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An Analysis of the Linearized Equations for Axisymmetric Deformations of Hyperelastic Cylinders

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Abstract: The equations describing the axisymmetric deformations of a cylindrical body composed of a hyperelastic isotropic material define a system in two variables (radius and height due to axisymmetry) of quasi-linear partial differential equations subject to nonlinear mixed boundary conditions. The author considers the boundary value problem of specifying the displacement of the lateral surface of the cylinder subject to zero normal stresses on the top and bottom. It is shown that this problem admits a trivial solution consisting of a uniform expansion or compression in the radial and height directions. The author studies the linearization of the full nonlinear equations about the trivial solution and constructs solutions for the resulting system of linear partial differential equations. As a consequence of these explicit representations, one gets the characteristic equation defining the eigenvalues of the linearized problem, which represent bifurcation points of the nonlinear system. The corresponding eigenfunctions can be classified into those that are symmetric about the $z = 0$ axis (midplane of the cylinder) representing either necked or barreled states of the cylinder and those that break this symmetry. For a class of Hadamard-Green type materials and all cylinder heights, the existence of eigenvalues for the symmetry-preserving and symmetry-breaking characteristic equations is shown.

1. INTRODUCTION

The study of the deformations of basic structures like plates, cylinders, bars, and so on is an important area in engineering; for example, in the design of bridges, planes, and cars. The ability to compute numerically or to predict the character of the solutions of the equations describing such deformations is thus an important practical problem. In this paper, we study the axisymmetric deformations of a cylindrical body composed of an isotropic hyperelastic material subject to a specified lateral surface displacement and to zero normal stresses on the top and bottom. The axisymmetry of the deformation reduces the full three-dimensional equations of nonlinear elasticity to a 2×2 system of quasi-linear partial differential equations for the radial and vertical displacements subject to some mixed boundary conditions, and parameterized by the applied loads and constitutive equations. The isotropy condition and some physically reasonable growth conditions (basically, that infinite expansions or compressions within the body be accompanied by the corresponding infinite stresses) allow us to construct a family of trivial solutions consisting of a uniform expansion or compression of the cylinder in the radial and vertical directions. The linearization about the trivial solution

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of the full nonlinear equations describing the deformation yields a 2×2 linear system of partial differential equations. We look for separable solutions of this system of partial differential equations and find that the solutions are given in terms of hyperbolic functions and Bessel functions of orders zero and one. With these explicit solutions, we are able to characterize the eigenvalues of the linearized problem which represent possible bifurcation points for nontrivial solutions of the nonlinear problem.

The problem we treat in this paper follows a hierarchy of problems in nonlinear elasticity beginning with the models of nonlinearly elastic bars analyzed, among others, by Antman [1], Antman and Carbone [5], and Owen [16]; nonlinearly elastic thin plates analyzed by Antman [2, 4], Negrón-Marrero [12], and Negrón-Marrero and Antman [13]; and cylinders by Sensenig [17]. Our results extend those in [13] for plates and generalize some of the results by [17], which are for linear stress-strain relations, to more general constitutive relations satisfying the strong ellipticity condition of nonlinear elasticity. The nonlinear elasticity model we use to describe the deformations of our cylindrical structure is based on those in Green and Zerna [10], Green and Adkins [9], Ogden [15], and Truesdell and Noll [21].

A related and extensively studied problem is that of the deformations of hyperelastic isotropic cylinders under uniaxial compression. We mention in particular the experimental work of Beatty and Dadras [6] and Beatty and Hook [7], and the analytical results in Davies [8] for a general class of materials satisfying the Baker-Ericksen inequalities, Simpson and Spector [19] for Hadamard-Green materials, Simpson and Spector [20] for a Blatz-Ko type material, and Wilkes [22], in which the cylinder is assumed to be infinite and composed of an incompressible material.

In Section 2, we introduce the equations and constitutive hypotheses describing the nonlinearly elastic axisymmetric deformations of cylinders. A full derivation of the equations is given in the appendix. In Section 3, we construct the family of trivial solutions and the linearization about it of the nonlinear equations of Section 2. We then construct explicit solutions by separation of variables. In Section 4, we consider a family of Hadamard-Green type materials. In this case, we can get an explicit representation for the trivial solution, which we then use to perform an asymptotic analysis of the eigenvalue equation for the linearized problem showing the existence of eigenvalues for all cylinder heights.

2. THE GOVERNING EQUATIONS

In this section, we present the equations describing the nonlinear deformations of a cylinder. A full derivation of the equations is given in the appendix. We consider a body occupying the cylindrical region in $\Re^3$ given by

$$\Omega = \{s\mathbf{e}_1(\phi) + z\mathbf{k} \mid s \in [0,1),\ \phi \in [0,2\pi),\ z \in (-h,h)\}, \tag{2.1}$$

where $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ is the standard basis for $\Re^3$ and $\mathbf{e}_1(\phi) = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$, $\mathbf{e}_2(\phi) = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$, and $\mathbf{e}_3(\phi) = \mathbf{k}$. We consider an axisymmetric deformation of the body of the form

$$\mathbf{p}(s,\phi,z) = r(s,z)\,\mathbf{e}_1(\phi) + \omega(s,z)\,\mathbf{k}. \tag{2.2}$$

The strains for our problem are given by the vector

$$\hat{w}(s,z) = (r_s(s,z), r_z(s,z), r(s,z)/s, \omega_s(s,z), \omega_z(s,z)). \tag{2.3}$$

The orientation-preserving condition of nonlinear elasticity implies that (r, ω) must satisfy the inequalities

$$r/s > 0, \quad r_s\omega_z - r_z\omega_s > 0. \tag{2.4}$$

For simplicity, let us write

$$(\nu, \gamma, \tau, p, \varepsilon) = (r_s, r_z, r/s, \omega_s, \omega_z). \tag{2.5}$$

Let $N(w(s,z))$, $T(\hat{w}(s,z))$, and $U(\hat{w}(s,z))$ be the normal components of the first Piola-Kirchhoff stress tensor in the radial, tangential, and vertical directions, respectively. Let $G(\hat{w}(s,z))$, $P(\hat{w}(s,z))$ be the shear components of stress in the $\mathbf{e}_1(\phi)\,\mathbf{k}$, $\mathbf{k}\mathbf{e}_1(\phi)$ directions, respectively. If we assume that the material of the cylinder is hyperelastic and isotropic, we get, among others, the following symmetry and even-odd conditions:

$$N(\nu, 0, \tau, 0, \varepsilon) = T(\tau, 0, \nu, 0, \varepsilon) \quad N(\nu, -\gamma, \tau, -p, \varepsilon) = N(\nu, \gamma, \tau, p, \varepsilon) \tag{2.6a,b}$$

$$\begin{aligned} G(\nu, -\gamma, \tau, -p, \varepsilon) &= -G(\nu, \gamma, \tau, p, \varepsilon) \\ T(\nu, -\gamma, \tau, -p, \varepsilon) &= T(\nu, \gamma, \tau, p, \varepsilon) \end{aligned} \tag{2.7a,b}$$

$$\begin{aligned} P(\nu, -\gamma, \tau, -p, \varepsilon) &= -P(\nu, \gamma, \tau, p, \varepsilon) \\ U(\nu, -\gamma, \tau, -p, \varepsilon) &= U(\nu, \gamma, \tau, p, \varepsilon). \end{aligned} \tag{2.8a,b}$$

The strong ellipticity condition from three-dimensional elasticity implies that

$$\begin{pmatrix} N_\nu & N_\gamma \\ G_\nu & G_\gamma \end{pmatrix} \quad \begin{pmatrix} N_\nu & N_p \\ P_\nu & P_p \end{pmatrix} \quad \begin{pmatrix} P_p & P_\varepsilon \\ U_p & U_\varepsilon \end{pmatrix} \quad \begin{pmatrix} G_\gamma & G_\varepsilon \\ U_\gamma & U_\varepsilon \end{pmatrix} \tag{2.9}$$

are positive definite and that

$$T_\tau > 0. \tag{2.10}$$

The physical requirements that an infinite expansive (compressive) stress be accompanied by a corresponding infinite elongation (compression) strain leads to the following growth conditions on the constitutive functions (N, G, T, P, U) :

$$T \to \pm\infty \text{ as } \begin{cases} \tau & \to \infty \\ \tau & \to 0^+ \end{cases} \tag{2.11a}$$

$$N \to \pm\infty \text{ as } \begin{cases} \nu & \to \infty \\ \nu & \text{is such that } \nu\varepsilon - \gamma p \to 0^+ \end{cases} \tag{2.11b}$$

$$G \to \pm\infty \text{ as } \begin{cases} \gamma & \to \infty \\ \gamma & \text{is such that } \nu\varepsilon - \gamma p \to 0^+ \end{cases} \tag{2.11c}$$

$$P \to \pm\infty \text{ as } \begin{cases} p \to \infty \\ p \text{ is such that } \nu\varepsilon - \gamma p \to 0^+ \end{cases} \tag{2.11d}$$

$$U \to \pm\infty \text{ as } \begin{cases} \varepsilon \to \infty \\ \varepsilon \text{ is such that } \nu\varepsilon - \gamma p \to 0^+ \end{cases} \tag{2.11e}$$

(In each of the limits above, the remaining four variables are fixed.)

The principle of virtual work and the fundamental lemma of the calculus of variations can be used to get the following boundary value problem for $(r(s,z), \omega(s,z))$:

$$\begin{cases} -(sN(\hat{w}(s,z)))_s + T(\hat{w}(s,z)) = sG(\hat{w}(s,z))_z \\ -(sP(\hat{w}(s,z)))_s - sU(\hat{w}(s,z))_z = 0, \quad (s,z) \in (0,1) \times (-h,h) \end{cases} \tag{2.12a,b}$$

$$r(0,z) = 0 \qquad r(1,z) = \lambda \qquad \forall z \tag{2.13a,b}$$

$$(sP(\hat{w}(s,z)))\,|_{s=0} = 0 \qquad P(\hat{w}(1,z)) = 0 \qquad \forall z \tag{2.14a,b}$$

$$G(\hat{w}(s,z)) = 0 = U(\hat{w}(s,z)) \qquad \forall s \in (0,1) \quad z = -h, h. \tag{2.15a,b}$$

The solution pair (r, ω) is required to satisfy the inequalities (2.4), and the constitutive functions (N, G, T, P, U) are required to satisfy the conditions (2.6)-(2.11).

3. THE LINEARIZED EQUATIONS

We consider a trivial deformation that is given by a uniform expansion or compression in the radial and vertical directions; that is, we seek solutions of the form

$$r(s,z) = \lambda s \quad \omega(s,z) = \varepsilon z \tag{3.1}$$

for some constant ε. In this case, the strains (2.3) reduce to $(\lambda, 0, \lambda, 0, \varepsilon)$. It is easy to check that the symmetries (2.6)-(2.8) imply that (2.12), (2.13), (2.14), and (2.15a) are satisfied. We get from equation (2.15b) that (λ, ε) must satisfy

$$U(\lambda, 0, \lambda, 0, \varepsilon) = 0. \tag{3.2}$$

It follows from (2.9c), (2.11e), and the (global) implicit function theorem that there exists a smooth function $\hat{\varepsilon} : (0, \infty) \to \Re$ such that

$$U(\lambda, 0, \lambda, 0, \hat{\varepsilon}(\lambda)) = 0. \tag{3.3a}$$

Moreover,

$$d\hat{\varepsilon}(\lambda)/d\lambda = -(U_\nu + U_\tau)/U_\varepsilon, \tag{3.3b}$$

where the arguments of U_ν and so on are $(\lambda, 0, \lambda, 0, \hat{\varepsilon}(\lambda))$. Hence, we take $\varepsilon = \hat{\varepsilon}(\lambda)$ in (3.1) and introduce the notation

$$N(\lambda) = N(\lambda, 0, \lambda, 0, \hat{\varepsilon}(\lambda)), \tag{3.4}$$

and so on. We now compute the linearization of the boundary value problem (2.12), (2.13), (2.14), and (2.15) about the trivial solution (3.1). For this purpose, the symmetry and even-odd conditions (2.6)-(2.8) are essential. For instance, from (2.6a) it follows that

$$N_\tau(\lambda) = T_\nu(\lambda),$$

and from (2.6b), we get that

$$N_\gamma(\lambda) = 0 = N_p(\lambda).$$

If we let (v, w) represent the variations in (r, ω), then the linearization of our boundary value problem about (3.1) is given by

$$\left\{ \begin{aligned} N_\nu(\lambda)[-(sv_s)_s + v/s] - sG_\gamma(\lambda)v_{zz} &= (N_\varepsilon(\lambda) + G_p(\lambda))sw_{sz} \\ -P_p(\lambda)(sw_s)_s - U_\varepsilon(\lambda)sw_{zz} &= (P_\gamma(\lambda) + U_\nu(\lambda))(sv)_{sz} \end{aligned} \right. \tag{3.5a,b}$$

$$v(0,z) = 0 = v(1,z) \quad \forall z \tag{3.6a,b}$$

$$\lim_{s\to 0^+} s\left[P_\gamma(\lambda)\, v_z + P_p(\lambda)\, w_s\right] = 0 \quad \forall z \tag{3.7a}$$

$$P_\gamma(\lambda)\, v_z(1,z) + P_p(\lambda)\, w_s(1,z) = 0 \quad \forall z \tag{3.7b}$$

$$G_\gamma(\lambda)\, v_z(s,\pm h) + G_p(\lambda)\, w_s(s,\pm h) = 0 \quad \forall s \tag{3.8a}$$

$$U_\nu(\lambda)\, v_s(s,\pm h) + U_\tau(\lambda)\left(\frac{v}{s}\right)(s,\pm h) + U_\varepsilon(\lambda)\, w_z(s,\pm h) = 0 \quad \forall s. \tag{3.8b}$$

We look for solutions of equations (3.5a,b) of the form

$$v(s,z) = a(z)\,\alpha(s), \qquad w(s,z) = b(z)\,\beta(s). \tag{3.9}$$

If we substitute (3.9) into (3.5), then we get

$$\begin{cases} N_\nu(\lambda)\left[-\left(s\alpha'(s)\right)' + \alpha(s)/s\right] a(z) - sG_\gamma(\lambda)\, a''(z)\,\alpha(s) \\ \qquad\qquad = \left(N_\varepsilon(\lambda) + G_p(\lambda)\right) s b'(z)\,\beta'(s) \\ -P_p(\lambda)\left(s\beta'(s)\right)' b(z) - U_\varepsilon(\lambda)\, s b''(z)\,\beta(s) \\ \qquad = \left(P_\gamma(\lambda) + U_\nu(\lambda)\right)\left(s\alpha(s)\right)' a'(z) \end{cases}. \tag{3.10}$$

To separate the variables, we take α, β to be solutions of the equations

$$-\left(s\alpha'(s)\right)' + \alpha(s)/s = sk_n^2\,\alpha(s) \qquad \alpha(0) = 0 = \alpha(1) \tag{3.11a}$$

$$-\left(s\beta'(s)\right)' = sk_n^2\,\beta(s) \qquad \lim_{s\to 0^+} s\beta'(s) = 0 = \beta'(1). \tag{3.11b}$$

In fact, $\alpha(s) = J_1(k_n s)$ and $\beta(s) = J_0(k_n s)$, where J_0 and J_1 are Bessel's functions of order zero and one, respectively, and k_n, $n \geq 0$, are the positive zeros of J_1. It now follows that the boundary conditions (3.6) and (3.7) are satisfied. Furthermore, the functions α, β satisfy the following identities:

$$\beta'(s) = -k_n\,\alpha(s) \qquad \left(s\alpha(s)\right)' = sk_n\,\beta(s). \tag{3.12}$$

If we combine (3.11) and (3.12), we can eliminate α, β from (3.10) to get the following boundary value problem for a, b:

$$\begin{cases} N_\nu(\lambda)k_n^2 a(z) - G_\gamma(\lambda)a''(z) = -(N_\varepsilon(\lambda) + G_p(\lambda))k_n b'(z) \\ P_p(\lambda)k_n^2 b(z) - U_\varepsilon(\lambda)b''(z) = (P_\gamma(\lambda) + U_\nu(\lambda))k_n a'(z) \end{cases} \tag{3.13}$$

$$G_\gamma(\lambda)a'(\pm h) - G_p(\lambda)k_n b(\pm h) = 0 \quad k_n U_\nu(\lambda)a(\pm h) + U_\varepsilon(\lambda)b'(\pm h) = 0. \tag{3.14}$$

With the substitution $(y_1(z), y_2(z), y_3(z), y_4(z)) = (a(z), b(z), a'(z), b'(z))$, the system (3.13) can be transformed to a homogeneous first-order system of ordinary differential equations with constant coefficients. The eigenvalues of the resulting coefficient matrix are the roots of the equation

$$\mu^4 - (A + B + CD)\mu^2 + AB = 0, \tag{3.15}$$

where

$$A = k_n^2 \frac{N_\nu(\lambda)}{G_\gamma(\lambda)} \quad B = k_n^2 \frac{P_p(\lambda)}{U_\varepsilon(\lambda)}$$

$$C = -k_n \frac{P_\gamma(\lambda) + U_\nu(\lambda)}{U_\varepsilon(\lambda)} \quad D = k_n \frac{N_\varepsilon(\lambda) + G_p(\lambda)}{G_\gamma(\lambda)}. \tag{3.16}$$

The roots of (3.15) are of the form $\pm\mu_1$, $\pm\mu_2$, where $\mu_1\mu_2 \neq 0$. The corresponding eigenvectors are given by

$$\left(\mu^2 - B,\ \mu C,\ \mu\left(\mu^2 - B\right), \mu^2 C\right)^t. \tag{3.17}$$

The case of repeated eigenvalues occurs when $(A + B + CD)^2 - 4AB = 0$. In this case, the eigenvalues are of the form $\pm\hat{\mu}$ of multiplicity two each with generalized eigenvectors given by

$$\left(\left[(1 - B)\mu^2 + B(A + 1)\right]/\mu,\ \mu^2 C,\ (2 - B)\mu^2 + AB,\ \mu C\left(\mu^2 + 1\right)\right)^t. \tag{3.18}$$

CASE I

When there are no repeated eigenvalues, the general solution of the system (3.13) is given by

$$\begin{cases} a(z) = (\mu_1^2 - B)(A_1 e^{\mu_1 z} + B_1 e^{-\mu_1 z}) + (\mu_2^2 - B)(A_2 e^{\mu_2 z} + B_2 e^{-\mu_2 z}) \\ b(z) = \mu_1 C(A_1 e^{\mu_1 z} - B_1 e^{-\mu_1 z}) + \mu_2 C(A_2 e^{\mu_2 z} - B_2 e^{-\mu_2 z}) \end{cases}, \tag{3.19}$$

where A_1, A_2, B_1, B_2 are constants to be determined. If we require that (3.19) satisfy the boundary conditions (3.14), we get a 4×4 system of linear equations for the constants A_1, A_2, B_1, B_2. The values of λ for which this system has nontrivial solutions define the eigenvalues of the linearization (3.5)-(3.8). A short computation shows that this system has nontrivial solutions if and only if λ satisfies one of the following equations:

$$\chi\delta \tanh(\mu_1 h) - \vartheta\phi \tanh(\mu_2 h) = 0 \tag{3.20a}$$

or

$$\chi\delta \tanh(\mu_2 h) - \vartheta\phi \tanh(\mu_1 h) = 0, \tag{3.20b}$$

where

$$\begin{aligned}
\chi &= \mu_1 \left(\mu_1^2 - B\right) G_\gamma(\lambda) - k_n \mu_1 C G_p(\lambda) \\
\vartheta &= \mu_2 \left(\mu_2^2 - B\right) G_\gamma(\lambda) - k_n \mu_2 C G_p(\lambda) \\
\phi &= -k_n \left(B U_\nu(\lambda) + \mu_1^2 P_\gamma\right) \\
\delta &= -k_n \left(B U_\nu(\lambda) + \mu_2^2 P_\gamma\right).
\end{aligned} \tag{3.21}$$

Equation (3.20a) corresponds to solutions of (3.19) with $A_1 = B_1, A_2 = B_2$, whereas (3.20b) corresponds to the case in which $A_1 = -B_1, A_2 = -B_2$. Note that (3.15) and (3.16) define μ_1, μ_2 as functions of λ, which in turn, on substitution into (3.20) and (3.21), define the characteristic equations for the eigenvalues of the linearized problem (3.5)-(3.8). For each root of k_n of J_1, we have a different pair of equations, the solutions of which, if any, we denote by $\{\lambda_{n1}, \lambda_{n2}, \ldots\}$. The corresponding eigenfunctions of (3.5)-(3.8) are given by

$$\left\{\begin{aligned}
v(s,z) &= J_1(k_n s)\,[\vartheta(\mu_1^2 - B)\sinh(\mu_2 h)\cosh(\mu_1 z) \\
&\qquad - \chi\,(\mu_2^2 - B)\sinh(\mu_1 h)\cosh(\mu_2 z)] \\
w(s,z) &= C J_0(k_n s)\,[\vartheta\mu_1 \sinh(\mu_2 h)\sinh(\mu_1 z) \\
&\qquad - \chi\,\mu_2 \sinh(\mu_1 h)\sinh(\mu_2 z)]
\end{aligned}\right. \tag{3.22a}$$

corresponding to (3.20a), and by

$$\left\{\begin{aligned}
v(s,z) &= J_1(k_n s)\,[\vartheta(\mu_1^2 - B)\cosh(\mu_2 h)\sinh(\mu_1 z) \\
&\qquad - \chi\,(\mu_2^2 - B)\cosh(\mu_1 h)\sinh(\mu_2 z)] \\
w(s,z) &= C J_0(k_n s)\,[\vartheta\mu_1 \cosh(\mu_2 h)\cosh(\mu_1 z) \\
&\qquad - \chi\,\mu_2 \cosh(\mu_1 h)\cosh(\mu_2 z)]
\end{aligned}\right. \tag{3.22b}$$

for the corresponding solutions of (3.20b).

CASE II

The case of repeated eigenvalues occurs when

$$(A+B+CD)^2-4AB=0. \tag{3.23}$$

In this case, the roots of (3.15) are $\pm\hat{\mu}$ of multiplicity two each where $\hat{\mu}^2=(A+B+CD)/2$. The general solution of the system (3.13) is now given by

$$\begin{cases} a(z)=\left(\hat{\mu}^2-B\right)\left(A_1e^{\hat{\mu}z}+A_2e^{-\hat{\mu}z}\right)+\left[(1-B)\hat{\mu}^2+B(A+1)\right] \\ \times\left(B_1e^{\hat{\mu}z}-B_2e^{-\hat{\mu}z}\right)/\hat{\mu}+z\left(\hat{\mu}^2-B\right)\left(B_1e^{\hat{\mu}z}+B_2e^{-\hat{\mu}z}\right) \\ b(z)=\hat{\mu}C\left(A_1e^{\hat{\mu}z}-A_2e^{-\hat{\mu}z}\right)+\hat{\mu}^2C\left(B_1e^{\hat{\mu}z}+B_2e^{-\hat{\mu}z}\right) \\ +\hat{\mu}Cz\left(B_1e^{\hat{\mu}z}-B_2e^{-\hat{\mu}z}\right) \end{cases} \tag{3.24}$$

where A_1, A_2, B_1, B_2 are constants to be determined. Applying the boundary conditions (3.14), we find that the eigenvalues of the linearized problem (3.5)-(3.8) are given by the solutions of one of the following equations:

$$\begin{aligned} &\left[\left(\hat{\mu}^2+1\right)C\left(G_\gamma(\lambda)U_\varepsilon(\lambda)\hat{\mu}^2+G_p(\lambda)U_v(\lambda)k_n^2\right)\right. \\ &\left.+G_\gamma(\lambda)U_v(\lambda)\left(\hat{\mu}^2-B\right)^2k_n+G_p(\lambda)U_\varepsilon(\lambda)\hat{\mu}^2C^2k_n\right] \\ &\times\sinh\left(\hat{\mu}h\right)\cosh\left(\hat{\mu}h\right) \;=\; \pm\chi\,\delta\,h. \end{aligned} \tag{3.25a,b}$$

Equation (3.25a) corresponds to solutions of (3.24) with $A_1=-A_2$, $B_1=B_2$, whereas (3.25b) corresponds to the case in which $A_1=A_2$, $B_1=-B_2$. The corresponding eigenfunctions are given by

$$\begin{cases} v(s,z)=J_1(k_ns)\left[\left\{-\left(\hat{\mu}^2-B\right)\left(\hat{A}\cosh\left(\hat{\mu}h\right)+h\chi\sinh\left(\hat{\mu}h\right)\right)\right.\right. \\ +\chi\dfrac{(1-B)\hat{\mu}^2+B(A+1)}{\hat{\mu}}\cosh\left(\hat{\mu}h\right)\Big\}\sinh\left(\hat{\mu}z\right) \\ \left.+z\left(\hat{\mu}^2-B\right)\chi\cosh\left(\hat{\mu}h\right)\cosh\left(\hat{\mu}z\right)\right] \\ w(s,z)=J_0(k_ns)\left[\left\{-\hat{\mu}C\left(\hat{A}\cosh\left(\hat{\mu}h\right)+h\chi\sinh\left(\hat{\mu}h\right)\right)\right.\right. \\ \left.+\hat{\mu}^2C\chi\cosh\left(\hat{\mu}h\right)\Big\}\cosh\left(\hat{\mu}z\right)+\hat{\mu}Cz\chi\cosh\left(\hat{\mu}h\right)\sinh\left(\hat{\mu}z\right)\right] \end{cases} \tag{3.26a}$$

for (3.25a) (with the + sign) and by

$$\begin{cases} v(s,z) = J_1(k_n s)\Big[\Big\{\Big(\hat{\mu}^2 - B\Big)\Big(\hat{C}\cosh(\hat{\mu}h) + h\delta\,\sinh(\hat{\mu}h)\Big) \\ -\delta\,\dfrac{(1-B)\hat{\mu}^2 + B(A+1)}{\hat{\mu}}\cosh(\hat{\mu}h)\Big\}\cosh(\hat{\mu}z) \\ -z\Big(\hat{\mu}^2 - B\Big)\,\delta\,\cosh(\hat{\mu}h)\sinh(\hat{\mu}z)\Big] \\ \\ w(s,z) = J_0(k_n s)\Big[\Big\{\hat{\mu}C\Big(\hat{C}\cosh(\hat{\mu}h) + h\delta\,\sinh(\hat{\mu}h)\Big) \\ -\hat{\mu}^2 C\delta\,\cosh(\hat{\mu}h)\Big\}\sinh(\hat{\mu}z) + \hat{\mu}Cz\delta\,\cosh(\hat{\mu}h)\cosh(\hat{\mu}z)\Big] \end{cases} \tag{3.26b}$$

for (3.25b), where

$$\begin{cases} \hat{A} = G_\gamma(\lambda)\Big((1-B)\hat{\mu}^2 + B(A+1) + \hat{\mu}^2 - B\Big) - G_p(\lambda)\hat{\mu}^2 Ck_n \\ \hat{C} = k_n U_\nu(\lambda)\,\dfrac{(1-B)\hat{\mu}^2 + B(A+1)}{\hat{\mu}} + U_\varepsilon(\lambda)\hat{\mu}C\Big(\hat{\mu}^2 + 1\Big). \end{cases} \tag{3.27}$$

Upon examining (3.16), we find that (3.23) is independent of k_n and h, the height of the cylinder, and thus acts as a constitutive restriction. The solutions λ of this equation, which are the possible eigenvalues of (3.5)-(3.8), are thus independent of k_n and h. Thus, equation (3.25), with these values of λ substituted, can be thought as defining the possible cylinder heights that would lead to some spectra. This situation is different for Case I, in which (3.20) in general has solutions λ for any given h.

4. EXISTENCE OF EIGENVALUES FOR HADAMARD-GREEN TYPE MATERIALS

Let **C** be the Cauchy-Green deformation tensor (cf. (6.13)) corresponding to the deformation (2.2). Let (I_C, II_C, III_C) be the principal invariants of **C** (cf. (6.16)). Then, the stored energy function describing the material behavior of the cylinder is given by (cf. (6.17))

$$\begin{aligned} g(I_C, II_C, III_C) &= A_1 I_C^{\alpha_1/2} + A_2\left(\frac{II_C}{III_C}\right)^{\frac{\alpha_2}{2}} + A_3 II_C^{\alpha_3/2} \\ &+ A_4\left(\frac{I_C}{III_C}\right)^{\frac{\alpha_4}{2}} + A_5 III_C^{\alpha_5/2} + A_6 III_C^{-\alpha_6/2}, \end{aligned} \tag{4.1}$$

where A_i are nonnegative and α_i are positive. If $(\lambda_1^2, \lambda_2^2, \lambda_3^2)$ are the eigenvalues of $\mathbf{C}$, then it follows that

$$I_C = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad II_C = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 \quad III_C = \lambda_1^2\lambda_2^2\lambda_3^2$$

$$\frac{II_C}{III_C} = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} \quad \frac{I_C}{III_C} = \frac{1}{\lambda_1^2\lambda_2^2} + \frac{1}{\lambda_1^2\lambda_3^2} + \frac{1}{\lambda_2^2\lambda_3^2}. \tag{4.2}$$

Thus, from (4.1) and (4.2), it follows that an infinite energy is required to produce an infinite expansion or compression of a fiber, surface, or volume element within the cylinder. Since

$$\begin{aligned} I_C &= \operatorname{tr}(\mathbf{C}) = \operatorname{tr}\left(\mathbf{F}^t\mathbf{F}\right) = \mathbf{F} : \mathbf{F} \\ II_C &= \frac{1}{2}\left(\operatorname{tr}(\mathbf{C})^2 - \operatorname{tr}\left(\mathbf{C}^2\right)\right) = \frac{1}{2}\left((\mathbf{F} : \mathbf{F})^2 - \mathbf{F}^t\mathbf{F} : \mathbf{F}^t\mathbf{F}\right), \end{aligned}$$

we have that with $A_2 = A_4 = 0$ and $\alpha_1 = \alpha_3 = 2$, (4.1) represents a Hadamard-Green type material. A variant of (4.1) was used by Antman [3] as a model of a stored energy function satisfying certain growth conditions that lead to the regularity of solutions for certain equilibrium problems for cylinder-like bodies. The use of the stored energy function (4.1) in terms of the principal invariants of $\mathbf{C}$ is better for numerical calculations than using a stored energy function in terms of the principal stretches $(\lambda_1, \lambda_2, \lambda_3)$ as is frequently used. This is due to the square root extraction process involved in getting the principal stretches that introduces a singularity in the Euler-Lagrange equations when some or all of the principal stretches coincide.

We now consider in more detail the case of a Hadamard-Green type material; that is, $A_2 = A_4 = 0$ and $\alpha_1 = \alpha_3 = 2$. In this case, (4.1) reduces to

$$g(I_C, II_C, III_C) = A_1 I_C + A_3 II_C + A_5 III_C^g + A_6 III_C^{-d}, \tag{4.3}$$

where we have set $g = \alpha_5/2$, $d = \alpha_6/2$. We now show that in this case, each of the equations (3.20a,b) has at least one solution for each k_n. We now can get an explicit form for equation (3.2). The principal invariants (I_C, II_C, III_C) at the trivial state $(\lambda, 0, \lambda, 0, \varepsilon)$ reduce to

$$I_C = 2\lambda^2 + \varepsilon^2 \quad II_C = \lambda^4 + 2\lambda^2\varepsilon^2 \quad III_C = \lambda^4\varepsilon^2. \tag{4.4}$$

(For simplicity, we let ε stand for $\hat{\varepsilon}(\lambda)$.) It follows now from (4.3), (6.22e), and (4.4) that (3.2) is given by

$$A_1\varepsilon^2 + 2A_3\lambda^2\varepsilon^2 + gA_5\left(\lambda^4\varepsilon^2\right)^g - dA_6\left(\lambda^4\varepsilon^2\right)^{-d} = 0. \tag{4.5}$$

If we multiply by $(\varepsilon^2)^d$, we get the following equation:

$$gA_5\lambda^{4g}\left(\varepsilon^2\right)^{g+d}+\left(A_1+2A_3\lambda^2\right)\left(\varepsilon^2\right)^{d+1}-dA_6\lambda^{-4d}=0. \tag{4.6}$$

Lemma 4.1. *Assume that $g=d+2$ for the stored energy function (4.3). Then, the solution ε of equation (3.2) is given by*

$$\left(\varepsilon^2\right)^{d+1}=\frac{-\left(A_1+2A_3\lambda^{2}\right)+\sqrt{\left(A_1+2A_3\lambda^{2}\right)^2+4gdA_5A_6\lambda^{8}}}{2gA_5\lambda^{4g}}. \tag{4.7}$$

Proof. This follows from the observation that (4.6) is a quadratic in $(\varepsilon^2)^{d+1}$ provided that $g=d+2$. Equation (4.7) is just the positive root of this quadratic.

From now on, we shall employ the notation $f(s)\sim g(s)$ as $s\longrightarrow s_0$ to denote

$$\lim_{s\longrightarrow s_0}\frac{f(s)}{g(s)}=1. \tag{4.8}$$

From Lemma 4.1, we now get

$$\varepsilon\sim\begin{cases}\left(\dfrac{A_6d}{A_1}\right)^{\frac{1}{2(d+1)}}\lambda^{\frac{-2d}{d+1}} & \lambda\longrightarrow 0^+\\[2ex] \left(\dfrac{A_6d}{A_5g}\right)^{\frac{1}{4(d+1)}}\lambda^{-2} & \lambda\longrightarrow\infty\end{cases}. \tag{4.9}$$

We shall need the following asymptotic formulas for III_C in (4.4):

$$\lambda^4\varepsilon^2\sim\begin{cases}\left(\dfrac{A_6d}{A_1}\right)^{\frac{1}{d+1}}\lambda^{\frac{4}{d+1}} & \lambda\longrightarrow 0^+\\[2ex] \left(\dfrac{A_6d}{A_5g}\right)^{\frac{1}{2(d+1)}}\left(1-c_1\lambda^{-2}\right)^{\frac{1}{d+1}} & \lambda\longrightarrow\infty\end{cases}, \tag{4.10}$$

where $c_1=A_3/\sqrt{gdA_5A_6}$.

From equation (4.5), it follows that

$$W_3(\lambda)\equiv gA_5\left(\lambda^4\varepsilon^2\right)^{g-1}-dA_6\left(\lambda^4\varepsilon^2\right)^{-d-1}=\frac{-\left(A_1+2A_3\lambda^2\right)}{\lambda^4}. \tag{4.11}$$

Working with this equation and (4.3), (4.4), and (6.22), we get

$$G_\gamma(\lambda)=2\left(A_1+A_3\lambda^2\right)=P_p(\lambda)\qquad P_\gamma(\lambda)=(\varepsilon/\lambda)\,P_p(\lambda) \tag{4.12a,b,c}$$

$$\begin{aligned} U_\nu(\lambda) &= -2(\varepsilon/\lambda)P_p(\lambda) \\ &+ 4\lambda^7\varepsilon^3\left(g(g-1)A_5\left(\lambda^4\varepsilon^2\right)^{g-2} + d(d+1)A_6\left(\lambda^4\varepsilon^2\right)^{-d-2}\right) \end{aligned} \tag{4.12d}$$

$$\begin{aligned} N_\nu(\lambda) &= \left(1-(\varepsilon/\lambda)^2\right)P_p(\lambda) \\ &+ 4\lambda^6\varepsilon^4\left(g(g-1)A_5\left(\lambda^4\varepsilon^2\right)^{g-2} + d(d+1)A_6\left(\lambda^4\varepsilon^2\right)^{-d-2}\right) \end{aligned} \tag{4.12e}$$

$$U_\varepsilon(\lambda) = 4\lambda^8\varepsilon^2\left(g(g-1)A_5\left(\lambda^4\varepsilon^2\right)^{g-2} + d(d+1)A_6\left(\lambda^4\varepsilon^2\right)^{-d-2}\right). \tag{4.12f}$$

Using (4.9) and (4.10), we get the following asymptotic expressions for the constitutive functions in (4.12):

$$G_\gamma(\lambda) \sim \begin{cases} 2A_1 & \lambda \to 0^+ \\ 2A_3\lambda^2 & \lambda \to \infty \end{cases} \qquad P_p(\lambda) \sim \begin{cases} 2A_1 & \lambda \to 0^+ \\ 2A_3\lambda^2 & \lambda \to \infty \end{cases} \tag{4.13a,b}$$

$$P_\gamma(\lambda) \sim \begin{cases} 2A_1\eta_1^{1/2}\lambda^{-\left(\frac{1+3d}{d+1}\right)} & \lambda \to 0^+ \\ 2A_3\eta_2^{1/4}\lambda^{-1} & \lambda \to \infty \end{cases} \tag{4.13c}$$

$$U_\nu(\lambda) \sim \begin{cases} 4dA_1\eta_1^{1/2}\lambda^{-\left(\frac{1+3d}{d+1}\right)} & \lambda \to 0^+ \\ 4W_{33}^{\infty}\eta_2^{3/4}\lambda & \lambda \to \infty \end{cases} \tag{4.13d}$$

$$N_\nu(\lambda) \sim \begin{cases} 2A_1(2d+1)\eta_1\lambda^{-2\left(\frac{1+3d}{d+1}\right)} & \lambda \to 0^+ \\ 2A_3\lambda^2 & \lambda \to \infty \end{cases} \tag{4.13e}$$

$$U_\varepsilon(\lambda) \sim \begin{cases} 4(d+1)A_1 & \lambda \to 0^+ \\ 4W_{33}^{\infty}\eta_2^{1/2}\lambda^4 & \lambda \to \infty \end{cases} \tag{4.13f}$$

where

$$\eta_1 = \left(\frac{A_6 d}{A_1}\right)^{\frac{1}{d+1}} \qquad \eta_2 = \left(\frac{A_6 d}{gA_5}\right)^{\frac{1}{d+1}}$$

$$W_{33}^{\infty} = g\,(g-1)\,A_5\eta_{\,2}^{\,d/2} + d\,(d+1)\,A_6\eta_{\,2}^{\,-(d+2)/2}. \tag{4.14}$$

Lemma 4.2. *Assume that $g = d+2$ for the stored energy function (4.3). Then, the roots μ_1, μ_2 of equation (3.15) have the following asymptotic representations:*

$$\mu_1 \sim \begin{cases} k_n\sqrt{\dfrac{2d+1}{2\,(d+1)}}\eta_{\,1}^{\,1/2}\lambda^{\,-\left(\frac{1+3d}{1+d}\right)} & \lambda \to 0^+ \\ k_n & \lambda \to \infty \end{cases}$$

$$\mu_2 \sim \begin{cases} k_n & \lambda \to 0^+ \\ k_n\left(\dfrac{A_3}{2W_{33}^{\infty}}\right)^{1/2}\eta_{\,2}^{-1/4}\lambda^{\,-1} & \lambda \to \infty. \end{cases} \tag{4.15}$$

Proof. From equation (3.15), it follows that

$$\mu_{\,1}^{\,2},\mu_{\,2}^{\,2} = \frac{1}{2}\left(A+B+CD \pm \sqrt{(A+B+CD)^2 - 4AB}\right). \tag{4.16}$$

If we combine (3.16) and (4.13), we get

$$A \sim \begin{cases} k_n^2\,(2d+1)\,\eta_1\lambda^{\,-2\left(\frac{1+3d}{1+d}\right)} & \lambda \to 0^+ \\ k_n^2 & \lambda \to \infty \end{cases} \tag{4.17a}$$

$$B \sim \begin{cases} \dfrac{k_n^2}{2\,(d+1)} & \lambda \to 0^+ \\ \dfrac{k_n^2}{2W_{33}^{\infty}}\eta_{\,2}^{\,-1/2}A_3\lambda^{-2} & \lambda \to \infty \end{cases} \tag{4.17b}$$

$$C \sim \begin{cases} -k_n\dfrac{(2d+1)}{2\,(d+1)}\eta_{\,1}^{\,1/2}\lambda^{\,-\left(\frac{1+3d}{1+d}\right)} & \lambda \to 0^+ \\ -k_n\eta_{\,2}^{\,1/4}\lambda^{\,-3} & \lambda \to \infty \end{cases} \tag{4.17c}$$

$$D \sim \begin{cases} k_n\,(2d+1)\,\eta_{\,1}^{\,1/2}\lambda^{\,-\left(\frac{1+3d}{1+d}\right)} & \lambda \to 0^+ \\ k_n\dfrac{2W_{33}^{\infty}}{A_3}\eta_{\,2}^{\,3/4}\lambda^{-1} & \lambda \to \infty \end{cases}. \tag{4.17d}$$

Combining (4.16) and (4.17), we get (4.15).

Theorem 4.3. *For the stored energy function (4.3), let $g = d+2$. Then, equations (3.20a,b) have at least a solution for each root k_n of J_1 and cylinder height h.*

Proof. Using the expressions (4.12), one can show that

$$G_\gamma^2(\lambda)\, U_\varepsilon^2(\lambda)\left((A+B+CD)^2 - 4AB\right) = \lambda^{-4}\left(\varepsilon^2-\lambda^2\right)^2 P_p(\lambda) \times \left(4\lambda^8\varepsilon^2 W_{33}(\lambda) - P_p(\lambda)\right)^2, \quad (4.18)$$

where

$$W_{33}(\lambda) = g(g-1)A_5\left(\lambda^4\varepsilon^2\right)^{g-2} + d(d+1)A_6\left(\lambda^4\varepsilon^2\right)^{-d-2}. \quad (4.19)$$

It follows from this and (4.16) that both μ_1^2 and μ_2^2 are real numbers. Furthermore, (4.12) and (4.18) can be used again in (4.16) so that μ_1^2 and μ_2^2 are positive. Hence, μ_1 and μ_2 are positive, and the functions on the left-hand side of equations (3.20a,b) are real valued functions of λ.

If we combine the expressions (4.13), (4.15), and (4.17), we get the following asymptotic formulas for (3.21):

$$\chi \sim \begin{cases} C_1\lambda^{-3\left(\frac{1+3d}{1+d}\right)} & \lambda \to 0^+ \\ C_2\lambda^2 & \lambda \to \infty \end{cases} \qquad \vartheta \sim \begin{cases} C_3\lambda^{-2\left(\frac{1+3d}{1+d}\right)} & \lambda \to 0^+ \\ -C_4\lambda^{-3} & \lambda \to \infty \end{cases} \quad (4.20a)$$

$$\phi \sim \begin{cases} -C_5\lambda^{-3\left(\frac{1+3d}{1+d}\right)} & \lambda \to 0^+ \\ -C_6\lambda^{-1} & \lambda \to \infty \end{cases} \qquad \delta \sim \begin{cases} -C_7\lambda^{-\left(\frac{1+3d}{1+d}\right)} & \lambda \to 0^+ \\ -C_8\lambda^{-1} & \lambda \to \infty \end{cases} \quad (4.20b)$$

where C are positive constants. Now, let

$$f_{\text{sym}}(\lambda) = \chi\delta\tanh(\mu_1 h) - \vartheta\phi\tanh(\mu_2 h)$$

$$f_{\text{asym}}(\lambda) = \chi\delta\tanh(\mu_2 h) - \vartheta\phi\tanh(\mu_1 h). \quad (4.21)$$

We then have for some positive constants D_1, D_2 that

$$f_{\text{sym}}(\lambda) \sim \begin{cases} -C_1C_7\lambda^{-4\left(\frac{1+3d}{1+d}\right)}\tanh\left(D_1\lambda^{-4\left(\frac{1+3d}{1+d}\right)}h\right) \\ +C_3C_5\lambda^{-5\left(\frac{1+3d}{1+d}\right)}\tanh(k_n h) & \lambda \to 0^+ \\ -C_2C_8\lambda\tanh(k_n h) - C_4C_6\lambda^{-4}\tanh\left(D_2\lambda^{-1}h\right) & \lambda \to \infty \end{cases}$$

Table 1. Coefficient and exponent values used in (4.1).

A_i		α_i	
i	*Value*	*i*	*Value*
1	10^{-4}	1	2
2	0	2	–
3	10^{-4}	3	2
4	0	4	–
5	10^{-6}	5	6
6	10^{-1}	6	4

$$\sim \begin{cases} C_3C_5\lambda^{-5\left(\frac{1+3d}{1+d}\right)}\tanh(k_n h) & \lambda \to 0^+ \\ -C_2C_8\lambda\tanh(k_n h) & \lambda \to \infty \end{cases} \tag{4.22a}$$

$$f_{\text{asym}}(\lambda) \sim \begin{cases} -C_1C_7\lambda^{-4\left(\frac{1+3d}{1+d}\right)}\tanh(k_n h) + C_3C_5\lambda^{-5\left(\frac{1+3d}{1+d}\right)} & \\ \times\tanh\left(D_1\lambda^{-4\left(\frac{1+3d}{1+d}\right)}h\right) & \lambda \to 0^+ \\ -C_2C_8\lambda\tanh\left(D_2\lambda^{-1}h\right) - C_4C_6\lambda^{-4}\tanh(k_n h) & \lambda \to \infty \end{cases}$$

$$\sim \begin{cases} C_3C_5\lambda^{-5\left(\frac{1+3d}{1+d}\right)}\tanh\left(D_1\lambda^{-4\left(\frac{1+3d}{1+d}\right)}h\right) & \lambda \to 0^+ \\ -C_2C_8D_2h & \lambda \to \infty. \end{cases} \tag{4.22b}$$

From (4.22), we get

$$f_{\text{sym}}(\lambda) \to \begin{cases} \infty & \lambda \to 0^+ \\ -\infty & \lambda \to \infty \end{cases} \qquad f_{\text{asym}}(\lambda) \to \begin{cases} \infty & \lambda \to 0^+ \\ \text{negative constant} & \lambda \to \infty \end{cases} \tag{4.23}$$

from which the stated result follows.

We close this section with a numerical example. For our calculations, we used the values shown in Table 1 for the exponents and coefficients with a cylinder of height $h = 0.2$. In Table 2, we show typical solutions of equations (3.20a,b) for the model (4 .1) and the values in Table 1. The computed eigenvalue λ^* corresponds to the first root of (3.20a,b) in each case, and μ_1, μ_2 are the computed roots of (3.15). As can be seen from the results, both sequences of computed eigenvalues seem to be converging to some value, possibly a common one. This appears to be reminiscent of the wrinkling phenomena observed in plain strain slab problems.

For $\delta > 0$ small, we can write

$$\begin{aligned} r(s,z) &= \lambda s + \delta v(s,z) + o(\delta) \qquad \omega(s,z) = \hat{\varepsilon}(\lambda)z + \delta w(s,z) + o(\delta) \\ \lambda &= \lambda^* + o(\delta) \end{aligned} \tag{4.24}$$

Table 2. Eigenvalues computed for (3.5)-(3.8) corresponding to the first six models.

	Symmetric Case (3.20a)			*Asymmetric Case (3.20b)*		
Root of J_1 (k_n)	λ^*	μ_1	μ_2	λ^*	μ_1	μ_2
3.8317	0.8374	20.76	3.832	1.3247	5.826	3.832
7.0156	0.9581	26.55	7.016	1.0878	18.69	7.016
10.173	0.9985	34.38	10.17	1.0344	31.19	10.17
13.324	1.0108	43.53	13.32	1.0210	42.34	13.32
16.471	1.0144	53.29	16.47	1.0173	52.87	16.47
19.615	1.0154	63.28	19.61	1.0163	63.14	19.62

where $r(s,z)$, $\omega(s,z)$ are solutions of the boundary value problem (2.12)-(2.15) and $v(s,z)$, $w(s,z)$ represent the eigenfunctions which are solutions of (3.5)-(3.8). Using the values of λ^* from Table 2, we can compute using equations (3.22) or (3.26) the eigenfunctions $v(s,z)$, $w(s,z)$, and, after dropping the $o(\delta)$ terms in (4.24), we obtain first-order approximations to the functions $r(s,z)$, $\omega(s,z)$. In Figures 1 and 2, we show examples of these first-order approximations giving the approximate shape of the deformed mid-cross section of the cylinder. (Full color pictures of the corresponding eigenfunctions in this example can be accessed at http://cuhwww.upr.clu.edu/~pnm/deformations/index.htm.)

5. CONCLUSIONS

The vertical component $w(s,z)$ of the eigenfunction (3.22a) is an odd function in z, whereas that of (3.22b) is even in z. Thus, the solutions λ of (3.20a) leading to the eigenfunction (3.22a) represent possible bifurcation points for the nonlinear system (2.12)-(2.15) of solutions that are symmetric with respect to the $z = 0$ plane. On the other hand, those solutions of (3.20b) leading to the eigenfunction (3.22b) represent possible bifurcation points of symmetry-breaking solutions. The same analysis shows that in Case II of the previous section, (3.26b) leads to symmetry-preserving branches of nontrivial solutions of (2.12)-(2.15) whereas (3.26a) gives the corresponding symmetry-breaking branches.

The stability analysis for necked states of nonlinearly elastic bars in tension was carried out in a complete manner by [16]. In particular, Owen shows that the only nontrivial stable solutions are the ones with a half-neck or draw representing the first bifurcating branch of solutions. The numerical work of Negrón-Marrero and Santiago-Figueroa [14] for nonlinearly elastic thin plates with thickness variations shows that there can be unstable solutions with a half-neck and stable multiple-necked solutions. Thus, we expect the stability analysis for our problem to be of a somewhat similar complex nature. However, a stability analysis of either the trivial solution (3.1) or (3.2), or the nontrivial branches of solutions to our problem, would be more physically meaningful among arbitrary deformations, not just axisymmetric ones. We shall pursue this question elsewhere.

Another issue related to our problem that is not covered in this paper is the formal analysis of the existence and disposition of the global branches of nontrivial solutions. This analysis is somewhat difficult in this case for two reasons. First, the singularity at $s = 0$ in equations (2.12) represents a serious obstacle in setting up our problem for the application

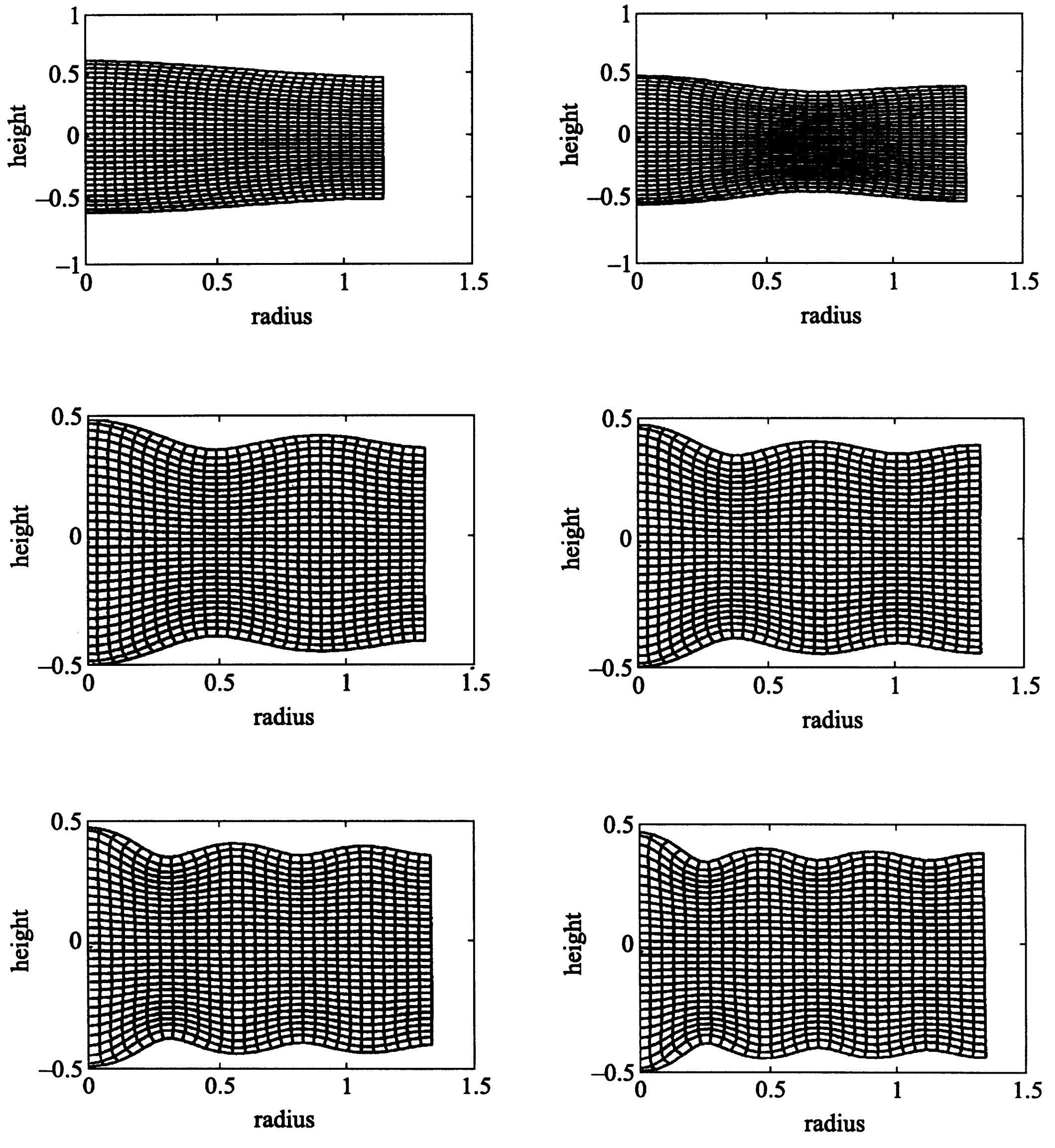

Fig. 1. Nonlinear deformations corresponding to the first six symmetric eigenmodes.

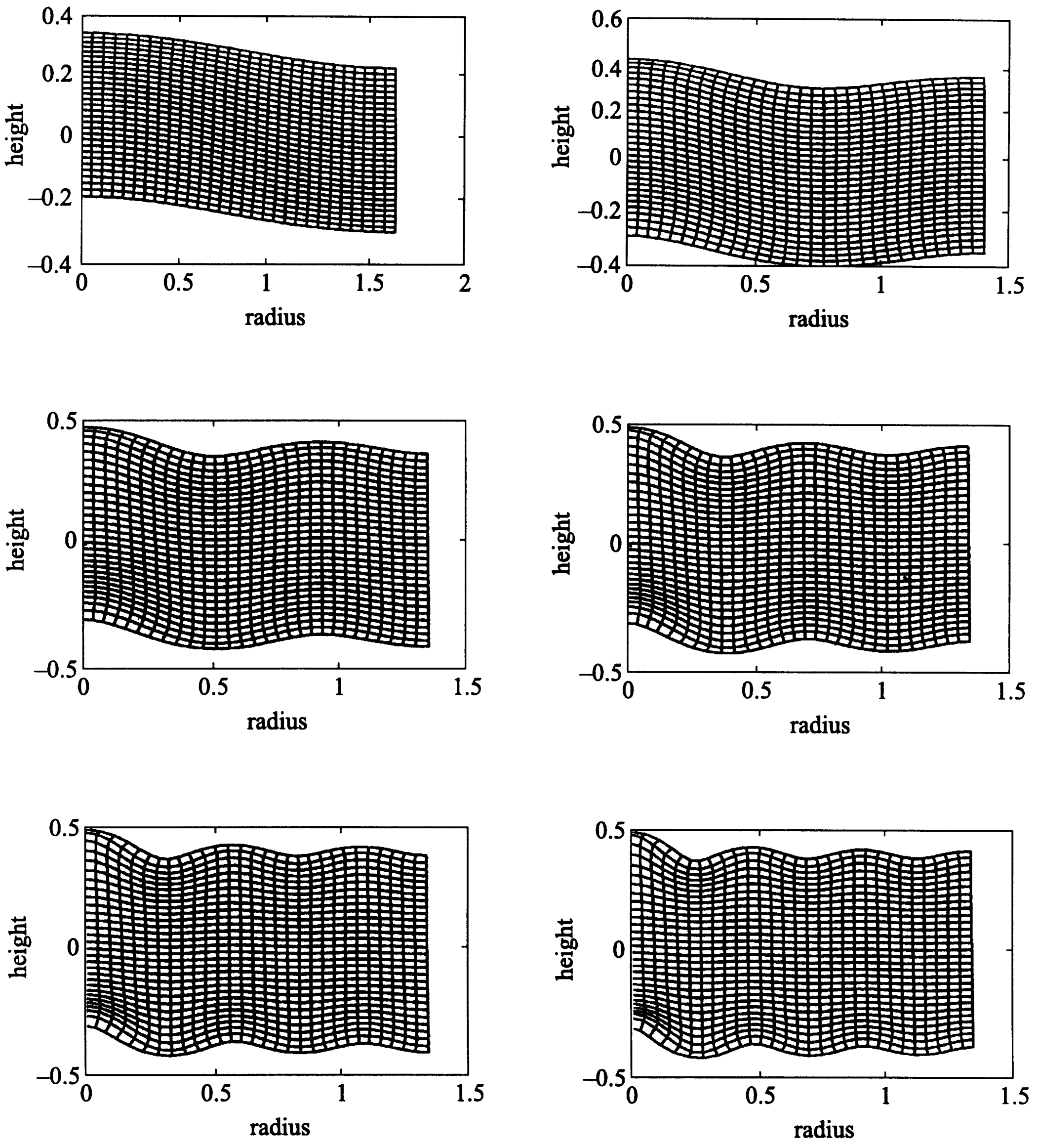

Fig. 2. Nonlinear deformations corresponding to the first six asymmetric eigenmodes.

of the standard theorems of bifurcation theory. The complexity of such an analysis can be appreciated in the somewhat simpler but still nontrivial problems treated in [2], Shih and Antman [18], and [13]. The second difficulty with such a global analysis has to do with the lack of global bifurcation theorems for nonlinear elasticity problems. The global continuation results in Healey and Simpson [11] have the potential of leading to such global bifurcation results.

6. APPENDIX

In this section, we present a full derivation of the equations given in Section 2 describing the nonlinear deformations of a cylinder. We consider a body occupying the cylindrical region in $\Re^3$ given by (2.1). Let

$$\eta\,(s,\phi\,,z)\equiv s\mathbf{e}_1\,(\phi\,)+z\mathbf{k},\,\forall\,(s,\phi\,,z)\in[0,1]\times[0,2\pi\,]\times[-h,h]\,, \tag{6.1}$$

so that $\eta([0,1]\times[0,2\pi]\times[-h,h])=\Omega$. We consider an axisymmetric deformation of the body of the form (2.2). Let $\mathbf{T}(\eta\,(s,\phi\,,z))$ represent the first Piola-Kirchhoff stress tensor. We assume that the top and bottom boundaries of the cylinder are stress free, that the outer edge at $s=1$ is displaced λ units, and that the center of the cylinder remains intact during the deformation. Thus, we consider the following boundary conditions:

$$\begin{gathered}\mathbf{T}\,(\eta\;\,(s,\phi\,,\pm h))\cdot\mathbf{k}=\mathbf{0}\quad\forall\,(s,\phi\,)\\ \\ r\,(0,z)=0\quad r\,(1,z)=\lambda\quad\forall z\\ (\mathbf{T}\,(\eta\;\,(1,\phi\,,z))\cdot\mathbf{e}_1)\cdot\mathbf{k}=\mathbf{0}\quad\forall\,(\phi\,,z)\,.\end{gathered}\tag{6.2}$$

The principle of virtual work states that the deformation $\mathbf{p}1$ must satisfy

$$\int_{\Omega}\mathbf{T}\,(\mathbf{y}):\left[\frac{\partial}{\partial\mathbf{y}}\delta\mathbf{p}\right]^{*}d\mathbf{y}-\int_{\partial\Omega}(\mathbf{T}\,(\mathbf{y})\cdot\mathbf{n}\,(\mathbf{y}))\cdot\delta\mathbf{p}\,d\sigma\;(\mathbf{y})=0 \tag{6.3}$$

for all virtual displacements $\delta\,\mathbf{p}$ satisfying appropriate boundary conditions. Here, $\mathbf{n}(\cdot)$ represents the outer normal to $\partial\Omega$, $\sigma\;(\cdot)$ represents a surface measure, and the asterisk over a tensor denotes its adjoint. From (2.1) and (2.2) and using the first boundary condition in (6.2), we can write (6.3) explicitly as

$$\int_0^1\int_0^{2\pi}\int_{-h}^{h}\Big[s\,(\mathbf{T}\,(\eta\,)\cdot\mathbf{e}_1\,(\phi\,))\cdot(\delta\mathbf{p})_s+(\mathbf{T}\,(\eta\,)\cdot\mathbf{e}_2\,(\phi\,))\cdot(\delta\mathbf{p})_\phi$$

$$+\,s\,(\mathbf{T}\,(\eta\,)\cdot\mathbf{k})\cdot(\delta\mathbf{p})_z\big]\,dzd\phi\,ds$$

$$-\int_{0}^{2\pi}\int_{-h}^{h}\left(\mathbf{T}\left(\eta\ (1,\phi,z)\right)\cdot\mathbf{e}_1\left(\phi\right)\right)\cdot\delta\mathbf{p}d\phi dz\ =\ 0 \tag{6.4}$$

where $\delta\mathbf{p} = \delta r\mathbf{e}_1 + \delta\omega\mathbf{k}$ and $\delta r(0,z) = 0 = \delta r(1,z)$ for all z. If we define

$$T_{ij}\ (s,\phi,z) = \left(\mathbf{T}\left(\eta\ (s,\phi,z)\right)\cdot\mathbf{e}_j\ (\phi)\right)\cdot\mathbf{e}_i\ (\phi)\,,\qquad i,j = 1,2,3 \tag{6.5}$$

then we can write (6.4) (omitting the arguments of T_{ij} and using the third boundary condition in (6.2)) as

$$\int_{0}^{1}\int_{0}^{2\pi}\int_{-h}^{h}\left[s\left(T_{11}\delta\ r_s\ + T_{31}\delta\ \omega_s\right) + T_{22}\delta\ r + s\left(T_{13}\delta\ r_z\ + T_{33}\delta\ \omega_z\right)\right]dsd\phi dz = 0 \tag{6.6}$$

for all smooth δ_r, $\delta\omega$ satisfying $\delta r(0,z) = 0 = \delta r(1,z)$ for all z. If we use integration by parts, then we can rewrite (6.6) as

$$\int_{0}^{1}\int_{0}^{2\pi}\int_{-h}^{h}\left\{\left[-\left(sT_{11}\right)_s + T_{22} - s\left(T_{13}\right)_z\right]\delta r + \left[-\left(sT_{31}\right)_s - s\left(T_{33}\right)_z\right]\delta\omega\right\}dsd\phi dz$$

$$+\int_{0}^{2\pi}\int_{-h}^{h}\left(sT_{11}\delta r + sT_{31}\delta\omega\right)|_0^1\ d\phi dz + \int_{0}^{1}\int_{0}^{2\pi}\left(sT_{13}\delta r + sT_{33}\delta\omega\right)|_{-h}^{h}\ dsd\phi = 0 \tag{6.7}$$

for all smooth δr, $\delta\omega$. The fundamental lemma of the calculus of variations now yields the following boundary value problem for $(r(s,z), \omega(s,z))$:

$$\begin{cases} -\left(sT_{11}\right)_s + T_{22} - s\left(T_{13}\right)_z = 0 & \\ -\left(sT_{31}\right)_s - s\left(T_{33}\right)_z = 0 & \forall\left(s,\phi,z\right)\in\Omega \end{cases}, \tag{6.8}$$

subject to

$$r\left(0,z\right) = 0\quad r\left(1,z\right) = \lambda\quad \forall z \tag{6.9a}$$

$$\left(sT_{31}\right)|_{s=0} = 0\quad \forall\left(\phi,z\right) \tag{6.9b}$$

$$T_{31} = 0\quad \forall\left(\phi,z\right)\quad s = 1 \tag{6.9c}$$

$$T_{13} = 0 = T_{33}\quad \forall\left(s,\phi\right),\quad z = -h,h. \tag{6.9d}$$

The deformation $\mathbf{p}$ with respect to Cartesian coordinates is given by

$$\mathbf{q}(\mathbf{y}) = \mathbf{p}\left(\eta^{-1}(\mathbf{y})\right) \qquad \mathbf{y} \in \Omega. \tag{6.10}$$

The deformation gradient is given by

$$\begin{aligned} \mathbf{F}(\eta\ (s,\phi,z)) &= \frac{\partial \mathbf{q}}{\partial \mathbf{y}}(\eta\ (s,\phi,z)) \\ &= r_s \mathbf{e}_1\mathbf{e}_1 + \omega_s \mathbf{k}\mathbf{e}_1 + (r/s)\,\mathbf{e}_2\mathbf{e}_2 + r_z \mathbf{e}_1\mathbf{k} + \omega_z \mathbf{k}\mathbf{k}. \end{aligned} \tag{6.11}$$

The strains for our problem are given by the vector

$$\hat{w}(s,z) = (r_s\ (s,z)\,,\ r_z\ (s,z)\,,\ r(s,z)/s,\ \omega_s(s,z),\ \omega_z\ (s,z)) \tag{6.12}$$

The Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^t\mathbf{F}$ is now given by the expression

$$\mathbf{C} = \left(r_s^2 + \omega_s^2\right)\mathbf{e}_1\mathbf{e}_1 + (r/s)^2\,\mathbf{e}_2\mathbf{e}_2 + (r_s r_z + \omega_s\omega_z)\,(\mathbf{e}_1\mathbf{k} + \mathbf{k}\mathbf{e}_1) + \left(r_z^2 + \omega_z^2\right)\mathbf{k}\mathbf{k}. \tag{6.13}$$

An easy computation reveals that the determinant of the deformation gradient (6.11) is given by

$$\det\mathbf{F} = (r/s)\,(r_s\omega_z - r_z\omega_s)\,. \tag{6.14}$$

Thus, the orientation-preserving condition $\det\mathbf{F} > 0$ implies that (r, ω) must satisfy the inequalities

$$r/s > 0 \qquad r_s\omega_z - r_z\omega_s > 0. \tag{6.15}$$

The principal invariants of $\mathbf{C}$ are given by

$$\begin{aligned} I_C &= r_s^2 + r_z^2 + \omega_s^2 + \omega_z^2 + (r/s)^2 \\ II_C &= (r/s)^2\left(r_s^2 + r_z^2 + \omega_s^2 + \omega_z^2\right) + (r_s\omega_z - r_z\omega_s)^2 \\ III_C &= (r/s)^2\,(r_s\omega_z - \omega_s r_z)^2\,. \end{aligned} \tag{6.16}$$

We assume that the body is composed of an isotropic hyperelastic material; that is, there exists a smooth function $g{:}(0,\infty)^3 \to \Re$ such that the stored energy of the body due to the deformation $\mathbf{F}$, denoted by $\hat{\Phi}(\mathbf{F})$, is given by

$$\hat{\Phi}(\mathbf{F}) = g\,(I_C, II_C, III_C) = \Phi\,(\hat{w}(s,z))\,. \tag{6.17}$$

It follows from (6.16) and (6.17) that

$$\begin{aligned}
\Phi\left(r_s, r_z, r/s, \omega_s, \omega_z\right) &= \Phi\left(\omega_z, r_z, r/s, \omega_s, r_s\right) \\
\Phi\left(r_s, r_z, r/s, \omega_s, \omega_z\right) &= \Phi\left(r_s, \omega_s, r/s, r_z, \omega_z\right) \\
\Phi\left(r_s, 0, r/s, 0, \omega_z\right) &= \Phi\left(r/s, 0, r_s, 0, \omega_z\right).
\end{aligned} \tag{6.18}$$

The first Piola-Kirchhoff stress tensor is now given explicitly by

$$\mathbf{T} = \frac{\partial \hat{\Phi}}{\partial \mathbf{F}} = \Phi_{,1}\mathbf{e}_1\mathbf{e}_1 + \Phi_{,2}\mathbf{e}_1\mathbf{k} + \Phi_{,3}\mathbf{e}_2\mathbf{e}_2 + \Phi_{,4}\mathbf{k}\mathbf{e}_1 + \Phi_{,5}\mathbf{k}\mathbf{k}. \tag{6.19}$$

From this equation, it follows that

$$T_{11} = \Phi_{,1} \quad T_{22} = \Phi_{,3} \quad T_{33} = \Phi_{,5} \quad T_{31} = \Phi_{,4} \quad T_{13} = \Phi_{,2} \tag{6.20}$$

which implies that the components of $\mathbf{T}$ in the representation (6.19) are independent of ϕ. To simplify the notation, we define

$$N = T_{11} \quad T = T_{22} \quad U = T_{33} \quad P = T_{31} \quad G = T_{13}. \tag{6.21}$$

Combining (6.17) and (6.20), we can get the explicit representation for the components of stress (6.21) as follows:

$$\begin{aligned}
N &= 2g_{,1}r_s + 2g_{,2}\left(\left(r/s\right)^2 r_s + \left(r_s\omega_z - \omega_s r_z\right)\omega_z\right) \\
&+ 2g_{,3}\left(r/s\right)^2\left(r_s\omega_z - \omega_s r_z\right)\omega_z
\end{aligned} \tag{6.22a}$$

$$\begin{aligned}
G &= 2g_{,1}r_z + 2g_{,2}\left(\left(r/s\right)^2 r_z - \left(r_s\omega_z - \omega_s r_z\right)\omega_s\right) \\
&- 2g_{,3}\left(r/s\right)^2\left(r_s\omega_z - \omega_s r_z\right)\omega_s
\end{aligned} \tag{6.22b}$$

$$\begin{aligned}
T &= 2g_{,1}\left(r/s\right) + 2g_{,2}\left(r/s\right)\left(r_s^2 + r_z^2 + \omega_s^2 + \omega_z^2\right) \\
&+ 2g_{,3}\left(r/s\right)\left(r_s\omega_z - \omega_s r_z\right)^2
\end{aligned} \tag{6.22c}$$

$$P = 2g_{,1}\omega_s + 2g_{,2}\left(\left(r/s\right)^2\omega_s - \left(r_s\omega_z - \omega_s r_z\right) r_z\right)$$

$$- \quad 2g_{,3}\,(r/s)^2\,(r_s\omega_z - \omega_s r_z)\,r_z \tag{6.22d}$$

$$\begin{aligned} U &= 2g_{,1}\omega_z + 2g_{,2}\left((r/s)^2\,\omega_z + (r_s\omega_z - \omega_s r_z)\,r_s\right) \\ &+ 2g_{,3}\,(r/s)^2\,(r_s\omega_z - \omega_s r_z)\,r_s\,. \end{aligned} \tag{6.22e}$$

For simplicity, let us write

$$(\nu\,,\gamma\,,\tau\,,p,\varepsilon) = (r_s\,,r_z\,,r/s,\omega_s\,,\omega_z)\,. \tag{6.23}$$

Hence, from the representations (6.22), which in turn are a consequence of the isotropy condition (6.17), we get the following symmetry properties:

$$N(\nu\,,\gamma\,,\tau\,,p,\varepsilon) = U(\varepsilon,\gamma\,,\tau\,,p,\nu\,) \qquad G(\nu\,,\gamma\,,\tau\,,p,\varepsilon) = G(\varepsilon,\gamma\,,\tau\,,p,\nu\,) \tag{6.24a,b}$$

$$T(\nu\,,\gamma\,,\tau\,,p,\varepsilon) = T(\varepsilon,\gamma\,,\tau\,,p,\nu\,) \qquad P(\nu\,,\gamma\,,\tau\,,p,\varepsilon) = P(\varepsilon,\gamma\,,\tau\,,p,\nu\,) \tag{6.24c,d}$$

$$G(\nu\,,\gamma\,,\tau\,,p,\varepsilon) = P(\nu\,,p,\tau\,,\gamma\,,\varepsilon) \qquad N(\nu\,,\gamma\,,\tau\,,p,\varepsilon) = N(\nu\,,p,\tau\,,\gamma\,,\varepsilon) \tag{6.25a,b}$$

$$T(\nu\,,\gamma\,,\tau\,,p,\varepsilon) = T(\nu\,,p,\tau\,,\gamma\,,\varepsilon) \qquad U(\nu\,,\gamma\,,\tau\,,p,\varepsilon) = U(\nu\,,p,\tau\,,\gamma\,,\varepsilon) \tag{6.25c,d}$$

$$N(\nu\,,0,\tau\,,0,\varepsilon) = T(\tau\,,0,\nu\,,0,\varepsilon) \qquad U(\nu\,,0,\tau\,,0,\varepsilon) = U(\tau\,,0,\nu\,,0,\varepsilon)\,. \tag{6.26a,b}$$

We also get the even-odd conditions (2.6b), (2.7), (2.8), and

$$N(-\nu\,,\gamma\,,\tau\,,p,-\varepsilon) = -N(\nu\,,\gamma\,,\tau\,,p,\varepsilon) \tag{6.27a}$$

$$G(-\nu\,,\gamma\,,\tau\,,p,-\varepsilon) = G(\nu\,,\gamma\,,\tau\,,p,\varepsilon) \tag{6.27b}$$

$$T(-\nu\,,\gamma\,,\tau\,,p,-\varepsilon) = T(\nu\,,\gamma\,,\tau\,,p,\varepsilon) \tag{6.27c}$$

$$P(-\nu\,,\gamma\,,\tau\,,p,-\varepsilon) = P(\nu\,,\gamma\,,\tau\,,p,\varepsilon) \tag{6.27d}$$

$$U(-\nu\,,\gamma\,,\tau\,,p,-\varepsilon) \;=\; -U(\nu\,,\gamma\,,\tau\,,p,\varepsilon)\,. \tag{6.27e}$$

The strong ellipticity condition from three-dimensional elasticity requires that

$$\mathbf{ab} : \frac{\partial \mathbf{T}}{\partial \mathbf{F}} : \mathbf{ab} > 0 \quad \forall \mathbf{ab} \neq \mathbf{00}. \tag{6.28}$$

From (6.19), (6.20), and (6.21), it follows that (6.28) implies (2.9) and (2.10). With the definitions (6.21), we get the boundary value problem (6.8)-(6.9), which can be written as (2.12)-(2.15).

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