Numer. Math. 58, 135–144 (1990)

Numerische
Mathematik

A numerical method for detecting singular minimizers of multidimensional problems in nonlinear elasticity

Pablo V. Negrón Marrero
University of Puerto Rico, Department of Mathematics,
Río Piedras, Puerto Rico 00931, USA and
Heriot-Watt University, Department of Mathematics, Riccarton,
Edinburgh EH14 4AS, Scotland

Received May 15, 1989/May 10, 1990

Summary. In this paper we describe and analyse a numerical method that detects singular minimizers and avoids the Lavrentiev phenomenon for three dimensional problems in nonlinear elasticity. This method extends to three dimensions the corresponding one dimensional method of Ball and Knowles.

Subject classification: AMS(MOS): 65N30; CR: G 1.8.

1 Introduction

The fundamental problem of the calculus of variations in three dimensions can be informally described as: minimize the integral

$$I(u)=\int_{\Omega} f(x, u(x), \nabla u(x))\,\mathrm{d}x \tag{1.1}$$

where $\Omega\subset\mathbb{R}^3$ and $u: \Omega\to\mathbb{R}^3$ belongs to an appropriate class of admissible functions A. (Here ∇u denotes the matrix of first partial derivatives of u.) We say that the integrand $f(x, u, z)$ is *regular* if $f\in C^2$ and

$$f_{z^i_\alpha z^j_\beta}(x, u, z)\zeta_\alpha\zeta_\beta\lambda^i\lambda^j>0 \qquad \forall\zeta,\ \lambda\in\mathbb{R}^3\backslash\{0\} \tag{1.2}$$

Minimizers of (1.1) satisfying (1.2) may be singular in the sense that they may be discontinuous or have unbounded derivatives at certain points. These minimizers typically do not satisfy the weak form of the Euler-Langrange equations

$$\int_{\Omega} [f_{z^i_\alpha} v^i_{,\alpha}+f_{u^i} v^i]\,\mathrm{d}x=0 \qquad \forall v\in C_0^\infty(\Omega) \tag{1.3}$$

where $v=(v^1, v^2, v^3)$. For these problems one usually has the following:

i) The *Lavrentiev phenomenon* [19, 12]:

(1.4)
$$\inf_{A \cap W^{1,\infty}(\Omega)} I(u) > \inf_{A} I(u)$$

ii) If $\{u_j\} \subset W^{1,\infty}(\Omega)$, $u_j \to u^*$ a.e., u^* the minimizer of (1.1), then

(1.5)
$$I(u_j) \to \infty \text{ as } j \to \infty .$$

Examples of one dimensional singular minimizers are given in [9, 10, 14] and in higher dimensions in [15, 17, 20, 6, 22–24, 7]. In all these higher dimensional examples the singular minimizer u^* is discontinuous. Examples with u^* continuous and ∇u^* unbounded are not known. These singular minimizers would be important as a model for the initiation of fracture or dislocation.

Properties (1.4) and (1.5) suggest that the usual finite element methods will fail both to approximate u^* and $I(u^*)$. Of course if one knows in advance the location and order of the singularity of u^* one can in principle adjust the basis functions to reflect this behaviour and in general one would get convergence to both u^* and $I(u^*)$. However such information about u^* is usually unavailable. In [8] a method based on a decoupling of the function u from its gradient is used to circumvent the Lavrentiev phenomenon and to detect one dimensional singular minimizers. We present here a generalization of this technique for the higher dimensional problem (1.1) in the context of nonlinear elasticity. Also motivated by the results in [13, 26] we give conditions that guarantee that the minimizers are $1-1$ a.e. (cf. (3.9)).

Notation. We let $M^{3\times 3}$ denote the space of real 3×3 matrices and $M_+^{3\times 3} = \{F \in M^{3\times 3}\colon \det F > 0\}$. For $F \in M^{3\times 3}$ we let $\operatorname{adj} F \in M^{3\times 3}$ be the transpose matrix of the cofactors of F. For $F \in M^{3\times 3}$ we take $|F|$ to be the euclidean norm of F thought as a vector in $\mathbb{R}^9$. $W^{k,p}(\Omega)$ denotes the Sobolev space of functions $u \in L^p(\Omega)$ with distributional derivatives up to order k which also belong to $L^p(\Omega)$ (see [1]).

2 Some definitions and basic results

Consider an elastic body which in a reference configuration occupies the bounded domain $\Omega \subset \mathbb{R}^3$ where the boundary $\partial\Omega$ is assumed sufficiently smooth. An *orientation preserving deformation* of the body is a mapping $u\colon \Omega \to \mathbb{R}^3$ such that

(2.1)
$$\det \nabla u(x) > 0 \quad \text{a.e.} \quad \text{in } \Omega.$$

Let $f\colon \bar{\Omega} \times M_+^{3\times 3} \to \mathbb{R}$ and $\psi\colon \bar{\Omega} \times \mathbb{R}^3 \to \mathbb{R}$ be continuous functions. Our *basic functional* is given by

(2.2)
$$I(u) = \int_{\Omega} (f(x, \nabla u(x)) + \psi(x, u(x)))\, \mathrm{d}x$$

where u belongs to some suitable class of functions satisfying (2.1). (In (2.2) f and ψ represent the mechanical stored energy and body force potential functions respectively.) Following [3, 4] we make the following assumptions on f and ψ:

(E1) f is *polyconvex*, i.e., there exists g continuous such that

$$f(x, F) = g(x, F, \operatorname{adj} F, \det F) \qquad \forall x \in \bar{\Omega},\ F \in M_+^{3\times 3},$$

where $g(x, \cdot, \cdot, \cdot)\colon M^{3\times 3} \times M^{3\times 3} \times (0, \infty) \to \mathbb{R}$ is convex.

(E2) There exist $K > 0$, C, $p \geqq 2$, $q \geqq p/(p-1)$, $r > 1$ such that

$$g(x, F, G, \delta) \geqq C + K(|F|^p + |H|^q + \delta^r)$$

for all $x \in \Omega$ and $(F, H, \delta) \in M^{3\times 3} \times M^{3\times 3} \times (0, \infty)$.

(E3) $g(x, F, H, \delta) \to \infty$ as $(F, H, \delta) \to \partial(M^{3\times 3} \times M^{3\times 3} \times (0, \infty))$.

(E4) There exist $b > 0$, γ, $a \in L^1(\Omega)$ such that

$$|\psi(x, u)| \leqq a(x) + b|u|^\gamma \qquad \forall x \in \bar{\Omega},\ u \in \mathbb{R}^3,$$

where $1 \leqq \gamma \leqq 3p/(3-p)$ if $p < 3$ and $\gamma \geqq 1$ if $p \geqq 3$.

Let $\bar{u} \in L^p(\partial\Omega_1)$ where $\partial\Omega_1 \subset \partial\Omega$ and $\operatorname{meas}(\partial\Omega_1) > 0$. Our *basic problem* is to minimize (2.2) over the set

$$A = \{u \in W^{1,p}(\Omega)\colon \operatorname{adj} \nabla u \in L^q(\Omega), \det \nabla u \in L^r(\Omega), \det \nabla U > 0 \text{ a.e.}, u|_{\partial\Omega_1} = \bar{u}\}. \tag{2.3}$$

We close this section with some results concerning the weak continuity of the "adj" and "det" functions. Recall that if $F \in M^{3\times 3}$, then

$$(\operatorname{adj} F)_{i\alpha} = 1/2\, \varepsilon_{ijk}\, \varepsilon^{\alpha\beta\gamma} F_{j\beta} F_{k\gamma}, \tag{2.4a}$$

$$\det F = 1/6\, \varepsilon_{ijk}\, \varepsilon^{\alpha\beta\gamma} F_{i\alpha} F_{j\beta} F_{k\gamma}. \tag{2.4b}$$

Note that if $F \in L^2(\Omega)$, then $\operatorname{adj} F \in L^1(\Omega)$; if $F \in L^2(\Omega)$ and $\operatorname{adj} F \in L^2(\Omega)$, then $\det F \in L^1(\Omega)$. By using and identity for adj ∇u and det ∇u in divergence form, Ball [3, 4] proved the following:

Proposition 2.1

a) *Let* $p \geqq 2$. *If* $u_k \rightharpoonup u$ *in* $W^{1,p}(\Omega)$, *then* adj $\nabla u_k \to$ adj ∇u *in the sense of distributions.*

b) *Let* $p \geqq 2$ *and* $q = p/(p-1)$. *If* $u_k \rightharpoonup u$ *in* $W^{1,p}(\Omega)$ *and* adj $\nabla u_k \rightharpoonup$ adj ∇u *in* $L^q(\Omega)$, *then* det $\nabla u_k \to$ det ∇u *in the sense of distributions.* □

As a consequence of this proposition one gets the following existence theorem for our basic problem (see [3, 4]).

Theorem 2.2. *Let* (E1)–(E4) *hold and assume that* A *as given by* (2.3) *is nonempty. Then the functional* (2.2) *attains a minimum on* A. □

Further results on the smoothness of the minimizers of this theorem are given in [25].

3 The decoupled and discretized problems

Following [8] we define the *decoupled functional* $I(u, v)$ by

(3.1) $$I(u, v) = \int_\Omega (f(x, v(x)) + \psi(x, u(x))\,\mathrm{d}x$$

where f and ψ are as in Sect. 2. We let

(3.2a) $$\hat{A} = \{u \in W^{1,p}(\Omega)\colon \operatorname{adj} \nabla u \in L^q(\Omega), \det \nabla u \in L^r(\Omega), u|_{\partial\Omega_1} = \bar{u}\},$$

(3.2b) $$B = \{v \in L^p(\Omega)\colon \operatorname{adj} v \in L^q(\Omega), \det v \in L^r(\Omega), \det v > 0 \text{ a.e.}\}.$$

The *decoupled problem* is to minimize (3.1) over $\hat{A} \times B$ subject to

(3.3a) $$\int_\Omega |\nabla u(x) - v(x)|^p\,\mathrm{d}x \leqq \varepsilon,$$

(3.3b) $$\int_\Omega |\operatorname{adj} \nabla u(x) - \operatorname{adj} v(x)|^q\,\mathrm{d}x \leqq \varepsilon,$$

where $\varepsilon > 0$ is given. The idea here is of course that as $\varepsilon \searrow 0$ we should get the basic problem of Sect. 2.

We introduce now a discretization of the decoupled problem. Let $\tilde{A}$ be as $\hat{A}$ in (3.2a) but without the boundary condition $u|_{\partial\Omega_1} = \bar{u}$. Let S^h and B^h be finite dimensional subsets of $\tilde{A}$ and B respectively. Let $\bar{u}_h \in \operatorname{Trace}(S^h)$ be such that $\bar{u}_h \rightharpoonup \bar{u}$ in $L^p(\partial\Omega_1)$. We set

(3.4) $$A^h = \{u \in S^h\colon u|_{\partial\Omega_1} = \bar{u}_h\}.$$

Remark. Note that we do not require that $\det \nabla u > 0$ a.e. for $u \in A^h$. This is because we want to be able to approximate any $u \in A$ by elements in A^h (see $(B1)$ below) and there is some danger that the approximations u_h need not satisfy the positivity of the determinant (see [5]).

We assume that

$(B1)$ $\forall u \in A\ \exists \{u_h\}$, $u_h \in A^h$, such that

$$\lim_{h \to 0} \int_\Omega |\nabla u_h - \nabla u|^p\,\mathrm{d}x = 0,$$

$$\lim_{h \to 0} \int_\Omega |\operatorname{adj} \nabla u_h - \operatorname{adj} \nabla u|^q\,\mathrm{d}x = 0.$$

$(B2)$ $\forall v \in L^\infty(\Omega)$ such that $\det v \geqq c > 0$ for some constant c, there exists $\{v_h\}$, $v_h \in B^h$, such that

$$\begin{aligned} |v_h(x)| &\leqq K \text{ a.e.} && \text{in } \Omega \quad \forall h,\\ \det v_h(x) &\geqq \bar{c} \text{ a.e.} && \text{in } \Omega \quad \forall h,\\ v_h(x) &\to v(x) \text{ a.e.} && \text{in } \Omega \quad \text{as } h \to 0, \end{aligned}$$

for some constants $\bar{c} > 0$ and K.

The *discretized decoupled problem* is to minimize (3.1) over $A^h \times B^h$ subject to (3.3). Let I_h^ε be the minimum of this problem which exists by the growth condition $(E2)$. Since this is a finite dimensional problem, there exists $(u_h^\varepsilon, v_h^\varepsilon) \in A^h \times B^h$ such that

$$(3.5) \qquad I_h^\varepsilon = \min_{A^h \times B^h} I(u, v) = I(u_h^\varepsilon, v_h^\varepsilon).$$

We now can prove the following extension of the corresponding results in [8] and [13].

Theorem 3.1. *Assume that $(E1)$–$(E4)$ and $(B1)$–$(B2)$ hold. Suppose there exists $u_1 \in A$ such that $I(u_1) < \infty$. Hence there exists a nondecreasing function γ: $(0, \infty) \to (0, \infty)$ such that*

$$(3.6) \qquad \lim_{\substack{h, \varepsilon \to 0 \\ 0 < h < \gamma(\varepsilon)}} I_h^\varepsilon = \inf_A I(u).$$

Let $h_j \to 0$, $\varepsilon_j \to 0$, $0 < h_j < \gamma(\varepsilon_j)$ and let $(u_{h_j}^{\varepsilon_j}, v_{h_j}^{\varepsilon_j})$ be a minimizing pair for $I(u, v)$ in $A^{h_j} \times B^{h_j}$ subject to (3.3) with $\varepsilon = \varepsilon_j$. Then there exists a subsequence (h_μ, ε_μ) of (h_j, ε_j) and a minimizer u^* of $I(u)$ over A such that as $\mu \to \infty$

$$(3.7\text{a}) \qquad u_{h_\mu}^{\varepsilon_\mu} \to u^* \quad \text{in } L^p(\Omega),$$

$$(3.7\text{b}) \qquad v_{h_\mu}^{\varepsilon_\mu} \rightharpoonup \nabla u^* \quad \text{in } L^p(\Omega),$$

$$(3.7\text{c}) \qquad \operatorname{adj} v_{h_\mu}^{\varepsilon_\mu} \rightharpoonup \operatorname{adj} \nabla u^* \quad \text{in } L^q(\Omega),$$

$$(3.7\text{d}) \qquad \det v_{h_\mu}^{\varepsilon_\mu} \rightharpoonup \det \nabla u^* \quad \text{in } L^r(\Omega).$$

Moreover if $p > 3$ and one includes in (2.3) and (3.2a) the additional constraint that

$$(3.8) \qquad \int_\Omega \det \nabla u \, dx \leqq \operatorname{vol} u(\Omega), \qquad (\text{cf. [13]}),$$

then the minimizer u^* is injective almost everywhere, i.e.

$$(3.9) \qquad \operatorname{card}(u^*)^{-1}(x') = 1 \quad \text{for almost all } x' \in u^*(\bar{\Omega}).$$

Proof. Let $\varepsilon > 0$ be given. There exists $\hat{u} \in A$ such that

$$(3.10) \qquad I(\hat{u}) < \inf_A I(u) + \varepsilon < \infty.$$

From $(E1)$ and $(E2)$ it follows that given $F \in M_+^{3 \times 3}$ there exists M such that if $|v| \geqq M$ or $0 < \det v \leqq 1/M$, then

$$(3.11) \qquad f(x, v) \geqq f(x, F) \quad \forall x \in \bar{\Omega}.$$

Define

$$(3.12) \qquad v_\delta(x) = \begin{cases} F & \text{if } |\nabla \hat{u}(x)| \geqq \delta \text{ or } \det \nabla \hat{u}(x) \leqq 1/\delta, \\ \nabla \hat{u}(x) & \text{otherwise.} \end{cases}$$

Note that $v_\delta \in L^\infty(\Omega)$ and $\det v_\delta \geqq c > 0$ for some constant c depending on δ. From (3.11) and (3.12) we have that if $\delta > M$, then

$$I(\hat{u}, v_\delta) \leqq I(\hat{u}). \tag{3.13}$$

Also

$$\int_\Omega |\nabla\hat{u} - v_\delta|^p \,\mathrm{d}x = \int_D |\nabla\hat{u} - F|^p \,\mathrm{d}x, \tag{3.14a}$$

$$\int_\Omega |\operatorname{adj} \nabla\hat{u} - \operatorname{adj} v_\delta|^q \,\mathrm{d}x = \int_D |\operatorname{adj} \nabla\hat{u} - \operatorname{adj} F|^q \,\mathrm{d}x, \tag{3.14b}$$

where $D = \{x : |\nabla\hat{u}(x)| \geqq \delta$ or $\det \nabla\hat{u}(x) \leqq 1/\delta\}$. Since $\nabla\hat{u} \in L^p(\Omega)$ and $\operatorname{adj} \nabla\hat{u} \in L^q(\Omega)$ and meas $D \to 0$ as $\delta \to \infty$ (by $(E2)$, $(E3)$, and $I(\hat{u}) \leqq \infty$), then for δ sufficiently large

$$2^{p-1} \int_\Omega |\nabla\hat{u} - v_\delta|^p \, dx < \varepsilon/2, \tag{3.15a}$$

$$2^{q-1} \int_\Omega |\operatorname{adj} \nabla\hat{u} - \operatorname{adj} v_\delta|^q \,\mathrm{d}x < \varepsilon/2. \tag{3.15b}$$

From $(B1)$ and $(B2)$ we get that there exist $u_h \in A^h$, $v_h \in B^h$ such that

$$\int_\Omega |\nabla u_h - \nabla\hat{u}|^p \,\mathrm{d}x, \quad \int_\Omega |\operatorname{adj} \nabla u_h - \operatorname{adj} \nabla\hat{u}|^q \,\mathrm{d}x \to 0 \quad \text{as } h \to 0, \tag{3.16a, b}$$

$$v_h \to v_\delta \text{ a.e.} \quad \text{in } \Omega \text{ as } h \to 0, \tag{3.16c}$$

$$|v_h(x)| \leqq K \text{ a.e.} \quad \text{in } \Omega \ \forall h, \tag{3.16d}$$

$$\det v_h(x) \geqq \bar{c} > 0 \text{ a.e.} \quad \text{in } \Omega \ \forall h. \tag{3.16e}$$

With $c = 2^{p-1}$ we have that

$$\int_\Omega |\nabla u_h - v_h|^p \,\mathrm{d}x \leqq c \int_\Omega |\nabla\hat{u} - v_\delta|^p \,\mathrm{d}x + c^2 \int_\Omega |\nabla u_h - \nabla\hat{u}|^p \,\mathrm{d}x + c^3 \int_\Omega |v_h - v_\delta|^p \,\mathrm{d}x \tag{3.17}$$

with a similar inequality for the adjoints.

Since $\nabla u_h \to \nabla\hat{u}$ in $L^p(\Omega)$ and $u_h|_{\partial\Omega_1} = \bar{u}_h \rightharpoonup \bar{u}$ in $L^p(\partial\Omega_1)$, we get that $u_h \to \hat{u}$ in $L^p(\Omega)$. Moreover from the imbedding theorems we get that

$$u_h \to \hat{u} \quad \text{in } L^\gamma(\Omega) \ (\text{cf. } (E4)). \tag{3.18}$$

From $(E4)$ we get that

$$L^\gamma(\Omega) \ni u \to \int_\Omega \psi(x, u(x)) \,\mathrm{d}x$$

is bounded and continuous (see [18]). Hence (3.18) implies that

$$\int_\Omega \psi(x, u_h(x))\,dx \to \int_\Omega \psi(x, \hat{u}(x))\,dx \quad \text{as } h \to 0. \tag{3.19}$$

From (3.16c, d, e) and the bounded convergence theorem, we get that

$$\int_\Omega f(x, v_h(x))\,dx \to \int_\Omega f(x, v_\delta(x))\,dx \quad \text{as } h \to 0. \tag{3.20}$$

Thus from (3.19) and (3.20) we get that

$$I(u_h, v_h) \to I(\hat{u}, v_\delta) \quad \text{as } h \to 0. \tag{3.21}$$

From (3.15)–(3.17) and the corresponding inequality for the adjoints, and (3.21) we get that there exists $\gamma(\varepsilon) \in (0, \infty)$ such that

$$I(u_h, v_h) \leqq I(\hat{u}, v_\delta) + \varepsilon, \tag{3.22a}$$

$$\int_\Omega |\nabla u_h - v_h|^p\,dx, \quad \int_\Omega |\operatorname{adj} \nabla u_h - \operatorname{adj} v_h|^q\,dx \leqq \varepsilon, \quad \forall h < \gamma(\varepsilon). \tag{3.22b, c}$$

Thus (3.10), (3.13) and (3.22) imply that

$$I_h^\varepsilon \leqq \inf_A I(u) + 2\varepsilon \quad \forall h < \gamma(\varepsilon) \tag{3.23}$$

Let $h_j, \varepsilon_j \to 0$ with $0 < h_j < \gamma(\varepsilon_j)$ and set $u_j = u_{h_j}^{\varepsilon_j}$, $v_j = v_{h_j}^{\varepsilon_j}$ where $I_{h_j}^{\varepsilon_j} = I(u_j, v_j)$. Now $(E2)$ and (3.23) imply that for a subsequence $\{v_\mu\}$ of $\{v_j\}$ we have that

$$v_\mu \rightharpoonup v^* \quad \text{in } L^p(\Omega) \tag{3.24a}$$

$$\operatorname{adj} v_\mu \rightharpoonup H^* \quad \text{in } L^q(\Omega), \tag{3.24b}$$

$$\det v_\mu \rightharpoonup \delta^* \quad \text{in } L^r(\Omega). \tag{3.24c}$$

Let $z_\mu = \nabla u_\mu - v_\mu$. Since $\varepsilon_\mu \to 0$, the constraint (3.3a) imply that

$$z_\mu \to 0 \quad \text{in } L^p(\Omega), \tag{3.25}$$

and thus from (3.24a) that $\nabla u_\mu \rightharpoonup v^*$ in $L^p(\Omega)$. Since $\{\nabla u_\mu\}$ is bounded in $L^p(\Omega)$ and $\{u_\mu\}$ is bounded in $L^p(\partial\Omega_1)$, and meas $(\partial\Omega_1) > 0$, then $\{u_\mu\}$ is bounded in $W^{1,p}(\Omega)$ (see [21]). Thus there exists $u^* \in W^{1,p}(\Omega)$ such that $v^* = \nabla u^*$ and (going to a subsequence if necessary)

$$u_\mu \to u^* \quad \text{in } L^p(\Omega), \tag{3.26a}$$

$$\nabla u_\mu \rightharpoonup \nabla u^* \quad \text{in } L^p(\Omega). \tag{3.26b}$$

Now the constraint (3.3b) and (3.24b) imply that adj $\nabla u_\mu \rightharpoonup H^*$ in $L^q(\Omega)$. This together with (3.26) and part (a) of Proposition (2.1) imply that

$$H^* = \operatorname{adj} \nabla u^* \text{ a.e.} \tag{3.27}$$

Thus

$$\text{adj}\,\nabla u_\mu \rightharpoonup \text{adj}\,\nabla u^* \quad \text{in } L^q(\Omega). \tag{3.28}$$

Using the definition (2.4b) one readily gets that

$$\det v_\mu = \det \nabla u_\mu + d_\mu, \tag{3.29a}$$

where by (3.25), (3.26b), (3.3b), and (3.28) we get that

$$d_\mu \to 0 \quad \text{in } L^1(\Omega). \tag{3.29b}$$

Thus (3.24c), (3.26), (3.28), (3.29), and part (b) of Proposition (2.1) imply that

$$\delta^* = \det \nabla u^* \text{ a.e.} \tag{3.20}$$

We have shown that for some $u^* \in W^{1,p}(\Omega)$,

$$v_\mu \rightharpoonup \nabla u^* \quad \text{in } L^p(\Omega), \tag{3.31a}$$

$$\text{adj}\, v_\mu \rightharpoonup \text{adj}\,\nabla u^* \quad \text{in } L^q(\Omega), \tag{3.31b}$$

$$\det v_\mu \rightharpoonup \det \nabla u^* \quad \text{in } L^r(\Omega). \tag{3.31c}$$

Now (3.26), (3.31), $(E1)$, and standard weak lower semicontinuity theorems (see [16]) imply that

$$I(u^*) \leqq \liminf_\mu I(u_\mu, v_\mu). \tag{3.32}$$

Since by (3.23) the right side of (3.32) is finite, we get from $(E3)$ that

$$\det \nabla u^* > 0 \text{ a.e.} \tag{3.33}$$

Also since Trace $(u_\mu) = \bar{u}_{h_\mu} \rightharpoonup \bar{u}$ in $L^p(\partial\Omega_1)$, we get from (3.26) that

$$u^*|_{\partial\Omega_1} = \bar{u}. \tag{3.34}$$

Thus $u^* \in A$ and we get from (3.23) and (3.32) that

$$I(u^*) = \inf_A I(u) = \lim_{\mu \to 0} I_{h_\mu}^{\varepsilon_\mu}.$$

Since the sequence h_j, $\varepsilon_j \to 0$, $0 < h_j < \gamma(\varepsilon_j)$ is arbitrary, we get that (3.6) holds.

If $p > 3$ and (2.3) and (3.2a) include (3.8), then the above proof goes through up to (3.34). One must still show that u^* satisfies (3.8). The proof of this is similar to that in [13] (Theorem 5) and thus we just sketch it. Since $u_\mu \rightharpoonup u^*$ in $W^{1,p}(\Omega)$, $p > 3$, by the compact imbedding theorems we get that $u_\mu \to u^*$ uniformly (taking a subsequence if necessary). Hence given $\varepsilon > 0$, there exists an open set θ_ε such that

$$u^*(\bar{\Omega}), \quad u_\mu(\bar{\Omega}) \subset \theta_\varepsilon \ \forall\ \mu \geqq \mu_0, \tag{3.35a}$$

$$\text{vol}(\theta_\varepsilon \setminus u^*(\bar{\Omega})) < \varepsilon. \tag{3.35b}$$

Moreover (3.26) and $p>3$ imply that (see [4])

$$\det \nabla u_\mu \rightharpoonup \det \nabla u^* \quad \text{in } L^{p/3}(\Omega). \tag{3.36}$$

Thus since u_μ satisfies (3.8) we get from (3.35) and (3.36) that

$$\int_\Omega \det \nabla u^* \, dx \leqq \operatorname{vol} u^*(\Omega) + \varepsilon$$

which gives that u^* satisfies (3.8) because ε is arbitrary. That u^* is injective a.e. follows from (3.33) and (3.8) (see [13]). □

Essentially this is the best theorem we can get because of the lack of better sequential weak lower semicontinuity theorems (that would yield (3.32)) capable of handling the singular behaviour in ($E3$). Unfortunetely this theorem cannot handle cavitation because such singular minimizers do not belong to the set A defined by (2.3) (see [6]). It is not clear whether condition ($B1$) holds for all $p \geqq 2$ and $q \geqq p/(p-1)$ but it can be easily checked if $2q \leqq p$ say. Extensions of the first part of Theorem (3.1) (not including (3.8)) to the case $p \geqq 1$ and $n \geqq 3$ can be done as in [2, 3, 11] by introducing the generalized "Adj" and "Det" functions. The injectivity almost everywhere under condition (3.8) can also be obtained under the weaker hypothesis $p>2$ (see [26]).

Acknowledgement. This research was supported by the Science and Engineering Research Council of the United Kingdon. I want to thank Prof. J.M. Ball for his valuable comments.

References

1. Adams, R.A.: Sobolev Spaces, Ed. New York: Academic Press 1975
2. Ball, J.M.: On the calculus of variations and sequentially weakly continuous maps. In: Ordinary and partial differential equations (Everitt, W.N., Sleeman, B.D. (eds.)), Lecture Notes in Mathematics, Vol. 564, pp. 13–25, Berlin New York: Springer 1976
3. Ball, J.M.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rat. Mech. Anal. **63**, 337–403 (1977)
4. Ball, J.M.: Constitutive inequalities and existence theorems in nonlinear elastostatics. In: Nonlinear analysis and mechanics: Heriot-Watt Symposium Vol. 1 (Knops, R.J. (Ed.)), London: Pitman 1977
5. Ball, J.M.: Global invertibility of Sobolev functions and the interpenetration of matter. Proc. Royal Soc. Edinburgh A**88**, 315–328 (1981)
6. Ball, J.M.: Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. Phil. Trans. Royal Soc. London A**306**, 557–611 (1982)
7. Ball, J.M.: Singular minimizers and their significance in elasticity, directions in partial differential equations, (Ed.). New York: Academic Press 1987
8. Ball, J.M., Knowles, G.: A numerical method for detecting singular minimizers. Numer. Math. **51**, 181–197 (1987)
9. Ball, J.M., Mizel, V.J.: Singular minimizers for regular one dimensional problems in the calculus of variations. Bull. Am. Math. Soc. New Ser. **11**, 143–146 (1984)
10. Ball, J.M., Mizel, V.J.: One dimensional variational problems whose minimizers do not satisfy the Euler-Langrange equation. Arch. Rat. Mech. Anal. **90**, 325–388 (1985)
11. Ball, J.M., Currie, J.C., Olver, P.J.: Null lagrangians, weak continuity, and variational problems of arbitrary order. J. Funct. Anal. **41**, 135–174 (1981)
12. Cesari, L.: Optimization – theory and applications, (Ed.). Berlin Heidelberg New York: Springer 1983

13. Ciarlet, P.G., Nečas, J.: Injectivity and self-contact in nonlinear elasticity. Arch. Rat. Mech. Anal. **96**, 171–188 (1986)
14. Davie, A.M.: Singular minimizers in the calculus of variations in one dimension. Arch. Rat. Mech. Anal. **101**, 161–177 (1988)
15. De Giorgi, E.: Un esempio di estremali discontinue per un problema variationale di tipo ellittico. Boll. Un. Mat. Ital. **4**, 135–137 (1968)
16. Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems. Princeton University Press, Princeton, New Jersey 1983
17. Giusti, E., Miranda, M.: Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni. Boll. Un. Mat. Ital. **1**, 219–225 (1968)
18. Krasnosel'ski, M.A.: Topological methods in the theory of nonlinear integral equations, New York: Macmillan 1964
19. Lavrentiev, M.: Sur quelques problèmes du calcul des variations. Ann. Math. Pure Appl. **4**, 7–28 (1926)
20. Maz'ya, V.G.: Examples of nonregular solutions of quasilinear elliptic equations with analytic coefficients. Funktsional. Anal. i Prilozhen **2**, 53–57 (1968)
21. Morrey, C.B.: Multiple integrals in the calculus of variations. (Ed.). Berlin Heidelberg New York: Springer 1966
22. Sivaloganathan, J.: Uniqueness of regular and singular equilibria for spherically symmetric problems of nonlinear elasticity. Arch. Rat. Mech. Anal. **96**, 97–136 (1986)
23. Sivaloganathan, J.: A field theory approach to the stability of radial equilibria in nonlinear elasticity. Math. Proc. Camb. Philos. Soc. **99**, 589–604 (1986)
24. Stuart, C.A.: Radially symmetric cavitation for hyperelastic materials. Ann. Inst. Henri Poincaré, Anal. Nonlinéaire **2**, 1–20 (1985)
25. Sverák, V.: Regularity properties of deformations with finite energy. Arch. Rat. Mech. Anal. **100**, 105–206 (1988)
26. Tang, Qi.: Almost everywhere injectivity in nonlinear elasticity (to appear)