, 0, \varepsilon)=0, \tag{4.4a}$$

$$U(k, k, 0, \varepsilon)=0. \tag{4.4b}$$

Solving (4.4b) as above, we obtain $\varepsilon=\hat{\varepsilon}(k)$. By putting this in the first equation we find that

$$\Omega(k, \gamma)\equiv N(k, k, 0, \hat{\varepsilon}(k))-\gamma=0. \tag{4.5}$$

This equation, however, can have multiple solutions if $\Omega(\cdot, \gamma)$ is non-monotone. One condition which avoids this non-monotonicity is the strengthened version of (2.11d) given by

$$\begin{pmatrix} N_v & N_\tau & N_\varepsilon \\ T_v & T_\tau & T_\varepsilon \\ U_v & U_\tau & U_\varepsilon \end{pmatrix} \text{ is positive definite,} \tag{4.6}$$

where the arguments of the functions are $(\tau, v, 0, \varepsilon)$. However, we will see in Section 5 that condition (4.6) implies that there is no bifurcation of non-trivial solutions from the trivial solution ($\varepsilon'=0$). Thus we do not require (4.6) but keep (2.11d), which gives the uniqueness of trivial solutions for the displacement boundary value problem and allows for multiple solutions of (4.5) (when $\Omega(\cdot, \gamma)$ is non-monotone). Henceforth, we consider only the displacement boundary value problem of Section 2.

5. Spectral properties of the linearised equations

We consider here the linearisation of the boundary value problem of Section 2 about the trivial solution of Section 4. By using (2.12) and (2.14) one finds that the linearised equations are

$$Lv\equiv(sv')'-v/s=-(N_\varepsilon^0/N_v^0)sw', \tag{5.1a}$$

$$(sw')'-(U_\varepsilon^0/P_p^0)sw=(U_v^0/P_p^0)(sv)', \quad 0<s<1, \tag{5.1b}$$

$$v(0)=0=v(1), \quad \lim_{s\to 0^+} sw'(s)=0=w'(1), \tag{5.2a,b,c,d}$$

where we used the notation (4.3b). (Note that among the consequences of (2.14) is that $U_v^0=U_\tau^0$.) By further differentiations giving a higher order boundary value problem for w (or v) one can obtain the solution of (5.1), (5.2) in terms of J_p, the Bessel function of the first kind of order p. In fact, if we let

$$A(\lambda)=(N_\varepsilon^0U_v^0-U_\varepsilon^0N_v^0)P_p^0N_v^0, \tag{5.3}$$

then (5.1), (5.2) has non-trivial solutions if and only if

$$A(\lambda)=\xi_k^2, \tag{5.4}$$

where

$$0<\xi_1<\xi_2<\xi_3<\dots \tag{5.5}$$

are the positive zeros of J_1. Moreover, if λ_k is a solution of (5.4), then (v_k, w_k) is a corresponding eigenpair of (5.1), (5.2) where

$$v_k(s)=J_1(A(\lambda_k)^{\frac{1}{2}}s)=J_1(\xi_k s), \tag{5.6a}$$

$$w_k(s)=-(\xi_k N_v^0/N_\varepsilon^0)J_0(\xi_k s). \tag{5.6b}$$

Note that when $A(\lambda)<0$, equation (5.4) has no solution. This shows that when condition (4.6) holds, the linearised problem has no eigenvalues and thus that there can be no bifurcation from the trivial solution for the boundary value problem of Section 2.

6. Global existence of bifurcating branches

In this section we recast the boundary value problem of Section 2 in the form required by the branching theorem of Section 3. The heart of the process rests on a careful exploitation of the isotropy conditions (2.14) to deal with the singularity at $s=0$ of the equations (2.4).

Expanding the derivatives in (2.4) and using (2.11a), we find that

$$s\rho''=(aP_p-bN_p)/\Delta, \tag{6.1a}$$

$$s\varepsilon''=(bN_v-aP_v)/\Delta, \tag{6.1b}$$

where

$$a=T-N-s[(\rho/s)'N_\tau+N_\varepsilon\varepsilon'], \tag{6.2a}$$

$$b=sU-P-s[(\rho/s)'P_\tau+P_\varepsilon\varepsilon'], \tag{6.2b}$$

$$\Delta=N_vP_p-N_pP_v; \tag{6.2c}$$

here the arguments of T, etc., are $(\rho/s, \rho', \varepsilon', \varepsilon)$. The linearisation of (6.1) about the trivial solution $(\lambda, \lambda, 0, \hat{\varepsilon}(\lambda))$ of Section 4 is given by (5.1). Thus we can write (6.1) as

$$(s\alpha')'-\alpha/s=-(N_\varepsilon^0/N_v^0 s\omega'+F_1. \tag{6.3a}$$

$$(s\omega')'-(U_\varepsilon^0/P_p^0)s\omega=(U_v^0/P_p^0)(s\alpha)'+F_2, \tag{6.3b}$$

where

$$\alpha=\rho-\lambda s,\quad \omega=\varepsilon-\hat{\varepsilon}(\lambda), \tag{6.4a,b}$$

$$F_1=(aP_p-bN_p)/\Delta-[\alpha/s-\alpha'-(N_\varepsilon^0/N_v^0)s\omega'], \tag{6.4c}$$

$$F_2=(bN_v-aP_v)/\Delta-[(U_\varepsilon^0/P_p^0)s\omega-\omega'+(U_v^0/P_{op})(s\alpha)']. \tag{6.4d}$$

We let $\mathbf{w}(s)$ be as in (2.1). Let

$$\mathbf{w}^*=(\rho', \rho/s, 0, \varepsilon), \tag{6.5a}$$

$$\hat{\mathbf{w}}=(\rho/s, \rho', 0, \varepsilon). \tag{6.5b}$$

Hence using Taylor's Theorem to expand T, N_τ, and N_ε about $\mathbf{w}^*$ and N about $\hat{\mathbf{w}}$

up to terms of order two, and by using (2.12a) and (2.14a) one finds that

$$a = (T_\tau(\mathbf{w}^*) - T_\nu(\mathbf{w}^*) + N_\tau(\mathbf{w}^*))(\rho/s - \rho') \\ - sN_\varepsilon(\mathbf{w}^*)\varepsilon' + A_{ij}(\mathbf{w})(w_i - w_i^*)(w_j - w_j^*) \tag{6.6a}$$

where the A_{ij}s are continuous functions of $\mathbf{w}$ and repeated indices are summed from 1 to 3. Similarly, we find that

$$b = sU(\mathbf{w}^*) + s(U_\tau(\mathbf{w}^*) - U_\nu(\mathbf{w}^*))(\rho/s - \rho') \\ -P_p(\mathbf{w}^*)\varepsilon' + B_{ij}(\mathbf{w})(w_i - w_i^*)(w_j - w_j^*), \tag{6.6b}$$

$$\Delta^{-1} = (P_p(\mathbf{w}^*)N_\nu(\mathbf{w}^*))^{-1} + \Delta_i(\mathbf{w})(w_i - w_i^*), \tag{6.6c}$$

where the B_{ij}s and Δ_is are continuous functions of $\mathbf{w}$. By substituting (6.6) into (6.4c,d) and using (2.12) and (2.14) to simplify the resulting expressions, one finds that F_1 and F_2 consist of sums of products of continuous functions of $\mathbf{w}$ with quadratic functions of

$$\alpha' - \alpha/s, \quad \omega', s^{\frac{1}{2}}\alpha', s^{\frac{1}{2}}\alpha, s^{\frac{1}{2}}\omega. \tag{6.7}$$

From the definitions of α and ω, the boundary conditions (2.5), (2.6), and (2.13) become

$$\alpha(0) = 0 = \alpha(1), \tag{6.8a,b}$$

$$\lim_{s\to 0^+} s\omega'(s) = 0 = \omega'(1). \tag{6.9a,b}$$

We define (u_1, u_2) by

$$s^{\frac{1}{2}}u_1(s) = (s\alpha')' - \alpha/s, \tag{6.10a}$$

$$s^{\frac{1}{2}}u_2(s) = (sw')' - (U_\varepsilon^0/P_p^0)sw. \tag{6.10b}$$

Let $g_1(s, t)$ and $g_2(s, t)$, respectively, be the Green's functions corresponding to the operators to the right of (6.10) subject respectively to the boundary conditions (6.8) and (6.9). Thus

$$g_1(s, t) = \tfrac{1}{2}\begin{cases} s(t - t^{-1}), & s < t, \\ t(s - s^{-1}), & t < s, \end{cases} \tag{6.11a}$$

$$g_2(s, t) = y_1'(a)^{-1}\begin{cases} (y_2(at)y_1'(a) - y_2'(a)y_1(at))y_1(as), & s < t, \\ (y_2(as)y_1'(a) - y_2'(a)y_1(as))y_1(at), & t < s, \end{cases} \tag{6.11b}$$

where

$$a^2 = U_\varepsilon^0/P_p^0, \tag{6.12}$$

$$y_1(s) = J_0(is), \quad y_2(s) = Y_0(is), \tag{6.13a,b}$$

where J_0 and Y_0 are the Bessel functions of the first and second kind of order zero, and $i^2 = -1$. Note that

$$y_1(s) \sim 1, \quad y_1'(s) \sim s/2 \quad \text{as} \quad s \to 0^+, \tag{6.14a,b}$$

$$y_2(s) \sim \log s, \quad y_2'(s) \sim s^{-1} \quad \text{as} \quad s \to 0^+. \tag{6.14c,d}$$

We define

$$G_i u = \int_0^1 g_i(s, t)t^{\frac{1}{2}}u(t)\,dt, \tag{6.15a}$$

$$G_i' u = \int_0^1 \frac{\partial g_i}{\partial s}(s, t)t^{\frac{1}{2}}u(t)\,dt, \quad i = 1, 2. \tag{6.15b}$$

Thus from (6.10), (6.11), and (6.15) we find that

$$\alpha = G_1 u_1, \quad \alpha' = G_1' u_1, \tag{6.16a,b}$$

$$\omega = G_2 u_2, \quad \omega' = G_2' u_2. \tag{6.17a,b}$$

Now (6.3), (6.7), (6.10), (6.16), (6.17) imply that

$$u_1 = -(N_\varepsilon^0/N_\nu^0) s^{\frac{1}{2}} G_2' u_2 + \hat{F}_1(u_1, u_2, \lambda), \tag{6.18a}$$

$$u_2 = (U_\nu^0/P_p^0)(s^{-\frac{1}{2}} G_1 u_1 + s^{\frac{1}{2}} G_1' u_1) + \hat{F}_2(u_1, u_2, \lambda), \tag{6.18b}$$

where $\hat{F}_1$ and $\hat{F}_2$ are sums of products of continuous functions of $s^{-1}G_1u_1$, $G_1'u_1$, $G_2'u_2$, G_2u_2 with quadratic functions of

$$s^{-\frac{1}{4}}(G_1'u_1 - s^{-1}G_1u_1),\ s^{-\frac{1}{4}}G_2'u_2,\ s^{\frac{1}{4}}G_1'u_1,\ s^{\frac{1}{4}}G_1u_1,\ s^{\frac{1}{4}}G_2u_2. \tag{6.19}$$

We now show that the right-hand side of (6.18) defines a compact operator from $C[0, 1]$ into itself. For this we shall need the following lemma due to Radon [19].

LEMMA 6.1. *Let* $\Gamma: [0, 1] \times [0, 1] \to [-\infty, \infty]$ *be such that* $\Gamma(s, \cdot)$ *is integrable for each* $s \in [0, 1]$. *Then*

$$u \in C[0, 1] \mapsto \int_0^1 \Gamma(\cdot, t) u(t)\, dt \in C[0, 1] \tag{6.20a}$$

is a compact linear mapping if and only if

$$\lim_{y \to s} \int_0^1 |\Gamma(y, t) - \Gamma(s, t)|\, dt = 0 \tag{6.20b}$$

for all $s \in [0, 1]$. □

Using this lemma, we can now prove:

PROPOSITION 6.2. *The operators* $s^{-\frac{1}{4}}(G_1' - s^{-1}G_1)$, G_1', $s^{-1}G_1$, $s^{-\frac{1}{4}}G_2'$, G_2 *and multiples of these with positive powers of s generate compact linear operators from* $C[0, 1]$ *into itself.*

Proof. We check only the compactness of $s^{-\frac{1}{4}}G_2'$. The other operators can be checked similarly. The kernel of this operator is

$$\Gamma(s, t) = \frac{\partial g_2}{\partial s}(s, t) s^{-\frac{1}{4}} t^{\frac{1}{2}}. \tag{6.21a}$$

The asymptotic estimate (6.14b) implies that $\Gamma(0, t) = 0$, except for $t = 0$. Thus from (6.11b) and (6.21a) we find that

$$\begin{aligned} &\lim_{s \to 0^+} \int_0^1 |\Gamma(s, t) - \Gamma(0, t)|\, dt \\ &\quad \leqq (a/y_1'(a)) \lim_{s \to 0^+} \Big\{ |y_2'(as) y_1'(a) - y_2'(a) y_1'(as)|\, s^{-\frac{1}{4}} \int_0^s |y_1(at)|\, t^{\frac{1}{2}}\, dt \\ &\quad + |y_1'(as)|\, s^{-\frac{1}{4}} \int_s^1 |y_2(at) y_1'(a) - y_2'(a) y_1(at)|\, t^{\frac{1}{2}}\, dt \Big\}. \end{aligned} \tag{6.21b}$$

The estimates (6.14) give

$$(a/y_1'(a))(y_2'(as)y_1'(a)-y_2'(a)y_1'(as))s^{-\frac{1}{4}} \sim s^{-\frac{5}{4}} \quad \text{as} \quad s\to 0^+,$$

$$\int_0^s y_1(at)t^{\frac{1}{2}}\,dt \sim (\tfrac{2}{3})s^{\frac{3}{2}} \quad \text{as} \quad s\to 0^+,$$

$$y_1'(as)s^{-\frac{1}{4}} \sim (a/2)s^{\frac{3}{4}} \quad \text{as} \quad s\to 0^+,$$

$$(y_2(at)y_1'(a)-y_2'(a)y_1(at))t^{\frac{1}{2}} \sim t^{\frac{1}{2}}\log t \quad \text{as} \quad t\to 0^+.$$

By using these estimates in (6.21b) we obtain (6.20b) for $s=0$.

Now from (6.11b) and (6.21a) we have that for $s>0$, $\Gamma(s,\cdot)$ is integrable, $\Gamma(y,\cdot)\to\Gamma(s,\cdot)$ as $y\to s$ except for $t=s$, and $\Gamma(y,\cdot)$ is uniformly bounded for y sufficiently close to s. Hence the Lebesque dominated convergence theorem implies that (6.20b) holds for $s>0$. This with the result for $s=0$ and Lemma 6.1 imply that $s^{-\frac{1}{4}}G_2'$ is compact from $C[0,1]$ into itself. □

We define

$$B = C[0,1]\times C[0,1], \tag{6.22a}$$

V = closure of those $(u_1,u_2)\in B$ such that (6.4a,b), (6.16), (6.17) satisfy (2.2), (6.22b)

$$G(u_1,u_2,\lambda) = \text{right-hand side of (6.18)}, \tag{6.23a}$$

$$F(u_1,u_2,\lambda) = (\hat{F}_1,\hat{F}_2), \tag{6.23b}$$

$$L(\lambda)(u_1,u_2) = G(u_1,u_2,\lambda) - F(u_1,u_2,\lambda). \tag{6.23c}$$

Then we have the following theorem which is the main result of this section.

THEOREM 6.3. *Let* (2.11), (2.12), *and* (2.14) *hold. Let* $\bar{\lambda}$ *be an eigenvalue of odd algebraic multiplicity with corresponding eigenpair* $(\bar{v},\bar{w})$ *of the linearised problem* (5.1), (5.2). *Then the boundary value problem* (2.4), (2.5), (2.6), (2.10) *and* (2.13) *has a connected set of solutions* $\mathscr{C}(\bar{\lambda})$ *containing* $(\rho=\bar{\lambda}s,\ \varepsilon=\hat{\varepsilon}(\bar{\lambda}),\bar{\lambda})$ *that: satisfies* (2.2) *always and is unbounded in* $(C^1[0,1])^2\times\mathbb{R}$; *or contains a point that violates* (2.2); *or contains a point of the form* $(\hat{\lambda}s,\hat{\varepsilon}(\hat{\lambda}),\hat{\lambda})$ *where* $\hat{\lambda}$ *is another eigenvalue of* (5.1), (5.2).

Proof. Since the displacement boundary value problem of Section 2 is equivalent to the integral equations (6.18), the result follows from the representation (6.19), Proposition 6.2, definitions (6.22) and (6.23), and the global branching theorem (Theorem 3.1). □

7. Stability analysis of the trivial solution

In this section we show that if the material of the plate is hyperelastic, so that the boundary value problem of Section 2 has a variational formulation, then for sufficiently large values of λ the trivial solution of Section 4 cannot be a local minimiser (in any sense) of the stored energy functional. Thus according to the energy criteria for stability, the trivial solution becomes unstable for these values of λ.

If the material of the plate is hyperelastic, then there exists a function $W: (0, \infty)^2 \times \mathbb{R} \times (0, \infty) \to \mathbb{R}$, the stored energy function (see Appendix), such that

$$T = W_\tau, \quad N = W_\nu, \quad P = W_p, \quad U = W_\varepsilon, \tag{7.1}$$

and the total stored energy of the plate is given by

$$I(\rho, \varepsilon) = \int_0^1 sW(\rho(s)/s, \rho'(s), \varepsilon'(s), \varepsilon(s))\, ds. \tag{7.2}$$

We consider the problem of minimising I over the set

$$H_\lambda = \{(\rho, \varepsilon) \in W^{1,1}(0, 1) \mid (2.2) \text{ holds almost everywhere.}, \\ \rho(0) = 0,\ \rho(1) = \lambda,\ I(\rho, \varepsilon) < \infty\} \tag{7.3}$$

for a given $\lambda > 0$. (Hence $W^{1,1}(0, 1)$ denotes the space of absolutely continuous functions on $[0, 1]$.)

A necessary condition for $(\rho_0, \varepsilon_0) \in H_\lambda$ to be a local minimum of I over H_λ is that $\delta^2 I(\rho_0, \varepsilon_0)$ (the second variation of I at (ρ_0, ε_0)) be positive semidefinite, i.e.

$$\delta^2 I(\rho_0, \varepsilon_0) \cdot (v, w) \geqq 0 \tag{7.4}$$

for all smooth (v, w) with $v(0) = 0 = v(1)$. Let $\rho_0 = \lambda s$, $\varepsilon_0 = \hat{\varepsilon}(\lambda)$ be the trivial solution of Section 4. Hence $(\rho_0, \varepsilon_0) \in H_\lambda$ and from (2.12), (7.1), and (7.2) we find that

$$\begin{aligned} \delta^2 I(\rho_0, \varepsilon_0) \cdot (v, w) = \int_0^1 s[&(T^0_\tau(v/s) + T^0_\nu v' + T^0_\varepsilon w)(v/s) \\ &+ (N^0_\tau(v/s) + N^0_\nu v' + N^0_\varepsilon w)v' \\ &+ (U^0_\tau(v/s) + U^0_\nu v' + U^0_\varepsilon w)w + P^0_p(w')^2]\, ds \end{aligned} \tag{7.5}$$

where the notation (4.3b) has been used. We try variations of the form

$$\bar{v} = J_1(\xi_1 s), \tag{7.6a}$$

$$\bar{w} = -(\xi_1 N^0_\nu / N^0_\varepsilon) J_0(\xi_1 s) \text{ (cf. (5.6))}. \tag{7.6b}$$

Since $J_1(\xi_1) = 0$ (see (5.5)) we have that $(\bar{v}, \bar{w})$ is a variation. A simple calculation now shows that

$$\delta^2 I(\rho_0, \varepsilon_0) \cdot (\bar{v}, \bar{w}) = (\text{positive constant}) \times (\xi_1^2 - A(\lambda)), \tag{7.7}$$

where $A(\lambda)$ is as in (5.3). Hence from (7.4) and (7.7) we obtain the following result.

THEOREM 7.1. *Let the material of the plate be hyperelastic so that* (7.1) *holds. Hence the trivial solution* (ρ_0, ε_0) *of Section* 4 *cannot be a local minimum* (*in any sense*) *of I over H_λ for those values of λ for which*

$$A(\lambda) > \xi_1^2. \quad \square \tag{7.8}$$

By refining the techniques in [**3**] so that they apply to hyperelastic materials, one can show that the problem of minimising (7.2) over (7.3) has a classical solution belonging to some weighted Sobolev space (the weight is given by the s

in front of W in (7.2)) characterised by the growth conditions on the stored energy function W. Thus combining this with Theory 7.1 one finds that for values of λ such that (7.8) holds, the problem of minimising (7.2) over (7.3) has a non-trivial classical solution.

By combining (5.3) and (7.1) we find that (7.8) is equivalent to

$$((W^0_{\nu\varepsilon})^2 - W^0_{\varepsilon\varepsilon}W^0_{\nu\nu})/W^0_{\rho\rho}W^0_{\nu\nu} > \xi_1^2. \tag{7.9}$$

Note that this does not necessarily represent a loss of convexity of W. In fact there is no convexity assumption of W which is only required to satisfy (2.11). However, as was suggested in Section 4, condition (7.9) implies the loss of convexity of any antiderivative with respect to k of the function $\Omega(k, \gamma)$ defined by (4.5).

8. Conclusions and remarks

The non-trivial solutions whose existence we established in Section 6 can be interpreted as necked states when the plate is in tension ($\lambda > 1$), and as bulged or barrelled states when the plate is in compression ($\lambda < 1$). These branches of non-trivial solutions exist as long as (2.2) holds, which in turn depends on the growth rates in (2.15)–(2.18). If an eigenvalue $\bar{\lambda}$ is simple with corresponding eigenpair $(\bar{v}, \bar{w})$, then by Theorem 3.1 the bifurcating branch $\mathscr{C}(\bar{\lambda})$ inherits the nodal properties (number of simple zeros in $(0, 1)$) of $(\bar{v}, \bar{w})$ in a neighbourhood of $(\bar{\lambda}s, \hat{\varepsilon}(\bar{\lambda}), \bar{\lambda})$. This number of simple zeros characterises the necked shape of the plate in this neighbourhood. A detailed disposition of the bifurcating branches could be obtained by specifying the growth rates in (2.15)–(2.18). However, we refrain from doing so, because such extreme behaviour is usually unobservable and our methods do not require them.

A more realistic boundary condition on ε at $s = 1$ is (after an appropriate scaling)

$$\varepsilon(1) = 1. \tag{8.1}$$

This means that the plate is welded at its edge, so that it can suffer no change in thickness. In this case the trivial solution of Section 4 does not exist. However, the analysis of Section 6 can be modified to show that our boundary value problem (with (8.1) instead of (2.13b)) is equivalent to an equation of the form $\mathbf{u} = G(\mathbf{u}, \lambda)$ with G compact and continuous. We thus obtain the existence of a branch of solutions $\mathscr{C}$ containing $(\rho = s,\ \varepsilon = 1,\ \lambda = 1)$ (see [**5, 11**]).

Our model does not account for dynamic and thermal effects which certainly would affect the growth and development of necks. The solutions of our problem do not have the periodic character which is the crucial ingredient to show that multiple necked states for bars in tension are physically unrealisable under the energy criteria (see [**14**]). Hence the question of whether or not multiple necked states for plates in tension are stable remains unanswered.

The eigenvalues $\{\lambda_k\}$ of the linearised problem of Section 5 are the solutions of equations (5.3), (5.4) which in general is a non-linear equation in λ. The corresponding eigenfunctions are given by equations (5.6). That these eigenvalues and eigenfunctions are the only possible ones can be seen by constructing four linearly independent solutions of (5.1) by the method of Frobenius, for example. An application of (5.2) now yields (5.3) to (5.6). To determine when these eigenvalues are simple one needs to verify conditions (3.3). From (5.6) one finds that (3.3a) clearly holds. An elementary but otherwise lengthy computation now shows that (3.3b) is equivalent to $dA(\lambda)/d\lambda \neq 0$ at the eigenvalue in question. Moreover, the disposition of these eigenvalues on the real line depends on the behaviour of the function $A(\cdot)$. For instance, if $A(\cdot)$ is strictly increasing and $A(\lambda) \to \infty$ as $\lambda \to \infty$, then the eigenvalues are simple and form a sequence $\{\lambda_k\}$ with $\lambda_k \to \infty$.

9. Appendix: derivation of the governing equations

In this section we give a derivation of the equilibrium equations of Section 2 from the three-dimensional theory of non-linear elasticity. We perform the analysis for hyperelastic materials because of its simplicity. A derivation without this assumption would be similar to the one given in [2] with the deformation $\mathbf{p}$ modified as below (cf. (9.3)) to allow for transverse deformations.

Notation. We let $\mathbb{E}^3$ denote the three-dimensional euclidean space whose elements will be denoted by lower case, bold face symbols. Let $\mathbf{a} \cdot \mathbf{b}$ denote the inner product of the vectors $\mathbf{a}$ and $\mathbf{b}$. We denote by Lin the (nine-dimensional) space of linear transformations from $\mathbb{E}^3$ into itself (second order tensors) whose elements will be denoted by upper case, bold-face symbols. The value of $\mathbf{A} \in \mathrm{Lin}$ at $\mathbf{u} \in \mathbb{E}^3$ will be denoted by $\mathbf{A} \cdot \mathbf{u}$. For $\mathbf{a}$, $\mathbf{b} \in \mathbb{E}^3$ we define the dyadic product $\mathbf{ab} \in \mathrm{Lin}$ by $(\mathbf{ab}) \cdot \mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$. We let $\mathrm{Lin}^+ = \{\mathbf{A} \in \mathrm{Lin} \mid \det \mathbf{A} > 0\}$ where "det" denotes the determinant.

Let $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be an orthonormal basis for $\mathbb{E}^3$. Let (s, ϕ) be polar coordinates for the $(\mathbf{i}, \mathbf{j})$ plane. We set

$$\mathbf{e}_1(\phi) = (\cos \phi)\mathbf{i} + (\sin \phi)\mathbf{j}, \tag{9.1a}$$

$$\mathbf{e}_2(\phi) = (-\sin \phi)\mathbf{i} + (\cos \phi)\mathbf{j}. \tag{9.1b}$$

The reference configuration of a circular plate of radius one and height $2h$ is

$$P = \{s\mathbf{e}_1(\phi) + z\mathbf{k} \mid s \in [0, 1),\ \phi \in [0, 2\pi),\ z \in (-h, h)\}. \tag{9.2}$$

Let $\mathbf{p}(s, \phi, z)$ be the position in the deformed configuration of the particle of P with cylindrical coordinates (s, ϕ, z). The axisymmetric compressions or expansions of a circular plate that can suffer extension and transverse deformations is given by

$$\mathbf{p}(s, \phi, z) = \rho(s)\mathbf{e}_1(\phi) + \beta(z)\varepsilon(s)\mathbf{k}, \tag{9.3}$$

where β is an odd, strictly increasing differentiable function. (It is reasonable to take $\beta(z) = z$ (see [2]).) Thus $(\rho(s), \varepsilon(s))$ is the radius and ratio of deformed to underformed height, respectively, in the deformed configuration of the cylindrical cross-section

$$C(s) = \{s\mathbf{e}_1(\phi) + z\mathbf{k} \mid \phi \in [0, 2\pi),\ z \in (-h, h)\} \tag{9.4}$$

in the reference configuration. We set

$$\nu(s) = \rho'(s), \quad \tau(s) = \rho(s)/s, \quad p(s) = \varepsilon'(s), \tag{9.5a,b,c}$$

where $\rho'(s) = d\rho/ds$, etc. The strains for our problem with their corresponding physical interpretations are given by (2.1).

We set

$$\mathbf{y} = s\mathbf{e}_1(\phi) + z\mathbf{k}, \quad \mathbf{q}(\mathbf{y}) = \mathbf{p}(s, \phi, z). \tag{9.6a,b}$$

The deformation gradient $\mathbf{F} = \partial\mathbf{q}/\partial\mathbf{y}$ is given according to (9.3) by

$$\mathbf{F} = \nu\mathbf{e}_1\mathbf{e}_1 + \beta p\mathbf{e}_3\mathbf{e}_1 + \tau\mathbf{e}_2\mathbf{e}_2 + \varepsilon\beta'\mathbf{e}_3\mathbf{e}_3, \tag{9.7}$$

which we require to satisfy the orientation preserving condition

$$\det \mathbf{F} = \tau\nu\varepsilon\beta' > 0. \tag{9.8}$$

Since $\beta' > 0$, we find that (2.2) suffices to ensure (9.8). The (left) Cauchy–Green strain tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^t$ takes the form

$$\mathbf{B} = \nu^2\mathbf{e}_1\mathbf{e}_1 + \nu p\beta(\mathbf{e}_1\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1) + \tau^2\mathbf{e}_2\mathbf{e}_2 + ((p\beta)^2 + (\varepsilon\beta')^2)\mathbf{e}_3\mathbf{e}_3. \tag{9.9}$$

The principal invariants of $\mathbf{B}$ are

$$I(\mathbf{B}) = \tau^2 + \nu^2 + p^2\beta^2 + (\varepsilon\beta')^2, \tag{9.10a}$$

$$II(\mathbf{B}) = \nu^2(\varepsilon\beta')^2 + (\nu^2 + p^2\beta^2 + (\varepsilon\beta')^2)\tau^2, \tag{9.10b}$$

$$III(\mathbf{B}) = \tau^2\nu^2(\varepsilon\beta')^2. \tag{9.10c}$$

The material of the plate is homogeneous and hyperelastic if there exists a smooth (stored energy) function Φ: $\mathrm{Lin}^+ \to \mathbb{R}$ such that with $\mathbf{F}$ given by (9.7), we have that

$$I(\rho, \varepsilon) = \int_P \Phi(F)\, d\mathbf{y} \tag{9.11}$$

represents the total stored energy corresponding to the deformation $\mathbf{q}$. Since P has the form (9.2), we can define the one-dimensional stored energy function $W(\tau, \nu, p, \varepsilon)$ by

$$W(\tau, \nu, p, \varepsilon) = \int_{-h}^{h} \int_0^{2\pi} \Phi(\mathbf{F})\, d\phi\, dz. \tag{9.12}$$

Thus combining (9.11) and (9.12) we obtain (7.2) and with the definitions (7.1) one easily finds that (2.4), (2.10) are just the Euler–Lagrange equations for (7.2) and that (2.8) are the corresponding natural boundary conditions.

We define the fourth order elasticity tensor by

$$\mathbf{A}(\mathbf{F}) = \partial^2\Phi/\partial\mathbf{F}^2. \tag{9.13}$$

We require that Φ satisfy the strong ellipticity condition

$$\mathbf{ab}: \mathbf{A}(\mathbf{F}): \mathbf{ab} > 0 \quad \forall(\mathbf{a}, \mathbf{b}) \neq (0, 0). \tag{9.14}$$

Now (7.1), (9.7), (9.12), and (9.13) imply that

$$a^2N_\nu + ab(N_p + P_\nu) + b^2P_p = \int_{-h}^{h} \int_0^{2\pi} (a\mathbf{e}_1 + b\beta\mathbf{e}_3)\mathbf{e}_1: \mathbf{A}(\mathbf{F}): (a\mathbf{e}_1 + b\beta\mathbf{e}_3)\mathbf{e}_1\, d\phi\, dz \tag{9.15}$$

for all a, $b \in \mathbb{R}$. Condition (9.14) now implies that (2.11a) holds. Similarly one obtains (2.11b,c).

If, in addition to the hyperelasticity condition (9.11), the material of the plate is isotropic, then (see [8]) there exists a smooth function $g(\cdot, \cdot, \cdot)$ such that

$$\Phi(\mathbf{F}) = g(I(\mathbf{B}), II(\mathbf{B}), III(\mathbf{B})) \tag{9.16}$$

where the Is are the principal invariants of $\mathbf{B} = \mathbf{F}\mathbf{F}^{T}$. By using (9.10), (9.12), and (9.16) one can easily verify that (2.12) and (2.14) hold.

Acknowledgment

This research was supported by the Science and Engineering Research Council of the United Kingdom.

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(*Issued* 12 *September* 1989)