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The Hanging Rope of Minimum Elongation for a Nonlinear Stress–Strain Relation

PABLO V. NEGRÓN-MARRERO
Department of Mathematics, University of Puerto Rico, Humacao, PR 00791-4300, U.S.A.
E-mail: pnm@www.uprh.edu

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Abstract. We consider the problem of determining the shape that minimizes the elongation of a rope that hangs vertically under its own weight and an applied force, subject to either a constraint of fixed total mass or fixed total volume. The constitutive function for the rope is given by a nonlinear stress–strain relation and the mass–density function of the rope can be variable. For the case of fixed total mass we show that the problem can be explicitly solved in terms of the mass density function, applied force, and constitutive function. In the special case where the mass–density function is constant, we show that the optimal cross-sectional area of the rope is as that for a linear stress–strain relation (Hooke's Law). For the total fixed volume problem, we use the implicit function theorem to show the existence of a branch of solutions depending on the parameter representing the acceleration of gravity. This local branch of solutions is extended globally using degree theoretic techniques.

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Key words: string, mass–density, nonlinear stress–strain relation, implicit function theorem, compact map.

The contributions of Clifford Truesdell to the development of the field of rational mechanics were vast. I met Professor Truesdell only briefly at a meeting on Theoretical Mechanics at Rutgers University in 1990. However, I am in debt to him for the legacy of his teachings in rational mechanics. The following paper is dedicated to his memory.

1. Introduction

The problem of the motion of a string under different types of boundary conditions and forces dates back to Euler [7] and Lagrange [14]. The problem we consider here is a variation of the *catenary* problem first proposed by Leonardo da Vinci. In particular, we consider the problem of a rope or string attached at one end, hanging vertically under its own weight, and subject to an applied force at the hanging end. (See Figure 1.) Instead of specifying a shape for the rope and determining its deformation, we consider the *optimal control* problem of determining the shape of

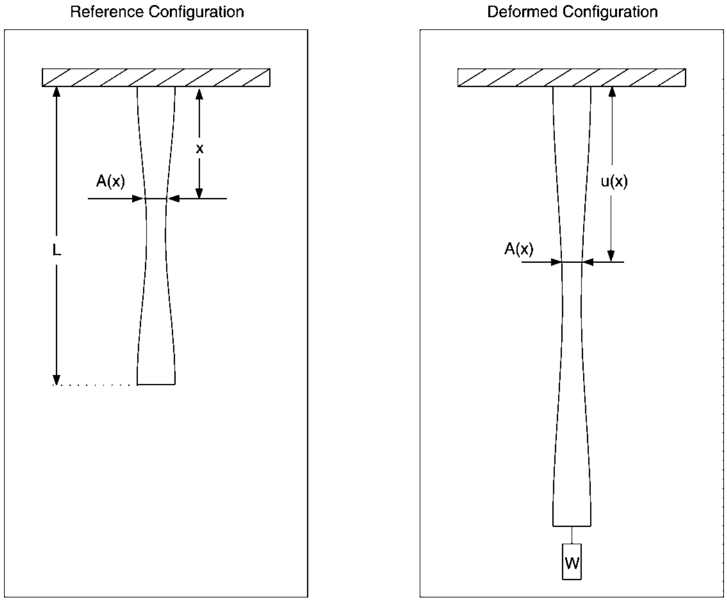

Figure 1. Geometry of deformation. In the reference configuration there is no gravity and we either have the total volume fixed to V or the total mass fixed to M.

the rope (given by its cross-sectional area function) that minimizes its elongation under the effect of its own weight and applied force, subject to either a constraint of fixed volume or fixed mass. This problem was treated in [20] for a linear stress–strain relation (Hooke's Law) and constant mass–density function. We generalize the results in [20] to materials that satisfy a general nonlinear stress–strain relation and with a variable mass–density function. The model for the string that we used is based on those in [3], to which we also refer for a detailed historical account of problems for strings.

A related problem to the one treated here is that, instead of hanging, the rope is now upside down and is thought of as a column. The question is, subject to the constraint of fixed volume: what shape should the column have in order to maximize its strength? This problem is equivalent to maximizing the first buckling mode of the system. We refer to [13, 5, 6] for further details on this problem.

In Section 2 we derive the equilibrium equations of the string and describe the constitutive assumptions on the material behavior. In Section 3 we characterize the problem of minimum elongation as one of the calculus of variations. For the problem of fixed total mass, we show in Section 4 that the corresponding Euler–Lagrange equations can be solved explicitly in terms of the mass–density function, applied force, and constitutive function (cf. (4.10)), and that this solution corresponds to a global weak minimizer of the total elongation functional. In the particular case where the mass–density function is constant, we find the surprising result that the corresponding cross-sectional area function is identical to the one found in [20] for the case of a linear stress–strain relation. The corresponding minimum elongation of the string, however, depends on the material response (cf. (4.15)).

In Section 5 we carry out the analysis for the constraint of fixed total volume. Since the problem is formulated as one for a functional in terms of a pseudo-antiderivative of the cross-sectional area function (cf. (3.3)), and the mass–density function may be variable, the volume constraint becomes an integral constraint. Thus we have an isoperimetric problem of the calculus of variations and the Euler–Lagrange equations involve an additional term proportional to a Lagrange multiplier. The Euler–Lagrange equations for this problem are then formulated as a smooth mapping between appropriate Banach spaces of smooth functions, using the gravitational parameter for continuation. We first study the problem with zero gravity and show that it has a solution which is unique and corresponds to a constant cross-sectional area function. We then show that the implicit function theorem is applicable to the problem with nonzero gravity to get the existence of a local curve of solutions. This result includes the case of small but negative gravitational parameter, which can be interpreted as the case of a gravitational field vertically upward or as if the rope would hang upside down. Finally we apply the classical Leray–Schauder degree for compact maps [16, 4, 2] to extend globally the local branch found via the Implicit Function Theorem. The uniqueness of solutions for the problem with zero gravity together with some a priori estimates on the solutions allow us to rule out two of the Rabinowitz alternatives for the global continuum. In addition we get that the problem has a solution for each nonnegative value of the gravitational parameter. We then show that for fixed values of the gravitational parameter, the solution obtained is a weak local minimizer of a modified version of the original variational formulation (cf. (5.25), (5.26)).

For any $m \geqslant 0$ we consider the spaces $C^m[0, L]$ consisting of functions v with m continuous derivatives in $[0, L]$ and with norm given by

$$\|v\|_{C^m[0,L]} = \sum_{k=0}^{m} \max_{0\leqslant x\leqslant L} \left|v^{(k)}(x)\right|.$$

We observe that $C^m[0, L]$ is a Banach space and for $m \geqslant 1$ it is compactly embedded into $C^{m-1}[0, L]$. That is, if $\{v_k\}$ is a bounded sequence in $C^m[0, L]$, then by the Arzelá–Ascoli's Theorem, $\{v_k\}$ has a subsequence that converges in $C^{m-1}[0, L]$.

2. The Equations of Equilibrium

2.1. GEOMETRY OF DEFORMATION

We consider a rope or string which in its reference configuration occupies the region Ω in $\mathbb{R}^3$. We let (x, y, z) represent Cartesian coordinates in Ω and assume that $[0, L] = \{x\colon (x, y, z) \in \Omega\}$ where the positive x axis is downward in the vertical direction. For any $x \in [0, L]$ we define the *cross-section* of Ω at x by

$$\Omega_x = \{(y, z)\colon (x, y, z) \in \Omega\}, \tag{2.1}$$

and let $A(x)$ be the area of Ω_x. We assume that the *cross-sectional area* function $A(\cdot)$ is positive and continuous on $[0, L]$ (see Figure 1). We consider a *one-dimensional* deformation of Ω given by

$$\mathbf{p}(x, y, z) = (u(x), y, z), \tag{2.2}$$

for some C^1 function $u(\cdot)$. (See Figure 1.) The requirement that an (infinitesimal) volume in the reference configuration cannot be reduced to zero by the deformation $\mathbf{p}$, implies that

$$u'(x) > 0, \quad \forall x \in [0, L]. \tag{2.3}$$

2.2. MECHANICAL RESPONSE

For any $x \in [0, L]$ we denote by $n(x)$ the force exerted by the material on $[0, x]$ on that on $[x, L]$ in a deformed configuration. We assume that the material of the rope has mass density per unit volume at x given by $\rho(x)$, where $\rho(\cdot)$ is a given positive continuously differentiable function. Hence the weight of the $[x, L]$ section of the rope is given by:

$$g \int_x^L \rho(\bar{x}) A(\bar{x})\, \mathrm{d}\bar{x}, \tag{2.4}$$

where g denotes the acceleration of gravity. Assuming that a force W is applied at $x = L$, the total force exerted on the section $[x, L]$ is

$$W + g \int_x^L \rho(\bar{x}) A(\bar{x})\, \mathrm{d}\bar{x}. \tag{2.5}$$

For equilibrium, the forces must balance at each $x \in [0, L]$, i.e.,

$$n(x) = W + g \int_x^L \rho(\bar{x}) A(\bar{x})\, \mathrm{d}\bar{x}. \tag{2.6}$$

We say that the material of the rope is *elastic and nonhomogeneous* if for some function $\widetilde{N}(\cdot, \cdot)$ we have that

$$n(x) = \widetilde{N}(u'(x), x). \tag{2.7}$$

The usual way to account for the lack of homogeneity is by taking

$$\widetilde{N}(u'(x), x) = A(x)\widehat{N}(u'(x)), \tag{2.8}$$

where $\widehat{N}\colon (0, \infty) \to \mathbb{R}$ satisfies:

A1. $\widehat{N}(\cdot)$ is a strictly increasing smooth function;
A2. $\widehat{N}(\nu) \to \infty$ as $\nu \to \infty$;
A3. $\widehat{N}(\nu) \to -\infty$ as $\nu \to 0^+$.

From properties A1–A3 it follows that $\widehat{N}$: $(0, \infty) \to \mathbb{R}$ has a smooth inverse $\hat{\nu}$: $\mathbb{R} \to (0, \infty)$. We further assume that

A4. $N \mapsto N^2\hat{\nu}_N(N)$ is strictly increasing on $[0, \infty)$;
A5. $N^2\hat{\nu}_N(N) \to \infty$ as $N \to \infty$.

One can easily check that condition A4 is equivalent to the strict convexity of the integrand in the functional giving the total elongation of the rope (cf. (3.4)).

If we combine (2.6), (2.7), and (2.8) we get that

$$\widehat{N}(u'(x)) = A(x)^{-1}\left[W + g\int_x^L \rho(\bar{x})A(\bar{x})\,\mathrm{d}\bar{x}\right]. \tag{2.9}$$

(See the Appendix for a derivation of this equation from the three-dimensional theory of elasticity.) Since the top of the rope is attached to a wall we have that

$$u(0) = 0. \tag{2.10}$$

We consider two types of additional constraints: we assume either that the total mass of the rope is a given constant M:

$$\int_0^L \rho(x)A(x)\,\mathrm{d}x = M, \tag{2.11}$$

or that the volume of the rope is a given constant V:

$$\int_0^L A(x)\,\mathrm{d}x = V. \tag{2.12}$$

For a constant mass–density function $\rho(\cdot)$ both constraints are equivalent.

3. Rope of Minimum Elongation

Note that we can write (2.9) as:

$$u'(x) = \hat{\nu}\left(A(x)^{-1}\left[W + g\int_x^L \rho(\bar{x})A(\bar{x})\,\mathrm{d}\bar{x}\right]\right). \tag{3.1}$$

Integrating now over $[0, L]$ and using (2.10), we get the following expression for the total elongation of the rope:

$$u(L) = \int_0^L \hat{\nu}\left(A(x)^{-1}\left[W + g\int_x^L \rho(\bar{x})A(\bar{x})\,\mathrm{d}\bar{x}\right]\right)\mathrm{d}x. \tag{3.2}$$

The problem then is to find a function $A(\cdot)$ that minimizes the above expression for $u(L)$ subject to either constraint (2.11) or (2.12).

Let

$$B(x) = \int_x^L \rho(\bar{x})A(\bar{x})\,\mathrm{d}\bar{x}. \tag{3.3}$$

Hence $B'(x) = -\rho(x)A(x)$ and we can write (3.2) as

$$u(L) = \int_0^L \hat{\nu}\left(-\frac{\rho(x)(W + gB(x))}{B'(x)}\right)\mathrm{d}x. \tag{3.4}$$

Note that condition A4 can be seen now to be equivalent to the strict convexity with respect to B' of the integrand in the above functional. Note that $B(L) = 0$ and either

$$B(0) = M, \tag{3.5}$$

if (2.11) holds, or

$$\int_0^L \frac{B'(x)}{\rho(x)}\,\mathrm{d}x = -V, \tag{3.6}$$

if (2.12) holds. Thus our problem now is to find a function $B(\cdot)$ that minimizes (3.4) subject to $B(L) = 0$ and either of the two constraints (3.5) or (3.6).

4. Fixed Total Mass

We now study the problem of minimizing (3.4) subject to (3.5) for an arbitrary mass density function $\rho(\cdot)$. More specifically, we study the problem

$$\min_{B\in\mathcal{X}} J(B), \tag{4.1}$$

where

$$J(B) = \int_0^L \hat{\nu}\left(-\frac{\rho(x)(W + gB(x))}{B'(x)}\right)\mathrm{d}x, \tag{4.2}$$

$$\mathcal{X} = \left\{B \in C^1[0, L]\colon\ B(0) = M,\ B(L) = 0,\ B'(x) < 0\ \forall x\right\}. \tag{4.3}$$

The Euler–Lagrange equations for this functional are given by:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\rho(x)(W + gB(x))}{B'(x)^2}\,\hat{\nu}_N\left(-\frac{\rho(x)(W + gB(x))}{B'(x)}\right)\right] + \frac{g\rho(x)}{B'(x)}\,\hat{\nu}_N\left(-\frac{\rho(x)(W + gB(x))}{B'(x)}\right) = 0, \quad 0 < x < L, \tag{4.4a}$$

$$B(0) = M, \qquad B(L) = 0. \tag{4.4b}$$

If we let

$$H(x) = -\frac{\rho(x)(W + gB(x))}{B'(x)}, \tag{4.5}$$

then a simple computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[H(x)^2\hat{\nu}_N(H(x))\big] = \rho(x)(W+gB(x))\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{H(x)}{B'(x)}\,\hat{\nu}_N(H(x))\right] + \left[\frac{\rho'(x)}{\rho(x)}\,H(x) - g\rho(x)\right]H(x)\hat{\nu}_N(H(x)).$$

If we multiply (4.4a) by $\rho(x)(W+gB(x))$, recall (4.5), and use the above identity, then we have that (4.4a) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[H(x)^2\hat{\nu}_N(H(x))\big] - \frac{\rho'(x)}{\rho(x)}\,H(x)^2\hat{\nu}_N(H(x)) = 0. \tag{4.6}$$

This equation can be easily integrated now to get that

$$H(x)^2\hat{\nu}_N(H(x)) = c\rho(x), \tag{4.7}$$

for some constant c. The left-hand side of this equation can be written as $h(H(x))$ where $h(N) = N^2\hat{\nu}_N(N)$. Thus (4.7) is equivalent to

$$\frac{W+gB(x)}{B'(x)} = -\frac{1}{\rho(x)}h^{-1}(c\rho(x)), \tag{4.8}$$

where h^{-1} is the inverse function of h which exists under hypotheses A4, A5. Equation (4.8) can be written as

$$B'(x) + \frac{g\rho(x)}{h^{-1}(c\rho(x))}\,B(x) = -\frac{W\rho(x)}{h^{-1}(c\rho(x))}.$$

Using the integrating factor

$$\mu(x) = \exp\left[g\int_0^x \frac{\rho(t)}{h^{-1}(c\rho(t))}\,\mathrm{d}t\right], \tag{4.9}$$

together with the boundary condition $B(L) = 0$, we conclude that

$$B(x) = \frac{W}{g}\left[\exp\left[g\int_x^L \frac{\rho(t)}{h^{-1}(c\rho(t))}\,\mathrm{d}t\right] - 1\right]. \tag{4.10}$$

It remains to determine the constant c. But using (4.10) we get that the boundary condition $B(0) = M$ is equivalent to

$$G(c) \equiv \frac{W}{g}\left[\exp\left[g\int_0^L \frac{\rho(t)}{h^{-1}(c\rho(t))}\,\mathrm{d}t\right] - 1\right] = M. \tag{4.11}$$

It follows from hypotheses A4, A5 that $h^{-1}(0) = 0$, h^{-1} is strictly increasing, and that $h^{-1}(s) \to \infty$ as $s \to \infty$. From these properties of h^{-1} it follows that G is strictly decreasing, $G(c) \to 0$ as $c \to \infty$, and $G(c) \to \infty$ as $c \to 0^+$. Thus equation (4.11) has a solution which is unique for each $M > 0$. It follows as well that (4.4) has a unique solution for each $M > 0$.

The above argument can be easily modified to show that the problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\rho(x)(W+g\Psi(x))}{\Psi'(x)^2}\,\hat{\nu}_N\left(-\frac{\rho(x)(W+g\Psi(x))}{\Psi'(x)}\right)\right] + \frac{g\rho(x)}{\Psi'(x)}\,\hat{\nu}_N\left(-\frac{\rho(x)(W+g\Psi(x))}{\Psi'(x)}\right) = 0, \quad a < x < L, \tag{4.12a}$$

$$\Psi(a) = B(a), \qquad \Psi(L) = 0, \tag{4.12b}$$

has a unique solution $\Psi(\,\cdot\,; a)$ for any $a \in [0, L)$, where B is the unique solution of (4.4). This fact can be used now to construct a stationary field for the functional (4.2). This together with condition A4 which implies the strict convexity with respect to B' of the integrand in (4.2), allow us to invoke Hilbert's Invariant Integral Theorem [18, Theorem 9.7] to get that the solution B of (4.4) is the unique (weak) minimizer of (4.2) on (4.3).

4.1. UNIFORM MASS DENSITY

In the case where ρ is constant we can determine explicitly the constant c in (4.11). In this case the integrand in (4.11) is a constant which we denote by K. Equation (4.11) now reduces to

$$\frac{W}{g}\left[\mathrm{e}^{gKL} - 1\right] = M,$$

which has solution $K = (1/Lg)\ln(1 + gM/W)$ that upon substitution into (4.10) yields

$$B(x) = \frac{W}{g}\left[\left(1 + \frac{gM}{W}\right)^{1-x/L} - 1\right]. \tag{4.13}$$

Since $A(x) = -B'(x)/\rho$, we get after simplification that

$$A(x) = \frac{W}{g\rho L}\ln\left(1 + \frac{gM}{W}\right)\cdot\left(1 + \frac{gM}{W}\right)^{1-(x/L)}. \tag{4.14}$$

Note that this function is decreasing, i.e., the minimum elongation is attained by tapering down the rope from top to bottom.* This is the same result obtained in [20] for the case $\hat{\nu}(\cdot)$ linear (Hooke's Law). If we substitute (4.13) into (3.4) we get that the total elongation of the rope is given by

$$u(L) = L\hat{\nu}\left(-\frac{\rho}{K}\right) = L\hat{\nu}\left(\frac{g\rho L}{\ln(1 + gM/W)}\right). \tag{4.15}$$

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* In the general case where ρ depends on x, the cross-sectional area function need not be monotone.

5. Fixed Total Volume

We now consider the problem of minimizing (3.4) subject to (3.6) for an arbitrary mass density function $\rho(\cdot)$. This problem can be formulated as

$$\min_{\mathcal{X}_V} J(B), \tag{5.1}$$

where J is like (4.2) and

$$\mathcal{X}_V = \left\{ B \in C^2[0, L]: \int_0^L \frac{B'(x)}{\rho(x)}\, \mathrm{d}x = -V, \right.$$
$$\left. B(L) = 0,\ B'(x) < 0,\ \forall x \right\}. \tag{5.2}$$

The first order necessary conditions for this problem are obtained by considering the extended functional

$$\widehat{J}(\lambda, B) = \int_0^L \left[\hat{\nu}\left(-\frac{\rho(x)(W + gB(x))}{B'(x)} \right) + \lambda \left(\frac{B'(x)}{\rho(x)} + \frac{V}{L} \right) \right] \mathrm{d}x, \tag{5.3}$$

over the set

$$\widehat{\mathcal{X}} = \left\{ (\lambda, B) \in \mathbb{R} \times C^2[0, L]:\ B(L) = 0,\ B'(x) < 0,\ \forall x \right\}, \tag{5.4}$$

and where λ is a *Lagrange multiplier*. By considering smooth variations w with $w(L) = 0$ one gets that the Euler–Lagrange equations for (5.3) and hence of (5.1) are given by:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\rho(x)(W + gB(x))}{B'(x)^2}\, \hat{\nu}_N\left(-\frac{\rho(x)(W + gB(x))}{B'(x)} \right) + \frac{\lambda}{\rho(x)} \right]$$
$$+ \frac{g\rho(x)}{B'(x)}\, \hat{\nu}_N\left(-\frac{\rho(x)(W + gB(x))}{B'(x)} \right) = 0, \quad 0 < x < L, \tag{5.5a}$$

$$\left[\frac{\rho(x)(W + gB(x))}{B'(x)^2}\, \hat{\nu}_N\left(-\frac{\rho(x)(W + gB(x))}{B'(x)} \right) + \frac{\lambda}{\rho(x)} \right]_{x=0} = 0, \tag{5.5b}$$

$$\int_0^L \frac{B'(x)}{\rho(x)}\, \mathrm{d}x = -V, \qquad B(L) = 0. \tag{5.5c}$$

Note that the multiplier λ is determined from the boundary condition (5.5b) and in fact must be negative. Since in this case it is not possible to obtain explicit solutions, we study the boundary value problem (5.5) using g as a continuation parameter.

Let $\mathcal{Y} = \mathbb{R}^2 \times C^2[0, L]$, $\mathcal{Z} = C^0[0, L] \times \mathbb{R}^2$, and

$$\mathcal{U} = \left\{ (g, \lambda, B) \in \mathcal{Y}:\ \lambda < 0,\ B(L) = 0,\ B'(x) < 0\ \forall x \right\}. \tag{5.6}$$

Equations (5.5) are now equivalent to $G(g, \lambda, B) = 0$ where $G\colon \mathcal{U} \to \mathcal{Z}$ is given by

$$G(g, \lambda, B) = \left(G_1(g, \lambda, B), G_2(g, \lambda, B), \int_0^L \frac{B'(x)}{\rho(x)}\, \mathrm{d}x + V \right), \tag{5.7}$$

where $G_1(g, \lambda, B)$ and $G_2(g, \lambda, B)$ are given by the left-hand sides of (5.5a) and (5.5b), respectively. A simplified version of the results in [19], which are for Schauder spaces, gives us that G_1, G_2 are twice continuously Fréchet differentiable, and since the other component of G is a twice differentiable linear functional of B, we conclude that:

LEMMA 5.1. *The function $G\colon \mathcal{U} \to \mathcal{Z}$ is twice continuously Fréchet differentiable and*

$$\begin{aligned}&\mathrm{D}_{(\lambda,B)}G(g,\lambda,B)\cdot(\gamma,v)\\&\quad=\left(\mathrm{D}_{(\lambda,B)}G_1(g,\lambda,B)\cdot(\gamma,v),\ \mathrm{D}_{(\lambda,B)}G_2(g,\lambda,B)\cdot(\gamma,v),\ \int_0^L\frac{v'(x)}{\rho(x)}\,\mathrm{d}x\right),\end{aligned}$$

where $\mathrm{D}_{(\lambda,B)}G_1(g,\lambda,B)\cdot(\gamma,v)$ *is given by*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\left[\rho(x)\left(\frac{g}{B'(x)^2}\,v(x)-2\frac{W+gB(x)}{B'(x)^3}\,v'(x)\right)\hat{\nu}_N\right.\\&\qquad\left.-\,\rho(x)^2\frac{W+gB(x)}{B'(x)^2}\left(\frac{g}{B'(x)}\,v(x)-\frac{W+gB(x)}{B'(x)^2}\,v'(x)\right)\hat{\nu}_{NN}+\frac{\gamma}{\rho(x)}\right]\\&\qquad-\frac{g\rho(x)\hat{\nu}_N}{B'(x)^2}\,v'(x)-\frac{g\rho(x)^2}{B'(x)}\left(\frac{g}{B'(x)}\,v(x)-\frac{W+gB(x)}{B'(x)^2}\,v'(x)\right)\hat{\nu}_{NN},\end{aligned}$$

and $\mathrm{D}_{(\lambda,B)}G_2(g,\lambda,B)\cdot(\gamma,v)$ *is given by*

$$\begin{aligned}&\left[\rho(x)\left(\frac{g}{B'(x)^2}\,v(x)-2\frac{W+gB(x)}{B'(x)^3}\,v'(x)\right)\hat{\nu}_N\right.\\&\qquad\left.-\,\rho(x)^2\frac{W+gB(x)}{B'(x)^2}\left(\frac{g}{B'(x)}\,v(x)-\frac{W+gB(x)}{B'(x)^2}\,v'(x)\right)\hat{\nu}_{NN}+\frac{\gamma}{\rho(x)}\right]_{x=0}\end{aligned}$$

and where the argument of $\hat{\nu}_N$ *and* $\hat{\nu}_{NN}$ *is* $-\rho(x)(W+gB(x))/B'(x)$.

5.1. LOCAL CONTINUATION

We now study the existence of solutions of (5.5) for small values of g.

LEMMA 5.2. *The equation $G(0, \lambda, B) = 0$ has a solution which is unique and for which the corresponding cross-sectional area function A is constant.*

Proof. If we set $g = 0$ in (5.5a) then we get that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\rho(x)W}{B'(x)^2}\,\hat{\nu}_N\left(-\frac{\rho(x)W}{B'(x)}\right)+\frac{\lambda}{\rho(x)}\right]=0,$$

i.e.,

$$\frac{\rho(x)W}{B'(x)^2}\,\hat{\nu}_N\left(-\frac{\rho(x)W}{B'(x)}\right)+\frac{\lambda}{\rho(x)}=\text{constant}.$$

The boundary condition (5.5b) with $g = 0$ implies that this "constant" must be equal to zero from which we conclude that

$$\left(\frac{\rho(x)W}{B'(x)}\right)^2 \hat{\nu}_N\left(-\frac{\rho(x)W}{B'(x)}\right) = -\lambda W.$$

(Note that (5.5b) implies that $\lambda < 0$.) Since the right-hand side of this equation is constant, it follows from hypothesis A4 that

$$-\frac{\rho(x)W}{B'(x)} = C,$$

for some positive constant C. (The volume constraint in (5.5c) implies that $C = WL/V$.) Thus $B'(x) = -\rho(x)W/C$ and since $B'(x) = -\rho(x)A(x)$ we get that $A(x) = W/C$, i.e., A is constant. □

Let (λ_0, B_0) be the solution pair of $G(0, \lambda, B) = 0$ given by the above lemma. We now have:

LEMMA 5.3. *The linear map* $\mathrm{D}_{(\lambda,B)}G(0, \lambda_0, B_0)$ *is a bijection from* $\mathbb{R} \times C^2[0, L]$ *into* $\mathcal{Z}$.

Proof. It follows from Lemma 5.1 that given any $(f, \alpha, \eta) \in \mathcal{Z}$, the equation

$$\mathrm{D}_{(\lambda,B)}G(0, \lambda_0, B_0) \cdot (\gamma, v) = (f, \alpha, \eta),$$

is equivalent to:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{B_0'(x)^2}\frac{\mathrm{d}}{\mathrm{d}N}\big(N^2\hat{\nu}_N(N)\big)\bigg|_{N=H} v'(x) + \frac{\gamma}{\rho(x)}\right] = f(x), \tag{5.8a}$$

$$\left[\frac{1}{B_0'(x)^2}\frac{\mathrm{d}}{\mathrm{d}N}\big(N^2\hat{\nu}_N(N)\big)\bigg|_{N=H} v'(x) + \frac{\gamma}{\rho(x)}\right]_{x=0} = \alpha, \tag{5.8b}$$

$$v(L) = 0, \qquad \int_0^L \frac{v'(x)}{\rho(x)}\,\mathrm{d}x = \eta, \tag{5.8c}$$

where $H = -\rho(x)W/B_0'(x)$, etc. Since the coefficient of $v'(x)$ in (5.8a) is positive by hypothesis A4, problem (5.8a), (5.8b), and the first equation in (5.8c) can be uniquely solved for v in terms of f, α, γ, where the dependence in γ is linear. (See [17].) Upon substitution of this expression for v into the second equation of (5.8c), we get a linear equation for γ, which can be uniquely solved. □

It follows now from Lemmas 5.1–5.3, and the implicit function theorem (see [15]) that:

THEOREM 5.4. *For small values of g, the problem* (5.5) *has a solution that depends continuously on g.*

5.2. GLOBAL CONTINUATION

In this section we carry a global analysis of solutions of (5.5) via Leray–Schauder degree theory. In order to apply the global continuation results in [16, 4, 2], we need to recast our problem in terms of a compact operator between appropriate Banach spaces. (It turns out that assumption A4 is crucial in this respect.) The local analysis of Section 5.1 is still valid in this setting and thus we just carry out the additional steps for the global analysis.

By an analysis similar to the one that leads to (4.6), we can get that (5.5a) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[H(x)^2\hat{\nu}_N(H(x))\big] - \frac{\rho'(x)}{\rho(x)}\,H(x)^2\hat{\nu}_N(H(x)) = \lambda\frac{\rho'(x)}{\rho(x)}\,(W + gB(x)),$$

which in turn is equivalent to:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{H(x)^2\hat{\nu}_N(H(x))}{\rho(x)}\right] = \lambda\,\frac{\rho'(x)}{\rho(x)^2}\,(W + gB(x)).$$

If we integrate this equation from 0 to x we get that

$$\frac{H(x)^2\hat{\nu}_N(H(x))}{\rho(x)} - \frac{H(0)^2\hat{\nu}_N(H(0))}{\rho(0)} = \lambda\int_0^x \frac{\rho'(t)}{\rho(t)^2}\,(W + gB(t))\,\mathrm{d}t. \tag{5.9}$$

A simple integration by parts shows that

$$\int_0^x \frac{\rho'(t)}{\rho(t)^2}\,(W + gB(t))\,\mathrm{d}t = \frac{W + gB(0)}{\rho(0)} - \frac{W + gB(x)}{\rho(x)} + g\int_0^x \frac{B'(t)}{\rho(t)}\,\mathrm{d}t.$$

Also from (5.5b) we have that

$$-H(0)^2\hat{\nu}_N(H(0)) = \lambda(W + gB(0)).$$

Using these two last identities we can conclude that (5.9) is equivalent to

$$\frac{H(x)^2\hat{\nu}_N(H(x))}{\rho(x)} = -\lambda\left[\frac{W + gB(x)}{\rho(x)} - g\int_0^x \frac{B'(t)}{\rho(t)}\,\mathrm{d}t\right]. \tag{5.10}$$

Since $\hat{\nu}_N > 0$, the boundary condition (5.5b) implies that $\lambda < 0$. Furthermore, since $B'(x) < 0$, it follows that the right-hand side of the above equation is positive. Let $h(N) = N^2\hat{\nu}_N(N)$. By hypothesis A4, this function for $N \geqslant 0$ has an inverse function $h^{-1}(\cdot)$. Thus after multiplying both sides by $\rho(x)$, the above equation is equivalent to

$$\frac{W + gB(x)}{B'(x)} = F(g, \lambda, B)(x), \tag{5.11}$$

where

$$F(g, \lambda, B)(x) = -\frac{1}{\rho(x)}\,h^{-1}\left(-\lambda\left[W + gB(x) - g\rho(x)\int_0^x \frac{B'(t)}{\rho(t)}\,\mathrm{d}t\right]\right). \tag{5.12}$$

Since $F(g, \lambda, B)(x) < 0$ for all x, we can write (5.11) as

$$B'(x) - \frac{g}{F(g, \lambda, B)(x)} B(x) = \frac{W}{F(g, \lambda, B)(x)}. \tag{5.13}$$

If we treat the coefficient and right-hand side in this equation as if they were known functions of x, then after using an appropriate integrating factor and the boundary condition $B(L) = 0$ we can write that

$$B = K_2(g, \lambda, B), \tag{5.14}$$

where

$$K_2(g, \lambda, B)(x) = \frac{W}{g}\left[\exp\left(-g\int_x^L \frac{\mathrm{d}t}{F(g, \lambda, B)(t)}\right) - 1\right]. \tag{5.15}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}x} K_2(g, \lambda, B)(x) = \frac{W}{F(g, \lambda, B)(x)} \exp\left(-g\int_x^L \frac{\mathrm{d}t}{F(g, \lambda, B)(t)}\right). \tag{5.16}$$

With this expression for B', the volume constraint in (5.5c) becomes

$$\lambda = K_1(g, \lambda, B), \tag{5.17}$$

where

$$K_1(g, \lambda, B) = V + \lambda + \int_0^L \frac{W}{\rho(x) F(g, \lambda, B)(x)} \times \exp\left(-g\int_x^L \frac{\mathrm{d}t}{F(g, \lambda, B)(t)}\right) \mathrm{d}x. \tag{5.18}$$

If we let $K = (K_1, K_2)$, then (5.5) is equivalent to

$$(\lambda, B) = K(g, \lambda, B). \tag{5.19}$$

Note that $K\colon \mathcal{E} \to \mathbb{R} \times C^1[0, L]$, where

$$\mathcal{E} = \{(g, \lambda, B) \in [0, \infty) \times (-\infty, 0) \times C^1[0, L]\colon B'(x) < 0\ \forall x\}. \tag{5.20}$$

The operator K need not be compact on the whole of $\mathcal{E}$ as the condition $B'(x) < 0$ and $\lambda < 0$ might be violated in the limit for some converging sequence. To deal with this possibility, we define for any $\delta > 0$ the open set

$$\mathcal{E}_\delta = \{(g, \lambda, B) \in \mathcal{E}\colon \lambda < -\delta,\ B'(x) < -\delta\ \forall x\}. \tag{5.21}$$

Note that $\mathcal{E} = \cup_{\delta>0}\mathcal{E}_\delta$. We now have:

LEMMA 5.5. *For each* $\delta > 0$ *and* $g \in [0, \infty)$*, the operator* $K(g, \cdot, \cdot)$ *that maps* $\{(\lambda, B)\colon (g, \lambda, B) \in \mathcal{E}_\delta\}$ *into* $\mathbb{R} \times C^1[0, L]$ *is compact.*

Proof. Since the mapping $f \mapsto \int_0^x f(t)\,\mathrm{d}t$ is compact from $C[0, L]$ into itself, it follows from (5.12) that $F(g, \cdot, \cdot)$ is compact from

$$\mathcal{E}_{\delta,g} = \{(\lambda, B)\colon\ (g, \lambda, B) \in \mathcal{E}_\delta\},$$

into $C[0, L]$. Furthermore, from (5.15), (5.16), and (5.18), we get that $K(g, \cdot, \cdot)$ is the composition of a continuous operator from $C[0, L]$ into $\mathbb{R} \times C^1[0, L]$ with the compact operator $F(g, \cdot, \cdot)$. Thus $K(g, \cdot, \cdot)$ is compact from $\mathcal{E}_{\delta,g}$ into $C^1[0, L]$. □

We need a few preliminary lemmas before invoking the global continuation results in [16, 4, 2].

LEMMA 5.6. *Let* $\{(g_j, \lambda_j, B_j)\}$ *be a sequence of solutions of* (5.19) *that converges to* (g, λ, B) *in* $\mathbb{R}^2 \times C^1[0, L]$ *with* $g \geqslant 0$. *Then* $B'(x) < 0$ *for all* x *and* $\lambda < 0$.

Proof. Since $B_j' < 0$ and $\lambda_j < 0$ for all j, it follows that $B' \leqslant 0$ and $\lambda \leqslant 0$. Assume that $B'(\bar{x}) = 0$ for some $\bar{x} \in [0, L]$. Then since $B_j'(\bar{x}) < 0$ for all j, we get that

$$\frac{W + g_j B_j(\bar{x})}{B_j'(\bar{x})} \to -\infty, \quad j \to \infty.$$

But from (5.11) and (5.12) we observe that

$$\frac{W + g_j B_j(\bar{x})}{B_j'(\bar{x})} \to -\frac{1}{\rho(\bar{x})} h^{-1}\left(-\lambda\left[W + gB(\bar{x}) - g\rho(\bar{x}) \int_0^{\bar{x}} \frac{B'(t)}{\rho(t)}\,\mathrm{d}t\right]\right),$$

which is finite and thus we get a contradiction. Hence $B'(x) < 0$ for all $x \in [0, L]$.

To argue that $\lambda < 0$, note that

$$\frac{W + g_j B_j(L)}{B_j'(L)} = \frac{W}{B_j'(L)} \to \frac{W}{B'(L)} < 0.$$

But if $\lambda = 0$, then (5.11) and (5.12) imply that

$$\frac{W + g_j B_j(L)}{B_j'(L)} \to -\frac{1}{\rho(L)} h^{-1}(0) = 0,$$

which again leads to a contradiction. □

LEMMA 5.7. *For any solution* (g, λ, B) *of* (5.19) *with* $g \geqslant 0$, *we have that*

$$\|B\|_{C[0,L]} \leqslant K, \tag{5.22a}$$

$$\|B'\|_{C[0,L]} \leqslant \frac{W\|\rho\|_{C[0,L]}}{h^{-1}(-\lambda W)} \exp\left(\frac{g}{h^{-1}(-\lambda W)} \int_0^L \rho(t)\,\mathrm{d}t\right), \tag{5.22b}$$

for some constant K *depending only on* ρ *and* V.

Proof. Since $B(L) = 0$, we have using an integration by parts that

$$B(x) = -\int_x^L B'(t)\,\mathrm{d}t = -\int_x^L \frac{B'(t)}{\rho(t)}\rho(t)\,\mathrm{d}t$$
$$= -\rho(x)\int_x^L \frac{B'(t)}{\rho(t)}\,\mathrm{d}t - \int_x^L \rho'(t)\int_x^L \frac{B'(\xi)}{\rho(\xi)}\,\mathrm{d}\xi\,\mathrm{d}t.$$

The volume constraint in (5.5c) and the fact that $B' < 0$ imply that

$$0 \leqslant -\int_x^L \frac{B'(t)}{\rho(t)}\,\mathrm{d}t \leqslant V. \tag{5.23}$$

This inequality together with the above expression for B gives the result (5.22a).

To get (5.22b), note that since $\lambda < 0$, $B' < 0$, and $B(L) = 0$, we get that

$$-\lambda\left[W + gB(x) - g\rho(x)\int_0^x \frac{B'(t)}{\rho(t)}\,\mathrm{d}t\right] \geqslant -\lambda W.$$

The result now follows from the representations (5.12), (5.16), and the fact that h^{-1} is strictly increasing. □

LEMMA 5.8. *Let* $\{(g_j, \lambda_j, B_j)\}$ *be a sequence of solutions of* (5.19) *with* $0 \leqslant g_j \leqslant R$ *for all* j *for some constant* R. *Then* $\{\lambda_j\}$ *satisfies that*

$$\liminf_j \lambda_j > -\infty, \qquad \limsup_j \lambda_j < 0.$$

Proof. If the first inequality does not hold, then $\{\lambda_j\}$ would have a subsequence, which we denote again by $\{\lambda_j\}$, such that $\lambda_j \to -\infty$. Since $h^{-1}(s) \to \infty$ as $s \to \infty$, we get from (5.22b), Lemma 5.7, that $B_j' \to 0$ in $C[0, L]$. But this is impossible because

$$\int_0^L \frac{B_j'(t)}{\rho(t)}\,\mathrm{d}t = -V, \quad \forall j. \tag{5.24}$$

Thus $\{\lambda_j\}$ must be bounded from below.

For the second inequality we argue again by contradiction. If $\{\lambda_j\}$ were not bounded away from zero, there would be a subsequence, which we denote again by $\{\lambda_j\}$, such that $\lambda_j \to 0$. It follows now from (5.22a), (5.23), and (5.12) that

$$c_j \leqslant F(g_j, \lambda_j, B_j)(x) < 0, \qquad \lim_{j\to\infty} c_j = 0.$$

Using this in (5.16) yields that

$$B_j'(x) \leqslant \frac{W}{c_j} < 0, \quad x \in [0, L].$$

Since $c_j \to 0$, the above inequality would contradict the volume constraint (5.24). Thus $\{\lambda_j\}$ must be bounded away from zero. □

LEMMA 5.9. *Let* $\{(g_j, \lambda_j, B_j)\}$ *be a sequence of solutions of* (5.19) *with* $0 \leqslant g_j \leqslant R$ *for all* j *for some constant* R. *Then* $\{(\lambda_j, B_j)\}$ *is bounded in* $\mathbb{R} \times C^1[0, L]$.

Proof. The result follows from Lemmas 5.7 and 5.8. □

We now have:

THEOREM 5.10. *Let* $\mathcal{C} \subset \mathcal{E}$ *be the connected component of solutions of* (5.5) *containing* $(0, \lambda_0, B_0)$ *where* (λ_0, B_0) *is given by Lemma* 5.2. *Then* $\mathcal{C}$ *is unbounded in* $\mathbb{R}^2 \times C^1[0, L]$ *and* (5.5) *has a solution for each* $g \geqslant 0$.

Proof. It follows from Lemma 5.5 and the results in [16, 4, 2], that $\mathcal{C}$ must satisfy at least one of the following alternatives:

(i) $\mathcal{C}$ is unbounded in $\mathbb{R}^2 \times C^1[0, L]$;
(ii) $\mathcal{C}$ contains a solution of the form $(0, \lambda^*, B^*)$ where $(\lambda^*, B^*) \neq (\lambda_0, B_0)$;
(iii) $\mathcal{C} \cap \partial\mathcal{E} \neq \emptyset$.

We can rule out alternative (ii) using Lemma 5.2, and alternative (iii) with Lemma 5.6. Thus (i) must hold and the result about the existence of solutions for each $g \geqslant 0$ follows from the unboundedness of $\mathcal{C}$ and Lemma 5.9. □

This result as stated, cannot be used to construct a consistent stationary field for the problem (5.1), basically because of the lack of uniqueness. We can however get a partial result for any fixed value of g. Note that condition A4 together with the fact that the constraint in (5.2) is linear in B' imply that the integrand in (5.3) is strictly convex in B'. Let B be a solution of (5.5) corresponding to the length value L. It follows now from the results in [18, Theorems 9.10, 9.23], that for ℓ sufficiently small, the solution B is a unique (weak) local minimizer of

$$J_\ell(v) = \int_0^\ell \hat{\nu}\left(-\frac{\rho(x)(W + gv(x))}{v'(x)}\right)\mathrm{d}x, \tag{5.25}$$

on the set

$$\mathcal{X}_\ell = \left\{v \in C^1[0, \ell]\colon \int_0^\ell \frac{v'(x)}{\rho(x)}\,\mathrm{d}x = \int_0^\ell \frac{B'(x)}{\rho(x)}\,\mathrm{d}x,\right.$$
$$\left. v(\ell) = B(\ell),\ v'(x) < 0,\ \forall x \in [0, \ell]\right\}. \tag{5.26}$$

Thus B gives a local minimum among configurations of a rope of length ℓ, with total volume equal to that of B in $[0, \ell]$, and with total mass at $x = \ell$ equal to $B(\ell)$.

6. Numerical Examples

In this section we present a typical family of constitutive functions that satisfies hypotheses A1–A5. In particular, we consider functions $\widehat{N}(\cdot)$ of the form:

$$\widehat{N}(\nu) = A_1\nu^{\alpha_1} - A_2\nu^{-\alpha_2}, \tag{6.1}$$

where $A_1 > 0$, $A_2 \geqslant 0$, $\alpha_1, \alpha_2 > 0$. This function clearly satisfies A1–A3 and thus has an inverse function $\hat{\nu}(\cdot)$ such that

$$\widehat{N}(\hat{\nu}(N)) = N, \quad N \in \mathbb{R}, \tag{6.2a}$$

$$\hat{\nu}(\widehat{N}(\nu)) = \nu, \quad \nu \in (0, \infty). \tag{6.2b}$$

If we differentiate (6.2a) with respect to N and solve for $\hat{\nu}_N(N)$, then we get that

$$h(N) = N^2 \hat{\nu}_N(N) = \frac{N^2}{\widehat{N}_\nu(\hat{\nu}(N))}.$$

If we let $N = \widehat{N}(\nu)$ in this expression, then we get from (6.2b) that

$$h(\widehat{N}(\nu)) = \frac{\widehat{N}^2(\nu)}{\widehat{N}_\nu(\nu)}. \tag{6.3}$$

Now A5 is equivalent to $h(\widehat{N}(\nu)) \to \infty$ as $\nu \to \infty$, which is satisfied by (6.1) for any $A_1 > 0$, $A_2 \geqslant 0$, $\alpha_1, \alpha_2 > 0$.

If we differentiate (6.3) with respect to ν, then we have that

$$h_N(\widehat{N}(\nu)) = \widehat{N}(\nu) \frac{2\widehat{N}_\nu^2(\nu) - \widehat{N}(\nu)\widehat{N}_{\nu\nu}(\nu)}{\widehat{N}_\nu^3(\nu)}. \tag{6.4}$$

Condition A4 requires that $N > 0$, which for (6.1) is equivalent to

$$\nu > \left(\frac{A_2}{A_1}\right)^{1/(\alpha_1+\alpha_2)}. \tag{6.5}$$

It follows from (6.4) now that A4 is equivalent to

$$2\widehat{N}_\nu^2(\nu) - \widehat{N}(\nu)\widehat{N}_{\nu\nu}(\nu) > 0, \tag{6.6}$$

provided ν satisfies (6.5). We now have:

PROPOSITION 6.1. *The constitutive function* (6.1) *satisfies condition* A4 *for any* $A_1 > 0$*,* $A_2 \geqslant 0$ *provided that* $\alpha_1 = \alpha_2 = \alpha > 0$ *or for any* $\alpha_1 > 0, \alpha_2 \geqslant 1$.

Proof. A direct calculation shows that for (6.1), inequality (6.6) is equivalent to

$$\alpha_1(\alpha_1 + 1)A_1^2\nu^{2\alpha_1} + \alpha_2(\alpha_2 - 1)A_2^2\nu^{-2\alpha_2} + (\alpha_1^2 + \alpha_2^2 + \alpha_2 + \alpha_1(4\alpha_2 - 1))A_1A_2\nu^{\alpha_1-\alpha_2} > 0,$$

provided ν satisfies (6.5). This inequality is automatically satisfied for any $\alpha_1 > 0$, $\alpha_2 \geqslant 1$. If $\alpha_1 = \alpha_2 = \alpha > 0$, the inequality is satisfied provided

$$\nu > \left[\frac{1-\alpha}{1+\alpha}\right]^{1/4\alpha} \left(\frac{A_2}{A_1}\right)^{1/2\alpha}.$$

Since the expression $(1-\alpha)/(1+\alpha)$ is less than 1 for $\alpha > 0$, this last inequality is satisfied provided ν satisfies (6.5). □

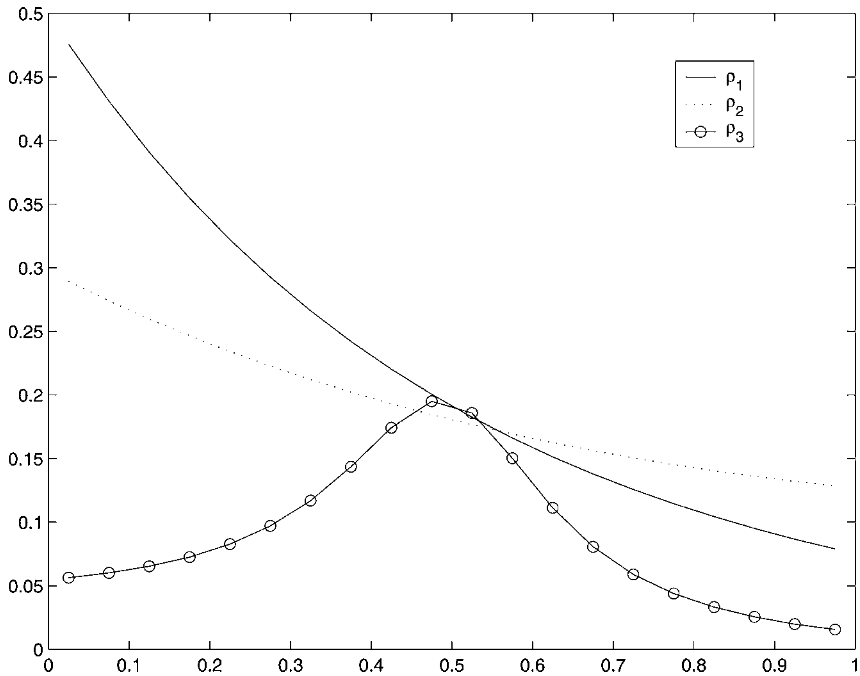

Figure 2. Computed mid-cross-sectional area functions for the density functions (6.7) and total fixed mass of 0.03.

We show now some numerical computations for the constitutive function (6.1) for the case $A_1 = 1$, $A_2 = 0$, and $\alpha_1 = 3$. We use variable density functions which along the axis of the rope are either increasing, decreasing or with an interior minimum. In particular, we consider:

$$\rho_1(x) = 0.1(1 + x), \tag{6.7a}$$

$$\rho_2(x) = 0.2 - 0.1x, \tag{6.7b}$$

$$\rho_3(x) = 5.0(x - 0.5)^2 + 0.1. \tag{6.7c}$$

We used the following values for the parameters L, W, g:

$$L = 1.0, \qquad W = 0.1, \qquad g = 9.8,$$

the units of which are in the metric system. We show in Figure 2 the computed cross-sectional area functions for the densities (6.7) and total fixed mass of 0.03. Note that for variable density functions, the area function need not be decreasing as is the case when the density is constant (cf. (4.14)). Note that for ρ_2 the rope is thinner at the beginning and fatter at the end as compared with the one for ρ_1 compensating in this way for the decrease in density. In Figure 3 we show the corresponding shape of the rope of minimum elongation for the case (6.7c) assuming circular cross sections. Similar results for the case of total fixed volume of 0.05 are shown in Figures 4 and 5.

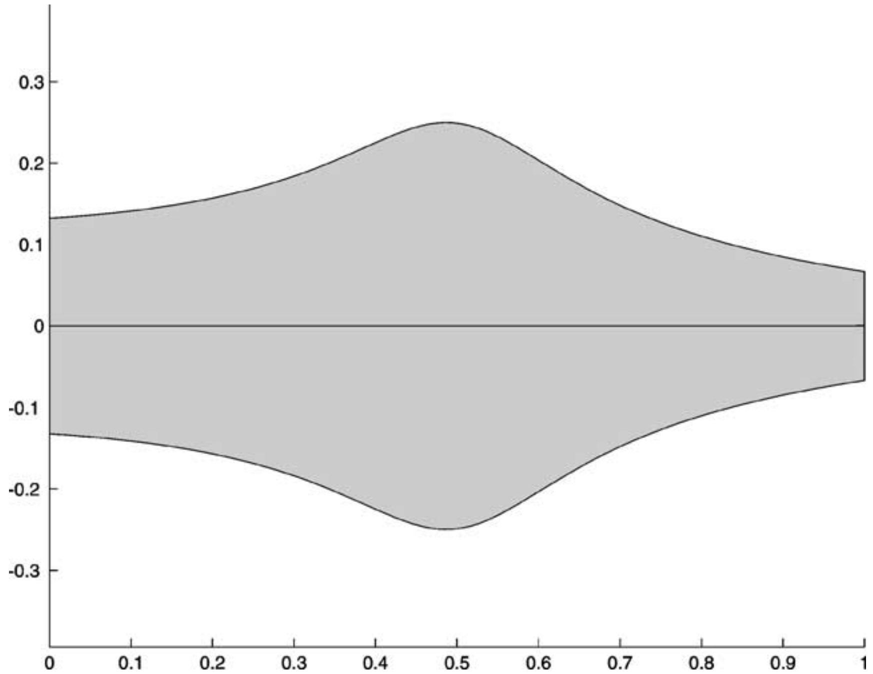

Figure 3. The shape of the rope of minimum elongation for the density function (6.7c) and total fixed mass of 0.03.

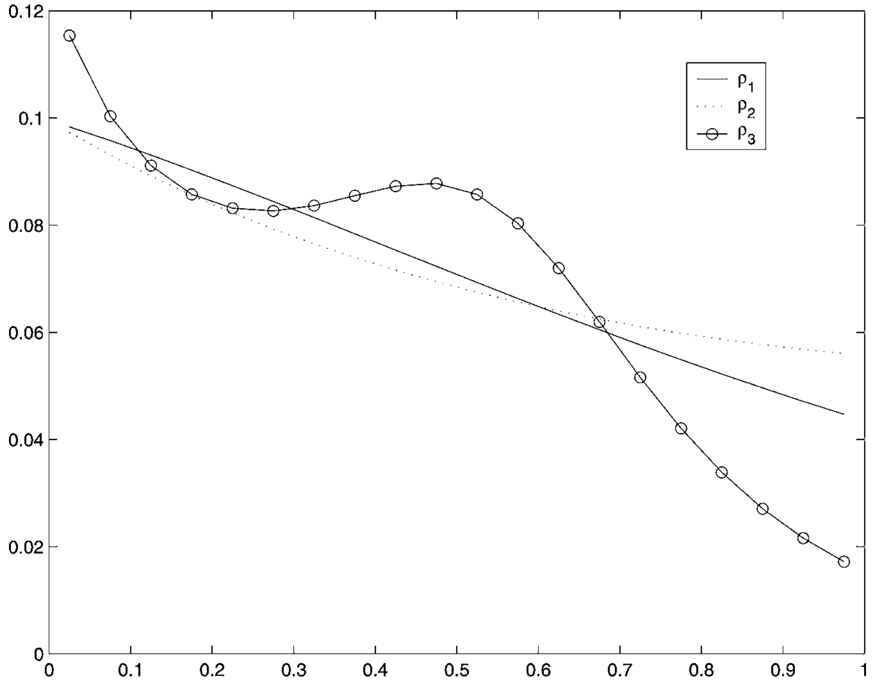

Figure 4. Computed mid-cross-sectional area functions for the density functions (6.7) and total fixed volume of 0.05.

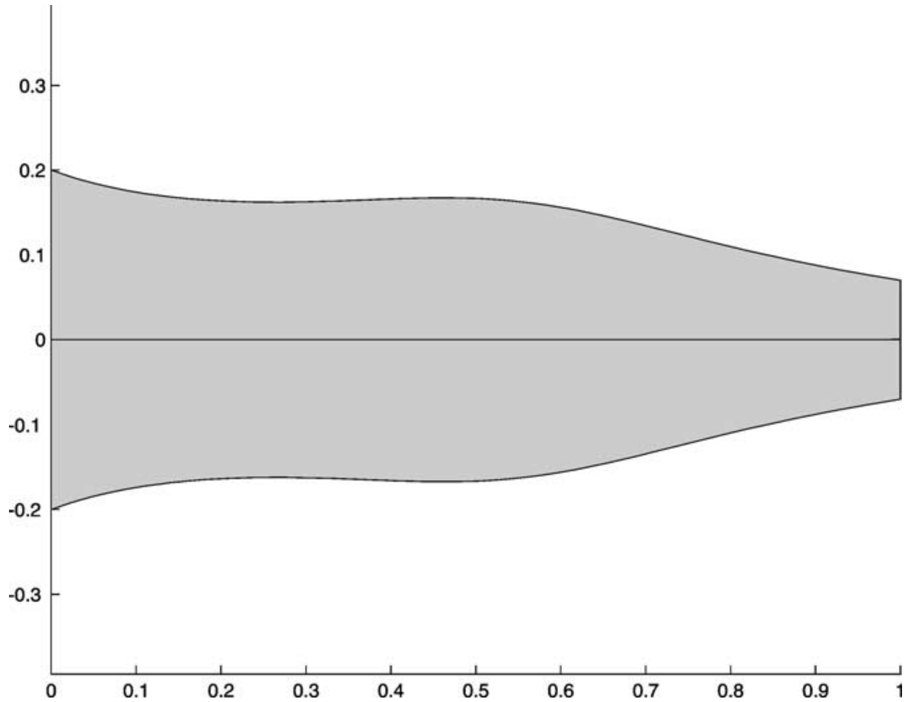

Figure 5. The shape of the rope of minimum elongation for the density function (6.7c) and total fixed volume of 0.05.

7. Conclusions

A problem perhaps more interesting from the practical point of view than the ones treated in this paper is that of minimizing the volume (thus minimizing the amount of material used) of the rope for a given length of the rope. The one-dimensional version of this problem can be treated similarly to the ones discussed here. Its three dimensional version has applications in the petroleum industry where long tubes from the top to the bottom of the sea (fixed length) need to be constructed with the least amount of material. In this case the tubes need to be hollow in order to transport various materials and there is the additional complication of the external water pressure.

The use of Leray–Schauder degree techniques in elasticity has a long and successful story that we will not try to review here. We refer to [3] for examples and its extensive literature review. However most of these applications of the Leray–Schauder degree have been limited to one dimensional problems, like the one treated in this paper, due to the complexity in transforming the equations of elasticity into an equivalent problem in terms of a compact operator. Not until recently, in [9], such a major enterprise was carried out for the three dimensional displacement problem of nonlinear elasticity. On the other hand, the use of a degree for proper Fredholm maps of index zero [8, 12] avoids the transformation of the original problem into one in terms of a compact operator but requires some a priori estimates on solutions of the linear problem and its spectrum. For the three dimensional mixed problem of nonlinear elasticity such spectral estimates were obtained in [11], and together with the estimates in [1] for elliptic systems, Healey and Simpson were able to apply a degree for proper Fredholm maps of index zero to get the existence of a global branch of solutions of this problem. This global continuum, in addition to the usual alternatives for such a continuum, may also “cease” to exist due to a failure of strong ellipticity, local injectivity, or the complementing condition. For specific materials with appropriate growth conditions, one can rule out termination due to a failure of strong ellipticity and local injectivity. (See, e.g., [10].)

Appendix

In this section we derive the model equations for the rope from the three-dimensional theory of elasticity. For the deformation (2.2) the deformation gradient is given by

$$\nabla \mathbf{p} = \begin{pmatrix} u'(x) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.1}$$

If we assume that the material of the body is *isotropic* and *hyperelastic*, then there exists a smooth *stored energy* function $\hat{\sigma}(\mathbf{F})$ of the form:

$$\hat{\sigma}(\mathbf{F}) = \sigma\left(\frac{1}{2}\mathbf{F}\cdot\mathbf{F}, \frac{1}{4}\mathbf{F}\mathbf{F}^t\cdot\mathbf{F}\mathbf{F}^t, \det\mathbf{F}\right),$$

where $\mathbf{F}\cdot\mathbf{H} = \operatorname{trace}(\mathbf{F}\mathbf{H}^t)$ and such that the (first) Piola–Kirchhoff stress tensor is given by

$$\mathbf{S}(\mathbf{F}) = \frac{\mathrm{d}\hat{\sigma}(\mathbf{F})}{\mathrm{d}\mathbf{F}} = \sigma_{,1}\mathbf{F} + \sigma_{,2}\mathbf{F}\mathbf{F}^t\mathbf{F} + (\det\mathbf{F})\sigma_{,3}\mathbf{F}^{-t}.$$

For (A.1) we have that

$$\frac{\mathbf{F}\cdot\mathbf{F}}{2} = \frac{(u'(x))^2+2}{2}, \qquad \frac{\mathbf{F}\mathbf{F}^t\cdot\mathbf{F}\mathbf{F}^t}{4} = \frac{(u'(x))^4+2}{4}, \qquad \det\mathbf{F} = u'(x). \tag{A.2}$$

It follows now that

$$\begin{aligned}\mathbf{S}(\nabla\mathbf{p}) = \operatorname{diag}\big(&u'(x)\sigma_{,1} + (u'(x))^3\sigma_{,2} + \sigma_{,3},\\ &\sigma_{,1} + \sigma_{,2} + u'(x)\sigma_{,3}, \sigma_{,1} + \sigma_{,2} + u'(x)\sigma_{,3}\big),\end{aligned} \tag{A.3}$$

where the arguments of $\sigma_{,1}$, etc., are given by (A.2). If we let $\mathbf{i}$ be a unit vector pointing in the positive x direction, then we have from (2.1) that the force exerted by the material on $[0, x]$ on that on $[x, L]$ is given by

$$-\int_{\Omega_x} \mathbf{S}(\nabla \mathbf{p}) \cdot \mathbf{i}\, \mathrm{d}s_x,$$

where $\mathrm{d}s_x$ denotes an element of area over Ω_x. If we let $\hat{\rho}(x, y, z)$ denote the mass-density per unit volume at (x, y, z) and we assume that a force per unit area $\widehat{W}\mathbf{i}$ is applied at the bottom of the rope, then we get that the total force on the material on $[x, L]$ is

$$g \int_x^L \int_{\Omega_\xi} \hat{\rho}(\xi, y, z)\mathbf{i}\, \mathrm{d}s_\xi\, \mathrm{d}\xi + \int_{\Omega_L} \widehat{W}\mathbf{i}\, \mathrm{d}s_L.$$

For equilibrium we must have that

$$\int_{\Omega_x} \mathbf{S}(\nabla \mathbf{p}) \cdot \mathbf{i}\, \mathrm{d}s_x = g \int_x^L \int_{\Omega_\xi} \hat{\rho}(\xi, y, z)\mathbf{i}\, \mathrm{d}s_\xi\, \mathrm{d}\xi + \int_{\Omega_L} \widehat{W}\mathbf{i}\, \mathrm{d}s_L,$$

which upon recalling (A.3) reduces to:

$$\begin{aligned} &A(x)\big(u'(x)\sigma_{,1} + (u'(x))^3\sigma_{,2} + \sigma_{,3}\big) \\ &\quad = g \int_x^L \int_{\Omega_\xi} \hat{\rho}(\xi, y, z)\, \mathrm{d}s_\xi\, \mathrm{d}\xi + W, \end{aligned} \tag{A.4}$$

where

$$A(x) = \int_{\Omega_x} \mathrm{d}s_x, \qquad W = \widehat{W} A(L).$$

If we let

$$\widehat{N}(u'(x)) = u'(x)\sigma_{,1} + (u'(x))^3\sigma_{,2} + \sigma_{,3},$$

and assume that $\hat{\rho}(x, y, z) = \rho(x)$, then (A.4) reduces to (2.9).

References

1. S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. *Comm. Pure Appl. Math.* **II**(17) (1964) 35–92.
2. J.C. Alexander and J.A. Yorke, The implicit function theorem and the global methods of cohomology. *J. Funct. Anal.* **21** (1976) 330–339.
3. S.S. Antman, *Nonlinear Problems of Elasticity*, Applied Mathematical Sciences 107. Springer, New York (1995).
4. F.E. Browder, On the continuity of fixed points under deformations of continuous mappings. *Summa Brazil. Mat.* **4** (1960) 183–191.
5. S.J. Cox, The shape of the ideal column. *Math. Intelligencer* **14**(1) (1992) 16–24.

6. S.J. Cox and M. McCarthy, The shape of the tallest column. *SIAM J. Math. Anal.* **29**(3) (1998) 547–554.
7. L. Euler, De motu corporum flexibilum. *Comm. Acad. Sci. Petrop.* **14** (1751) 182–196.
8. C.C. Fenske, Extensio gradus ad quasdam applicationes Fredholmii. *Mitt. Math. Seminar Giessen* **121** (1976) 65–70.
9. T.J. Healey, Global continuation in displacement problems of nonlinear elastostatics via the Leray–Schauder degree. *Arch. Rational Mech. Anal.* **152** (2000) 273–282.
10. T.J. Healey and P. Rosakis, Unbounded branches of classical injective solutions to the forced displacement problem in nonlinear elastostatics. *J. Elasticity* **49** (1997) 65–78.
11. T.J. Healey and H.C. Simpson, Global continuation in nonlinear elasticity. *Arch. Rational Mech. Anal.* **143** (1998) 1–28.
12. H. Kielhöfer, Multiple eigenvalue bifurcation for Fredholm mappings. *J. Reine Angew. Math.* **358** (1985) 104–124.
13. J.B. Keller and F.I. Niordson, The tallest column. *J. Math. Mech.* **16** (1966) 433–446.
14. J.L. Lagrange, Application de la méthode exposée précédente à la solution de différens problèmes de dynamique. *Misc. Tour.* $\mathbf{2}_{(2)}$ (1762) 196–298.
15. S. Lang, *Real Analysis*, 2nd edn. Addison-Wesley, Reading, MA (1983).
16. J. Leray and J. Schauder, Topologie et équations fonctionelles. *Ann. Sci. École Norm. Sup.* **3**(51) (1934) 45–78.
17. I. Stakgold, *Green's Functions and Boundary Value Problems*. Wiley, New York (1979).
18. J.L. Troutman, *Variational Calculus with Elementary Convexity*. Springer, New York (1983).
19. T. Valent, *Boundary Value Problems of Finite Elasticity*, Springer Tracts in Natural Philosophy. Springer, New York (1988).
20. G. Verma and J. Keller, Hanging rope of minimum elongation. *SIAM Rev.* (1983) 369–399.