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Singular global bifurcation problems for the buckling of anisotropic plates

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Contents

This paper treats a variety of unexpected pathologies that arise in the global bifurcation analysis of axisymmetric buckled states of anisotropic plates. The geometrically exact plate theory used accounts for flexure, extension and shear. The nonlinear constitutive functions have very general form. As a consequence of the anisotropy the trivial solution may depend discontinuously on the load parameter. Accordingly, the equations for the bifurcation problem have the same character, so that bifurcating branches of solutions become disconnected as the load parameter crosses values at which discontinuities occur. The anisotropy furthermore implies that the governing equations have a singular behaviour much worse than that for isotropic plates. Consequently, a variety of novel constructions are required to demonstrate the validity of the essential results upon which global bifurcation theory stands. (These results include the compactness of certain operators and the uniqueness of solutions of initial value problems for singular ordinary differential equations.) It is shown that in regions of solution-parameter space in which the equations depend continuously on the load parameter there exist connected global branches of solution pairs that have detailed nodal properties inherited from eigenfunctions of the linearized problem. Moreover, these nodal properties are preserved across gaps occurring where discontinuities occur. The methodology used to show this result actually supports constructive methods for finding disconnected branches.

1. Introduction

In this paper we furnish a detailed global description of the qualitative behaviour of axisymmetric buckled states of anisotropic nonlinearly elastic circular plates under edge thrust. Our theory of plates is geometrically exact in the sense that no geometric quantity (such as the sine of an angle) is ever approximated (by a linear or cubic expression). Moreover, we use a very general class of nonlinear constitutive equations describing the response of the plate in flexure, compression and shear.

The anisotropy of the plate, which is compatible with the axisymmetry, is of a sort that is observed in cast metals (see the photographs in Walker (1958)) or that can be produced by disposing reinforcing fibres either circumferentially or radially. An extreme case of a plate with radial reinforcement is what we call the Taylor plate, which offers no resistance to circumferential tension or to flexure about its rays. (This model is a plate-theoretic analogue of Taylor's (1919) membrane theory for parachutes. We shall use it for illustration.)

The presence of anisotropy radically alters the nature of the governing quasi-linear system of ordinary differential equations from that for isotropic plates. First, the trivial solution may depend discontinuously on the load parameter λ. In fact, for a certain class of materials there is a threshold such that if λ is below this threshold, then normal components of stress at the centre of the plate are zero, whereas if λ exceeds this threshold, then these components jump to $-\infty$ (Antman & Negrón-Marrero 1987); a synopsis and extension of this work is given in §3. The buckling problem inherits the discontinuous dependence on λ from the trivial solution. Secondly, the anisotropy changes the way the independent radial variable s appears in the equations and thus changes the nature of the singularity. Now the corresponding equations for an isotropic plate have the usual polar singularity at $s = 0$ (which happens to present very serious challenges to analysis, cf. Antman (1978)). But the differences between the equations for isotropic and anisotropic plates are so marked that to handle the latter we must develop a battery of new techniques to formulate and analyse an appropriate set of integral equations that support global bifurcation theory. The effects of the singularity at the origin are most pronounced in our studies of the linearized eigenvalue problems, the compactness of nonlinear integral operators, and the preservation of nodal properties.

When the trivial solutions depend discontinuously on the load parameter λ, the bifurcating solution branches for the buckling problem are typically disconnected at the values of λ at which the trivial solutions jump. (The plate would suffer a snap-buckling at such loads, wholly unexpected in plate theory. The mathematical disposition of the branches, however, is quite different from that found for shells that snap.) In §8 we prove that connected solution branches are characterized by a specific nodal structure (designed to accommodate the singularities of the problem), which is inherited from the eigenfunctions of the linearization about the trivial solution. In §9 we show that these nodal properties are actually preserved across gaps in the branches. To this end we embed our constitutive equations into a one-parameter family containing those of an isotropic material. We then apply two-parameter global continuation theory to the resulting family of problems.

There is an extensive literature on anisotropic plates and shells (see, for example, Ambartsumian 1967, 1974; Carrier 1944; Hoff 1981; Lekhnitskii 1957; Reissner 1958; Steele & Hartung 1965, and works cited therein). These works are all devoted to the study of linear problems. Thus some of our results for linear problems (needed for the subsequent nonlinear analysis) are prefigured by theirs, but none of our nonlinear results are. Even in the realm of linear theory, our equations are far richer than theirs because our trivial state is one with finite compression. Consequently, our linearized equations depend nonlinearly and possibly discontinuously on the parameter λ. Moreover, shear deformation, accounted for here, but often ignored in engineering plate theories, can greatly alter the spectrum of the linearized problem.

There have been several studies of bifurcation problems involving non-smooth operators (see, for example, Stuart 1976; Stuart & Toland 1980). Stuart's (1976) paper is the only one known to us that accounts for discontinuous dependence on the parameter λ; the structure of his problem differs considerably from ours. Because our nonlinear integral equations involve compact operators (and because the spectrum of the linearized problem consequently consists entirely of eigenvalues) we do not have to appeal to some of the more arcane developments in bifurcation theory to carry out our analysis; the pathologies that remain are sufficiently challenging.

2. Formulation of the governing nonlinear boundary-value problem

Geometry of deformation

Let $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ be a fixed right-handed orthonormal basis for the euclidean 3-space $\mathbb{E}^3$. Let (s,ϕ) be polar coordinates for the $(\boldsymbol{i},\boldsymbol{j})$-plane. We set

$$\boldsymbol{e}_1(\phi) = \cos\phi\boldsymbol{i}+\sin\phi\boldsymbol{j}, \quad \boldsymbol{e}_2(\phi) = -\sin\phi\boldsymbol{i}+\cos\phi\boldsymbol{i}, \quad \boldsymbol{e}_3 = \boldsymbol{k}. \tag{2.1}$$

An *axisymmetric configuration* of a circular plate that can suffer flexure, extension and shear is determined by a pair of vector functions:

$$[0,1]\times\mathbb{R}\ni(s,\phi)\mapsto\boldsymbol{r}(s,\phi),\boldsymbol{b}(s,\phi), \tag{2.2}$$

with $\boldsymbol{r}(s,\cdot)$ and $\boldsymbol{b}(s,\cdot)$ having period 2π and with

$$\boldsymbol{r}(s,\phi)\cdot\boldsymbol{e}_2(\phi) = 0, \quad \boldsymbol{b}(s,\phi)\cdot\boldsymbol{e}_2(\phi) = 0, \quad |\boldsymbol{b}(s,\phi)| = 1. \tag{2.3}$$

In characterizing the configuration of a plate by functions defined on the disc

$$\{s\boldsymbol{e}_1(\phi): s\in[0,1], \phi\in[0,2\pi]\}, \tag{2.4}$$

we are giving the notion of *plate* an intrinsically two-dimensional sense. To interpret the significance of the variables $\boldsymbol{r}$ and $\boldsymbol{b}$, we may regard the plate as a three-dimensional body with reference configuration

$$\{s\boldsymbol{e}_1(\phi)+z\boldsymbol{k} : 0 \leqslant s \leqslant 1, 0 \leqslant \phi \leqslant 2\pi, |z| \leqslant Z\}, \tag{2.5}$$

where $2Z$ is the thickness of the plate. For simplicity we take Z to be constant. The midplane of the plate is just the plane in (2.5) defined by $z = 0$. Let $\boldsymbol{p}(s,\phi,z)$ be the deformed position of the material point $s\boldsymbol{e}_1(\phi)+z\boldsymbol{k}$ of (2.5). Then one of many conceivable interpretations of $\boldsymbol{r}$ and $\boldsymbol{b}$ is obtained by regarding $\boldsymbol{p}$ as constrained to have the form

$$\boldsymbol{p}(s,\phi,z) = \boldsymbol{r}(s,\phi)+\omega(s,\phi,z)\,\boldsymbol{b}(s,\phi), \tag{2.6}$$

where ω is a prescribed function and $\omega(s,\phi,\cdot)$ is an odd, strictly increasing, continuously differentiable function. Thus $\boldsymbol{r}(s,\phi)$ may be interpreted as giving the deformed position of the material point with reference position $s\boldsymbol{e}_1(\phi)$ (on the midplane), and $\boldsymbol{b}(s,\phi)$ as giving the deformed configuration

$$z \mapsto \boldsymbol{r}(s,\phi)+\omega(s,\phi,z)\,\boldsymbol{b}(s,\phi)$$

of the fibre $z \mapsto s\boldsymbol{e}_1(\phi)+z\boldsymbol{k}$ (through $s\boldsymbol{e}_1(\phi)$ that is perpendicular to the midplane). For the axisymmetric deformations we treat, we assume that $\omega_\phi = 0$. In consonance with our assumption that the plate be uniform we also assume that $\omega_s = 0$. We set:

$$\left.\begin{aligned} \boldsymbol{a}(s,\phi) &\equiv \cos\theta(s)\,\boldsymbol{e}_1(\phi)+\sin\theta(s)\,\boldsymbol{k}, \\ \boldsymbol{b}(s,\phi) &\equiv -\sin\theta(s)\,\boldsymbol{e}_1(\phi)+\cos\theta(s)\,\boldsymbol{k}, \end{aligned}\right\} \tag{2.7}$$

$$\rho(s) \equiv \boldsymbol{r}(s,\phi)\cdot\boldsymbol{e}_1(\phi), \quad \zeta(s) \equiv \boldsymbol{r}(s,\phi)\cdot\boldsymbol{k}, \tag{2.8a, b}$$

$$\boldsymbol{r}_s(s,\phi) \equiv \nu(s)\,\boldsymbol{a}(s,\phi)+\eta(s)\,\boldsymbol{b}(s,\phi), \tag{2.9}$$

$$\tau(s) \equiv s^{-1}\rho(s), \tag{2.10}$$

$$\sigma(s) \equiv s^{-1}\sin\theta(s), \tag{2.11}$$

$$\mu(s) \equiv \theta'(s). \tag{2.12}$$

The strains for our problem are

$$\boldsymbol{w} \equiv (w_1, w_2, w_3, w_4, w_5) \equiv (\tau, \nu, \eta, \sigma, \mu), \tag{2.13}$$

where τ is the stretch of a circular fibre, ν accounts for the stretch of a radial fibre, η accounts for the shear between a vertical and a radial fibre, σ measures flexure about a radial fibre, and μ measures flexure about a circular fibre. Note that (2.8)–(2.12) imply that all the geometrical variables for our problem may be found from (ν, η, θ).

That the deformation of (2.5) defined by (2.3), (2.6), and $\omega_\phi = 0$ be continuously differentiable requires that

$$\rho(0) = 0, \quad \theta(0) = 0, \quad \eta(0) = 0. \tag{2.14a, b, c}$$

We fix the displacement to within a rigid motion by setting

$$\zeta(0) = 0. \tag{2.14d}$$

That this deformation preserve orientation (i.e. have a positive jacobian) requires that

$$\tau(s) > \omega(Z)\,|\sigma(s)|, \quad \nu(s) > \omega(Z)\,|\mu(s)|. \tag{2.15}$$

We denote the set of $\boldsymbol{w}$s satisfying (2.15) by $\mathscr{W}$; $\mathscr{W}$ is convex.

The requirement that the edge $s = 1$ of (2.5) be constrained to be parallel to $\boldsymbol{k}$ yields the boundary condition

$$\theta(1) = 0. \tag{2.16}$$

We adopt (2.14)–(2.16) in general (even when (2.6) is not operative).

Mechanics

Let $\boldsymbol{n}^1(s_0, \phi_0)$ and $\boldsymbol{m}^1(s_0, \phi_0)$ denote the resultant contact force and contact couple per unit reference length of the circle $\phi \mapsto s_0\,\boldsymbol{e}_1(\phi)$ exerted across this circular section at the material point with coordinates (s_0, ϕ_0). Let $\boldsymbol{n}^2(s_0, \phi_0)$ and $\boldsymbol{m}^2(s_0, \phi_0)$ denote the resultant contact force and contact couple per unit length of the ray $s \mapsto s\boldsymbol{e}_1(\phi_0)$ exerted across this section at (s_0, ϕ_0). (Hence we are giving the notion of plate its intrinsic two-dimensional interpretation.) That these resultants be axisymmetric means that they have representations of the form

$$\boldsymbol{n}^1(s,\phi) = \hat{N}(s)\,\boldsymbol{a}(s,\phi) + \hat{H}(s)\,\boldsymbol{b}(s,\phi), \quad \boldsymbol{n}^2(s,\phi) = \hat{T}(s)\,\boldsymbol{e}_2(\phi), \tag{2.17a, b}$$

$$\boldsymbol{m}^1(s,\phi) = -\hat{M}(s)\,\boldsymbol{e}_2(\phi), \quad \boldsymbol{m}^2(s,\phi) = \hat{\Sigma}(s)\,\boldsymbol{a}(s,\phi). \tag{2.17c, d}$$

Conditions (2.17) require that several components of the resultants be zero. The only such requirement perhaps lacking an immediate physical interpretation is that $\boldsymbol{m}^2 \cdot \boldsymbol{b} = 0$. Were this component not zero, then it would tend to bend radial fibres about $\boldsymbol{b}$ and therefore out of their natural $(\boldsymbol{e}_1, \boldsymbol{k})$-plane. Conditions (2.17) are really restrictions on the material response saying that the constitutive equations, to be described below, must be such that (2.17) follows from (2.3). A three-dimensional interpetation of this question, based upon (2.6), is given by Antman (1978, §10).

We assume that a normal pressure of intensity $\lambda g(\rho(1))$ units of force per reference length is applied to the edge $s = 1$:

$$\boldsymbol{n}^1(1,\phi) = -\lambda g(\rho(1))\,\boldsymbol{e}_1(\phi), \tag{2.18a}$$

which is equivalent to

$$\hat{N}(1) = -\lambda g(\rho(1)), \quad \hat{H}(1) = 0, \tag{2.18b, c}$$

by virtue of (2.7), (2.16), (2.17*a*).

If the pressure has intensity λ units of force per deformed length, then $g(\rho) = \rho$, whereas if it has intensity λ units of force per reference length, then $g(\rho) = 1$. We thus assume that

$$g(\rho) > 0, \quad g'(\rho) \geqslant 0 \quad \text{for} \quad \rho > 0. \tag{2.19}$$

We assume that g is three times continuously differentiable.

We assume that there are no other externally applied forces acting on the plate. Then by summing forces and moments on a typical annular sector lying between radii s and 1, we obtain the equilibrium equations

$$s\hat{N}(s) = -\left[\lambda g(\rho(1)) + \int_s^1 \hat{T}(t)\,\mathrm{d}t\right]\cos\theta(s), \tag{2.20a}$$

$$s\hat{H}(s) = \left[\lambda g(\rho(1)) + \int_s^1 \hat{T}(t)\,\mathrm{d}t\right]\sin\theta(s), \tag{2.20b}$$

$$s\hat{M}(s) = \hat{M}(1) - \int_s^1 \{t[\eta(t)\hat{N}(t) - \nu(t)\hat{H}(t)] + \hat{\Sigma}(t)\cos\theta(t)\}\,\mathrm{d}t. \tag{2.20c}$$

Equations (2.20a, b) imply that

$$\hat{H}\cos\theta = -\hat{N}\sin\theta. \tag{2.21}$$

Differentiating (2.20) with respect to s, we recover the classical form of the equilibrium equations

$$[s\hat{N}(s)]' = s\hat{H}(s)\,\theta'(s) + \hat{T}(s)\cos\theta(s), \tag{2.22a}$$

$$[s\hat{H}(s)]' = -s\hat{N}(s)\,\theta'(s) - \hat{T}(s)\sin\theta(s), \tag{2.22b}$$

$$[s\hat{M}(s)]' - \hat{\Sigma}(s)\cos\theta(s) = s[\eta(s)\hat{N}(s) - \nu(s)\hat{H}(s)]. \tag{2.23}$$

We can replace (2.22) with (2.21) and

$$(s\hat{N}/\cos\theta)' = \hat{T}, \tag{2.24}$$

which comes from (2.20a).

Constitutive equations

We assume that the material of the plate is homogeneously elastic by requiring that there be three times continuously differentiable functions

$$\mathscr{W} \ni \mathbf{w} \mapsto T(\mathbf{w}), N(\mathbf{w}), H(\mathbf{w}), \Sigma(\mathbf{w}), M(\mathbf{w}) \tag{2.25}$$

such that $$\hat{T}(s) = T(\mathbf{w}(s)) = T(\tau(s), \nu(s), \eta(s), \sigma(s), \mu(s)), \text{ etc.} \tag{2.26}$$

We require that the constitutive functions $T, \ldots, M$ satisfy the monotonicity conditions:

$$\frac{\partial(N,H,M)}{\partial(\nu,\eta,\mu)}, \frac{\partial(T,\Sigma)}{\partial(\tau,\sigma)} \quad \text{are positive-definite.} \tag{2.27a, b}$$

These conditions are a plate-theoretic analogue of the strong ellipticity condition of the three-dimensional theory (cf. Antman 1978, §10). Among the consequences of (2.27) is that an increase in ν is accompanied by an increase in N.

We regard (2.27a) as the fundamental constitutive inequality. From time to time we use other less fundamental inequalities. In certain auxiliary equations these inequalities typically prohibit instances of non-uniqueness, which lack compelling physical importance; their treatment would require rather obvious but aggravatingly tedious adjustments in the exposition. Among these other constitutive inequalities is the requirement that:

$$\frac{\partial(T,N)}{\partial(\tau,\nu)} \text{ is positive-definite when } \quad \eta=\sigma=\mu=0, \tag{2.28}$$

which is used in §3. Condition (2.28) ensures that $(T(\cdot,\nu,0,0,0))$ is invertible. It roughly says that a change in τ has more effect on T than it has on N.

To appreciate further conditions of this sort, we consider the deformation of a rectangular block with edges parallel to the (x,y,z)-axes. If we fix the length in the x-direction, increase the length in the y-direction, and apply zero force to the faces perpendicular to the z-axis, we might expect that the tensions in both the x- and y-directions increase. If we identify ν and τ with the stretches in the x- and y-directions respectively, this argument would imply that

$$N_\tau > 0, \tag{2.29}$$

which implies that

$$N_\tau + N_\nu > 0. \tag{2.30}$$

In the sequel, when we impose further such constitutive restrictions, we do not bother supplying motivations like these.

We require that the material meet the following minimal restrictions on its symmetry:

$$T, N, \Sigma, M \text{ are even in } \eta,\ H \text{ is odd in } \eta, \tag{2.31}$$

$$T, N, H \text{ are unchanged under } (\sigma,\mu)\to(-\sigma,-\mu), \tag{2.32}$$

$$\Sigma, M \text{ change sign under } (\sigma,\mu)\to(-\sigma,-\mu). \tag{2.33}$$

These conditions ensure that deformed states come in mirror images.

We finally impose compatible growth conditions ensuring that extreme values of the strains are accompanied by extreme values of the corresponding resultants:

$$\begin{Bmatrix} T(\mathbf{w}) \\ N(\mathbf{w}) \end{Bmatrix} \to -\infty \text{ as } \begin{Bmatrix} \tau \searrow \omega(Z)\,|\sigma| \\ \nu \searrow \omega(Z)\,|\mu| \end{Bmatrix} \text{ if } \begin{Bmatrix} \nu \\ \tau \end{Bmatrix} \text{ is bounded above} \tag{2.34a}$$

and if η, σ, μ are bounded,

$$\begin{Bmatrix} T(\mathbf{w}) \\ N(\mathbf{w}) \end{Bmatrix} \to \infty \quad \text{as} \quad \begin{Bmatrix} \tau \\ \nu \end{Bmatrix} \to \infty \quad \text{if} \quad \begin{Bmatrix} \nu - \omega(Z)\,|\mu| \\ \tau - \omega(Z)\,|\sigma| \end{Bmatrix} \text{ has a positive lower bound} \tag{2.34b}$$

and if η, σ, μ are bounded,

$$H(\mathbf{w}) \to \pm\infty \quad \text{as} \quad \eta \to \pm\infty \quad \text{if} \quad (\tau,\nu,\sigma,\mu) \text{ lies in a compact subset of}$$
$$\{(\tau,\nu,\sigma,\mu) : \tau > \omega(Z)\,|\sigma|, \nu > \omega(Z)\,|\mu|\}, \tag{2.35}$$

$$\begin{Bmatrix} \Sigma(\mathbf{w}) \\ M(\mathbf{w}) \end{Bmatrix} \to \pm\infty \quad \text{as} \quad \begin{Bmatrix} \sigma \\ \mu \end{Bmatrix} \to \pm \begin{Bmatrix} \tau/\omega(Z) \\ \nu/\omega(Z) \end{Bmatrix} \quad \text{if} \quad \begin{Bmatrix} (\nu,\eta,\mu) \\ (\tau,\eta,\mu) \end{Bmatrix} \text{ lies in a compact subset of}$$
$$\begin{Bmatrix} \{(\nu,\eta,\mu) : \nu > \omega(Z)\,|\mu|\} \\ \{(\tau,\eta,\sigma) : \tau > \omega(Z)\,|\sigma|\} \end{Bmatrix}. \tag{2.36}$$

For simplicity, we assume that the natural state is stress free, so that

$$T(1,1,0,0,0) = 0 = N(1,1,0,0,0). \tag{2.37}$$

The material is hyperelastic if there exists a (four times continuously differentiable) stored-energy function W such that

$$T = W_{\tau}, \quad N = W_{\nu}, \quad H = W_{\eta}, \quad \Sigma = W_{\sigma}, \quad M = W_{\mu}. \tag{2.39}$$

Consequently,
$$T_{\nu} = N_{\tau}, \quad \Sigma_{\mu} = M_{\sigma}, \quad \text{etc.} \tag{2.39}$$

There are compelling thermodynamic arguments supporting hyperelasticity. We do not insist that the material be hyperelastic, however, primarily because our methods do not require it and secondly because hyperelasticity can be lost in certain approximation schemes, which could include those by which plate theories are constructed from three-dimensional theories.

Boundary value problem

We seek classical solutions of the strain-configuration equations (2.7)–(2.12), the equilibrium equations (2.21), (2.23) and (2.24), the constitutive equations (2.26), and the boundary conditions (2.14), (2.16) and (2.18). Such solutions must satisfy the strict inequalities (2.15). All these equations and side conditions constitute our boundary value problem.

The Taylor plate

An extreme case of anisotropy, useful for illustrative purposes, is furnished by the Taylor plate, which has no circumferential tensile or flexural strength. It is accordingly defined by the constitutive restrictions that

$$T = 0 = \Sigma, \tag{2.40}$$

and that N, H, M be independent of τ and σ. For such a plate, the boundary value problem is greatly simplified. The Taylor plate may be regarded as consisting of an infinite array of radially disposed rods. In fact, the governing equations for a Taylor plate correspond to those for a rod with a very singular non-uniformity that makes it infinitely strong at $s = 0$. It is thus the plate-theoretic analogue of the model for membranes that Taylor (1919) used to describe the behaviour of parachutes.

3. Trivial solutions

We now describe the unbuckled states of the plate. These correspond to the trivial solutions of our boundary value problem, the analysis of which is far from trivial. Our development consists in a brief distillation and a considerable extension of the work of Antman & Negrón-Marrero (1987).

An unbuckled state is one for which there is neither bending nor shearing, i.e. one for which

$$\theta = \eta = \sigma = \mu = 0. \tag{3.1}$$

In such a state
$$\nu = \rho', \quad \tau = \rho/s, \tag{3.2}$$

by (2.8)–(2.10). Thus (2.31)–(2.33) reduce (2.21)–(2.24), (2.26) to the single equation

$$[sN(\rho/s, \rho', 0, 0, 0)]' = T(\rho/s, \rho' 0, 0, 0), \tag{3.3}$$

reduce (2.14*a*) and (2.18*b*) to

$$\rho(0) = 0, \quad N(\rho(1), \rho'(1), 0, 0, 0) = -\lambda g(\rho(1)), \tag{3.4a, b}$$

and reduce (2.15) to

$$\rho(s)/s > 0, \quad \rho'(s) > 0. \tag{3.5}$$

Conditions (2.27*a*) and (2.34) imply that $N(\tau, \cdot, 0, 0, 0)$ has a three times continuously differentiable inverse $\nu^{\#}(\tau, \cdot)$. We define

$$T^{\#}(\tau, n) \equiv T(\tau, \nu^{\#}(\tau, n), 0, 0, 0). \tag{3.6}$$

Then (3.2)–(3.5) is equivalent to

$$(s\tau)' = \nu^{\#}(\tau, n), \quad (sn)' = T^{\#}(\tau, n), \tag{3.7a, b}$$

$$s\tau(s) \to 0 \quad \text{as} \quad s \to 0, \quad n(1) = -\lambda g(\tau(1)), \tag{3.8}$$

$$\tau(s) > 0 \quad \text{for} \quad s > 0. \tag{3.9}$$

In this section we study classical solutions $\rho_0(\cdot\,;\lambda) \in C^0([0,1]) \cap C^2((0,1])$ of the boundary value problem (3.3)–(3.5), or equivalently, classical solutions $\tau_0(\cdot\,;\lambda)$, $N^0(\cdot\,;\lambda) \in C^1((0,1])$ of (3.7)–(3.9). (Solutions $\rho_0(\cdot\,;\lambda)$ of the specialization of (2.20), (2.26) to the trivial state, subject to (3.4*a*) and (3.5), are weak solutions of (3.3)–(3.5). It can be shown that if mild growth restrictions are imposed on T and N, then a weak solution ρ_0 in an appropriate Sobolev space determined by these growth conditions is actually a classical solution in the sense just described (cf. Antman 1983).)

Let

$$s = e^{\xi - 1}, \quad \tau(s) = \tau_*(\xi), \quad n(s) = n_*(\xi). \tag{3.10}$$

Then (3.7) is equivalent to the autonomous system

$$\dot{\tau} = \nu^{\#}(\tau, n) - \tau, \quad \dot{n} = T^{\#}(\tau, n) - n, \quad -\infty < \xi < 1, \tag{3.11}$$

in which the superposed dot denotes the derivative with respect to ξ and in which we have dropped the asterisks from τ and n. Thus we can study (3.11) subject to the transformed versions of (3.8) by phase-plane methods.

The vertical isoclines of (3.11) consist of those points (τ, n) for which

$$\tau = \nu^{\#}(\tau, n), \tag{3.12a}$$

or equivalently,

$$n = N(\tau, \tau, 0, 0, 0), \tag{3.12b}$$

and the horizontal isoclines consist of those points (τ, n) for which

$$n = T^{\#}(\tau, n), \tag{3.13a}$$

or equivalently,

$$n = N(\tau, \nu, 0, 0, 0), \quad n = T(\tau, \nu, 0, 0, 0). \tag{3.13b}$$

For $n \leqslant 0$, conditions (2.28), (2.30), (2.34), (2.37) imply that (3.12) is equivalent to an equation of the form

$$\tau = v(n), \tag{3.14}$$

with v increasing from 0 to 1 as n increases from $-\infty$ to 0, and that (3.13) is equivalent to an equation of the form

$$\tau = h(n), \tag{3.15}$$

with $h(n) \to 0$ as $n \to -\infty$, $h(0) = 1$. The functions v and h are three times continuously differentiable because T and N are.

Let

$$\left.\begin{aligned} \mathscr{N}^h &\equiv \{-\infty < n < 0 : h(n) > v(n)\}, \\ \mathscr{N} &\equiv \{-\infty < n \leqslant 0 : h(n) = v(n)\}, \\ \mathscr{N}^v &\equiv \{-\infty < n < 0 : v(n) > h(n)\}. \end{aligned}\right\} \tag{3.16}$$

Note that $0 \in \mathscr{N}$. Because v and h are continuous, $\mathscr{N}$ is closed while $\mathscr{N}^h$ and $\mathscr{N}^v$ are open. $\mathscr{N}^h$ and $\mathscr{N}^v$, one of which could be empty, can therefore be decomposed as unions of a countable number of disjoint open intervals. The singular points of (3.11) occur at the intersections of the horizontal and vertical isoclines, i.e. they are points of $\{(\tau, n) : n \in \mathscr{N}, \tau = h(n)\}$. Those particular trajectories of (3.11) that everywhere lie between the horizontal and vertical isoclines are confined to a curve (unique by virtue of (2.28)) with equation

$$\tau = f(n). \tag{3.17}$$

(For an isotropic plate v and h coincide and form a curve of singular points of (3.11). The analysis of (3.11) in this case is elementary.) The roles of (3.14)–(3.17) are illustrated in figure 1. Antman & Negrón-Marrero (1987) show that the only trajectories that correspond to solutions of the boundary value problem (3.7)–(3.10) are those that lie on $\tau = f(n)$, that begin at a singular point, and that terminate at $(\bar{\tau}(\lambda)), \bar{n}(\lambda))$, the solution of

$$n = -\lambda g(\tau), \quad \tau = f(n), \tag{3.18}$$

which is unique by (2.19). From these considerations follows

THEOREM 1. *Let* (2.28), (2.34), (2.37) *hold. Then for* $\lambda \geqslant 0$ *problem* (3.3), (3.4) *has a unique solution* $\rho_0(\cdot, \lambda) \in C^0([0,1]) \cap C^2((0,1])$ *and equivalently, problem* (3.7)–(3.9) *has a unique solution* $\tau_0(\cdot, \lambda)$, $N^0(\cdot, \lambda) \in C^1((0,1])$. *If* $\bar{n}(\lambda) \in \mathscr{N}$, *then* $\rho_0(s, \lambda) = s\bar{\tau}(\lambda)$. *If* $\bar{n}(\lambda) \in \mathscr{N}^h$ *then* $\bar{n}(\lambda)$ *belongs to a component open interval* (a, b) *of* $\mathscr{N}^h$, $N^0(\cdot\,;\lambda)$ *strictly decreases from* $N^0(0;\lambda) = a$ *to* $N^0(1;\lambda) = \bar{n}(\lambda)$, *and* $\rho_0(s;\lambda) = sf(N^0(s;\lambda))$. *If* $\bar{n}(\lambda) \in \mathscr{N}^v$, *then* $\bar{n}(\lambda)$ *belongs to a component open interval* (c, d) *of* $\mathscr{N}^v$, $N^0(\cdot\,;\lambda)$ *strictly increases from* $N^0(0;\lambda) = d$ *to* $N^0(1;\lambda) = \bar{n}(\lambda)$, *and* $\rho_0(s,\lambda) = sf(N^0(s;\lambda))$ (*see figure* 1).

It is important to observe the following significant and typical results. If $\mathscr{N}^h$ has a component open interval of the form $(b, 0)$, then $N^0(0;\lambda) = 0$ for all λ such that $\bar{n}(\lambda) \in (b, 0)$. Thus the centre of the plate is stress-free for a range of boundary

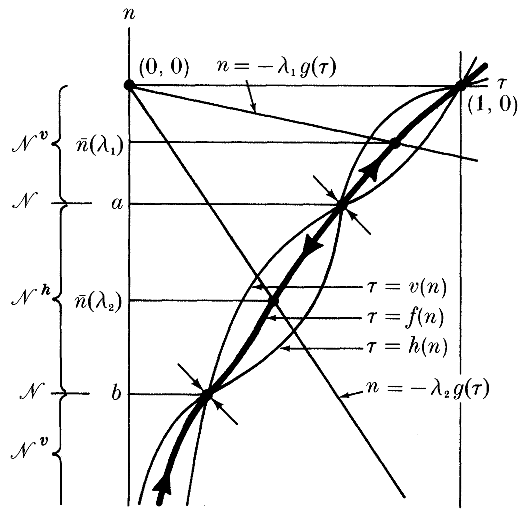

FIGURE 1. Phase portrait of (3.11) showing only the isoclines and the trajectories on the curve $\tau = f(n)$. Here the horizontal and vertical isoclines intersect transversally at $n = 0, a, b$. The singular points $(1, 0)$ and $(b, f(b))$ are attractive nodes and $(a, f(a))$ is a saddle point.

pressures. In particular, if $\mathcal{N}^h = (-\infty, 0)$, then the centre of the plate is stress-free for all pressures λ. If $\mathcal{N}^v$ has a component open interval of the form $(d, 0)$, then $N^0(0;\lambda) = d$ for all λ such that $\bar{n}(\lambda) \in (d, 0)$. Thus the smallest amount of pressure on the boundary causes the stress at the centre to jump from 0 to a non-zero value at which it remains while λ increases up to the value at which $\bar{n}(\lambda) = d$. In particular, if $\mathcal{N}^v = (-\infty, 0)$, then the normal stresses at the centre equal $-\infty$ for all $\lambda > 0$. Similarly, if $\mathcal{N}^v$ has a component open interval of the form $(-\infty, b)$, then for large enough λ, the normal stresses at the centre equal $-\infty$. These remarks exemplify that if $\mathcal{N}^h \neq (-\infty, 0)$, then the solutions of (3.3), (3.4) do not depend continuously on $\lambda \in [0, \infty)$.

For any constitutive function, such as N_ν, we set

$$N_\nu^0(s;\lambda) \equiv N_\nu(\tau_0(s;\lambda),\ \nu_0(s;\lambda), 0, 0, 0), \quad \text{etc.,} \tag{3.19}$$

where $\tau_0(s;\lambda) \equiv \rho_0(s;\lambda)/s$, $\nu_0(s;\lambda) \equiv \rho_0'(s;\lambda)$. Let

$$A_1(s;\lambda) \equiv \frac{T_\nu^0(s;\lambda) - N_\tau^0(s;\lambda)}{2N_\nu^0(s;\lambda)}; \quad B_1(s;\lambda) \equiv \left[A_1(s;\lambda)^2 + \frac{T_\tau^0(s;\lambda)}{N_\nu^0(s;\lambda)}\right]^{\frac{1}{2}}, \tag{3.20}$$

$$\alpha_1(\lambda) \equiv A_1(0;\lambda), \beta_1(\lambda) \equiv B_1(0;\lambda).$$

Note that $A_1 = 0$ for hyperelastic plates.

By studying the perturbation of (3.11) about singular points we readily find

THEOREM 2. *Let the hypotheses of theorem 1 hold. If $-\infty < N^0(0;\lambda) < 0$, then the limits of $\tau_0(s;\lambda)$ and $\nu_0(s;\lambda)$ as $s \to 0$ exist and are equal and positive. Moreover,*

$$\alpha_1(\lambda) + \beta_1(\lambda) \geqslant 1, \tag{3.21a}$$

or equivalently,

$$T_\tau^0(0;\lambda) + T_\nu^0(0;\lambda) \geqslant N_\tau^0(0;\lambda) + N_\nu^0(0;\lambda). \tag{3.21b}$$

In this case there are numbers $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$, *such that*

$$\begin{Bmatrix}\rho_0(s;\lambda)\\ N^0(s;\lambda)\end{Bmatrix} = \begin{Bmatrix}s\tau_0(0;\lambda)\\ N^0(0;\lambda)\end{Bmatrix} + \begin{Bmatrix}A(\lambda)\\ C(\lambda)\end{Bmatrix} s^{\alpha_1(\lambda)+\beta_1(\lambda)} + \begin{Bmatrix}B(\lambda)\\ D(\lambda)\end{Bmatrix} s^{2[\alpha_1(\lambda)+\beta_1(\lambda)]} + o(s^{2[\alpha_1(\lambda)+\beta_1(\lambda)]}) \quad as \quad s\to 0. \tag{3.22}$$

If $\alpha_1(\lambda)+\beta_1(\lambda) = 1$, *then all terms on the right side of* (3.22), *except the first, vanish. The expansions for* ρ_0' *and* ρ_0'' *are given by the obvious derivatives of* (3.22).

If the material is hyperelastic, then (3.21) reduces to $T^0_\tau(0;\lambda) \geqslant N^0_\nu(0;\lambda)$. When the strict inequality of (3.21) holds, we say that the plate is circularly reinforced at the centre. Note that the $(\tau_0(0;\lambda), N^0(0;\lambda))$ in (3.22) is a saddle point for (3.11), as figure 1 shows. At the nodes of (3.11), an inequality like the negation of (3.21) holds, but such singular points are not possible values for (τ, n) at the centre of the plate.

For those problems for which $(\tau_0(s;\lambda), N^0(s;\lambda)) \to (0,-\infty)$ as $s\to 0$, we also have

$$\nu_0(s;\lambda)\to 0, \quad T^0(s;\lambda)\to -\infty \quad \text{as} \quad s\to 0. \tag{3.23a, b}$$

Were (3.23a) not true, then ν_0 would have a positive lower bound a. Equation (3.7a) would then yield $s\tau_0(s) \geqslant as$, a contradiction. The limit (3.23b) follows from that for τ_0, from (3.23a), and from condition (2.34a).

Now we seek a representation like (3.22) when $(\tau_0(0;\lambda), N^0(0;\lambda)) = (0,-\infty)$. As figure 1 suggests, this point is a saddle point. There are a variety of ways to introduce new variables in whose corresponding phase portraits this singular point no longer is at infinity. One way to do this is to use (3.10) to replace (3.3) with the autonomous system

$$\dot\tau = \nu - \tau, \tag{3.24a}$$

$$\dot\nu = (N_\nu)^{-1}[T - N + (\tau-\nu)N_\tau], \tag{3.24b}$$

where the arguments of T, N and their derivatives are $(\tau,\nu,0,0,0)$. We know from (3.23) that (3.24) admits the solution $(\tau,\nu) = (0,0)$. We cannot linearize (3.24) about this singular point because we do not know that the indeterminate forms appearing in (3.24b) have limits as $(\tau,\nu)\to(0,0)$. We are not, however, interested in all solutions of (3.24) near $(0,0)$, but only in $(\tau_0(\cdot\,;\lambda), \nu_0(\cdot\,;\lambda))$, which satisfies our boundary value problem. We know that this solution corresponds to a trajectory leaving $(\tau,\nu) = (0,0)$ on a separatrix tangent to the line $\nu = \tau$. On this separatrix the right-hand side of (3.24b) consists of ratios of functions of any parameter for the separatrix, say ξ or τ. Thus the functions appearing on the right-hand side of (3.24b) and some of their derivatives with respect to τ and ν may have well-defined limits along the separatrix. In this case we can linearize (3.24) about $(\tau,\nu) = (0,0)$ in the direction of the separatrix. We find that this linearization is the following system for (τ_1,ν_1)

$$\dot\tau_1 = -\tau_1 + \nu_1, \tag{3.25a}$$

$$\dot\nu_1 = [\lim(T_\tau/N_\nu) + U]\tau_1 + [\lim(T_\nu - N_\tau)/N_\nu - 1 + V]\nu_1, \tag{3.25b}$$

where
$$U = \lim \frac{(\tau-\nu)\,[N_\nu N_{\tau\tau} - N_\tau N_{\tau\nu}] + (N-T)\,N_{\nu\tau}}{(N_\nu)^2}, \tag{3.25c}$$

$$V = \lim \frac{(\tau-\nu)\,[N_\nu N_{\tau\nu} - N_\tau N_{\nu\nu}] + (N-T)\,N_{\nu\nu}}{(N_\nu)^2}, \tag{3.25d}$$

where the limits are taken as $(\tau,\nu) \to (0,0)$ along the line $\nu = \tau$.

A reasonable set of sufficient conditions ensuring that these limits exist is that

$$\left.\begin{aligned} T(\tau,\nu,0,0,0) &= -\phi\Phi\tau^{-\phi-1}\nu^{-\chi} - \Gamma\tau^{-\gamma-1} + \ldots, \\ N(\tau,\nu,0,0,0) &= -\chi\Phi\tau^{-\phi}\nu^{-\chi-1} - \Delta\nu^{-\delta-1} + \ldots, \end{aligned}\right\} \tag{3.26}$$

where $\phi,\chi,\gamma,\delta,\Phi,\Gamma,\Delta$ are numbers satisfying $\phi,\chi,\gamma,\delta > 0$, $\Phi,\Gamma,\Delta \geqslant 0$, $\Phi+\Gamma > 0$, $\Phi+\Delta > 0$. By (3.26) we mean that any term accounted for by the ellipses must be negligible with respect to one of the visible terms in any limit by which $(\tau,\nu)\to 0$. This convention applies not only to T and N but also to their first two derivatives with respect to τ and ν. If (3.26) holds, then we find that $U = 0 = V$ in (3.25) and that the remaining terms in (3.25*b*) have well-defined limits. We can then solve the resulting equations for (τ_1,ν_1) and use this solution to construct the leading terms of the expansion of $\rho_0(\cdot\,;\lambda)$ about $s = 0$:

$$\rho_0(s;\lambda) = As^{\alpha_1(\lambda)+\beta_1(\lambda)} + o(s^{\alpha_1(\lambda)+\beta_1(\lambda)}) \quad \text{as} \quad s \to 0, \tag{3.27}$$

where $\alpha_1(\lambda)$ and $\beta_1(\lambda)$ are defined as limits of (3.20) as $s\to 0$. Of course, (3.27) is the analogue of (3.22). We are requiring that (3.21*a*) hold in this limiting sense so that $(\tau,\nu) = (0,0)$ has the character of a saddle point, which is necessary to ensure the existence of a solution $(\tau_0(\cdot\,;\lambda), N^0(\cdot\,;\lambda))$ of (3.11) with $(\tau_0(0,\lambda), N^0(0;\lambda)) = (0,-\infty)$.

But when (3.26) holds, it is simpler to substitute it into (3.3) and seek solutions of the form $\rho(s) = \Omega s^\omega + o(s^\omega)$, where Ω and $\omega = \alpha_1(\lambda)+\beta_1(\lambda)$ are positive constants to be determined. If we do this, we find that

$$\begin{aligned} &[\chi(\phi+\chi+1)\,(\omega-1)+\phi\omega-\chi]\,\omega^{-(\chi+1)}\Omega^{-(\phi+\chi)}\Phi s^{-(\phi+\chi)\,(\omega-1)} \\ &\qquad + [(\delta+1)\,(\omega-1)-1]\,\omega^{-(\delta+1)}\Omega^{-\delta}\Delta s^{-\delta(\omega-1)} + \Omega^{-\gamma}\Gamma s^{-\gamma(\omega-1)} + \ldots = 0. \end{aligned} \tag{3.28}$$

If, for example, $\Gamma = 0 = \Delta$, then (3.28) implies that

$$\omega \equiv \alpha_1(\lambda)+\beta_1(\lambda) = \frac{\chi+\chi(\phi+\chi+1)}{\phi+\chi(\phi+\chi+1)}. \tag{3.29}$$

The requirement that the strict form of (3.21*a*) hold is thus equivalent to the inequality that $\chi > \phi$.

These results show that the treatment of problems for which $N(0;\lambda) = -\infty$ depends upon the fine structure of the constitutive functions. Rather than seeking a stultifying exhaustiveness, we content ourselves with illustrating the main ideas by restricting our attention to the Taylor plate. For it, (3.7) reduces to

$$(s\tau') = \nu^{\#}(n), \quad (sn)' = 0. \tag{3.30}$$

The solution of (3.30) satisfying (3.8) is readily found to be

$$\tau(s) = s^{-1}\int_0^s \nu^{\#}(-ct^{-1})\,\mathrm{d}t, \quad n(s) = -cs^{-1}, \tag{3.31}$$

where the constant c satisfies

$$c = \lambda g\left(\int_0^1 \nu^{\#}(-ct^{-1})\right)\mathrm{d}t. \tag{3.32}$$

Conditions (2.19), (2.27*a*), (2.34) ensure that (3.32) has a unique positive solution for c when $\lambda > 0$. Representation (3.31) is particularly simple when $g = 1$. The phase portrait of the reduced form of (3.11) is readily found.

We now describe some further detailed properties of the solutions, which are useful in determining the spectral properties of the linearization of the full boundary value problem about the trivial solution. Let us assume the further constitutive restrictions

$$0 < T_\nu(\boldsymbol{w}), N_\tau(\boldsymbol{w}) < N_\nu(\boldsymbol{w}), T_\tau(\boldsymbol{w}) \quad \text{for} \quad \boldsymbol{w} = (\tau, \nu, 0, 0, 0). \tag{3.33}$$

Then a direct computation shows that

$$h'(n) > 0, \tag{3.34}$$

from which it follows that

$$f'(n) > 0. \tag{3.35}$$

We now show that

$$\rho_0' \equiv \nu^{\#}(\tau_0, N^0) < 1. \tag{3.36}$$

We consider the locus of points for which $\nu^{\#}(\tau, n) = 1$; it is equivalent to

$$n = N(\tau, 1, 0, 0, 0). \tag{3.37}$$

Because $N(\tau, 1, 0, 0, 0) > N(\tau, \tau, 0, 0, 0)$ for $\tau < 1$ by (2.27*a*), the curve (3.37) lies between the vertical isocline (3.12*b*) and the line $n = 0$ for $\tau < 1$. We prove (3.36) merely by showing that (3.37) lies above the horizontal isocline, and hence strictly above the curve (3.17) for $\tau < 1$. Were (3.37) to intersect the horizontal isocline at $\tau^* < 1$, then (3.13*b*) would imply that $N(\tau^*, 1, 0, 0, 0)$ must equal $T(\tau^*, 1, 0, 0, 0)$, which is inconsistent with (3.33).

We now show that

$$N^0_\lambda(s;\lambda) < 0 \quad \text{for} \quad 0 < s \leqslant 1. \tag{3.38}$$

This result is obvious when λ increases in an open interval for which $\bar{n}(\lambda)$ lies in a corresponding open interval of $\mathscr{N}$. We accordingly confine our attention to half open intervals of the λ-axis for which $\bar{n}(\lambda) \in [a, b)$ or $\bar{n}(\lambda) \in (c, d]$ where (a, b) and (c, d) are component open intervals of $\mathscr{N}^h$ and $\mathscr{N}^v$ respectively. We limit our attention to the case that $N^0(0;\lambda) > -\infty$. From (3.7), (3.8) we find that

$$s\tau'_{0\lambda} = (\nu^{\#}_\tau - 1)\,\tau_{0\lambda} + \nu^{\#}_n N^0_\lambda, \quad sN^{0\prime}_\lambda = T^{\#}_\tau\,\tau_{0\lambda} + (T^{\#}_n - 1)\,N^0_\lambda, \tag{3.39a, b}$$

where the arguments of the derivatives of $\nu^{\#}$ and $T^{\#}$ are $\tau_0(s;\lambda)$, $N^0(s;\lambda)$. The phase portrait (figure 1) shows that the initial point on a solution trajectory is unchanged, whereas the terminal point moves down the curve (3.17) as λ increases through the half-open intervals described above. Thus

$$\tau_{0\lambda}(0;\lambda) = 0, \quad N^0_\lambda(0;\lambda) = 0, \tag{3.40}$$

$$\tau_{0\lambda}(1;\lambda) < 0, \quad N^0_\lambda(1;\lambda) < 0. \tag{3.41}$$

Now $N^0_\lambda(\cdot\,;\lambda)$ cannot have a double zero at a positive value of s, for if so, the uniqueness theorem for initial value problems would imply that (3.39) has the unique solution $\tau_{0\lambda} = 0 = N^0_\lambda$, which cannot satisfy (3.41). Were $N^0_\lambda(\cdot\,;\lambda)$ to be positive at some point on $(0,1)$, then there would be an interval $[s_1, s_3] \subset (0,1)$ with $N^0_\lambda(\cdot\,;\lambda)$ being positive on $[s_1, s_3)$, having a local maximum at s_1, vanishing at s_3, and having a negative derivative at s_3. It would then follow from (3.39), (2.28), (3.33) that

$$\tau_{0\lambda}(s_1;\lambda) > 0, \quad \tau_{0\lambda}(s_3;\lambda) < 0, \quad \tau'_{0\lambda}(s_3;\lambda) > 0. \tag{3.42}$$

Then there would be an $s_2 \in (s_1, s_3)$ at which $\tau_{0\lambda}$ would have a negative local minimum, whence

$$\tau_{0\lambda}(s_2;\lambda) < 0, \quad \tau'_{0\lambda}(s_2;\lambda) = 0. \tag{3.43}$$

But then (3.39) would imply the contradiction that $0 = (\nu^{\#}_\tau - 1)\,\tau_{0\lambda} + \nu^{\#}_n N^0_\lambda > 0$ at s_2. Thus (3.38) must hold.

The direct use of phase-plane methods is not possible if T or N depend explicitly on s, as happens for plates of variable thickness or for plates with a non-homogeneous material response. Such problems can be analysed by using the homotopy invariance of the Leray–Schauder degree to construct solutions by continuation methods. The rather intricate analysis needed to support this approach was carried out by Negrón-Marrero (1985). It relies on techniques that we exploit in §6. An alternative procedure might be developed by using the ideas of the first part of §10.

4. The linearized buckling problem

We use the notation introduced in (3.19), (3.20). We also set

$$A_3(s;\lambda) \equiv \frac{\Sigma^0_\mu(s;\lambda) - M^0_\sigma(s;\lambda)}{2M^0_\mu(s;\lambda)}, \quad B_3(s;\lambda) \equiv \left[A_3(s;\lambda)^2 + \frac{\Sigma^0_\sigma(s;\lambda)}{M^0_\mu(s;\lambda)}\right]^{\frac{1}{2}}, \tag{4.1}$$

$$\alpha_3(\lambda) \equiv A_3(0;\lambda), \quad \beta_3(\lambda) \equiv B_3(0;\lambda).$$

Note that $A_3 = 0$ for hyperelastic materials. The linearization of the boundary value problem (2.8), (2.10), (2.24), (2.21), (2.23), (2.14), (2.16), (2.18) about the trivial solution is

$$\rho'_1 = \nu_1, \quad \zeta'_1 = \rho'_0(s;\lambda)\,\theta_1 + \eta_1, \quad \tau_1 = \rho_1/s, \tag{4.2}$$

$$[sN^0_\nu(s;\lambda)\,\rho'_1 + N^0_\tau(s;\lambda)\,\rho_1]' - T^0_\nu(s;\lambda)\,\rho'_1 - T^0_\tau(s;\lambda)\,\rho_1/s = 0, \tag{4.3}$$

$$H^0_\eta(s;\lambda)\,\eta_1 = -N^0(s;\lambda)\,\theta_1, \tag{4.4}$$

$$\begin{aligned}[sM^0_\mu(s;\lambda)\,\theta'_1]' - 2M^0_\mu(s;\lambda)\,A_3(s;\lambda)\,\theta'_1 + M^0_\sigma(s;\lambda)'\,\theta_1 + M^0_\mu(s;\lambda)\,[A_3(s;\lambda)^2 - B_3(s;\lambda)^2]\,\theta'_1/s \\ = s[N^0(s;\lambda) - \rho'_0(s;\lambda)\,H^0_\eta(s;\lambda)]\,\eta_1 = -sQ(s;\lambda)\,\theta_1, \end{aligned} \tag{4.5a}$$

with
$$Q(s;\lambda) \equiv N^0(s;\lambda)\,[N^0(s;\lambda)/H^0_\eta(s;\lambda) - \rho_0'(s;\lambda)], \tag{4.5b}$$

$$\rho_1(0) = 0, \quad \zeta_1(0) = 0, \quad \eta_1(0) = 0, \quad \theta_1(0) = 0, \tag{4.6}$$

$$N^0_\nu(1;\lambda)\,\rho_1'(1) + N^0_\tau(1;\lambda)\,\rho_1(1) = -\lambda g'(\rho_0(1;\lambda))\,\rho_1(1), \tag{4.7}$$
$$\eta_1(1) = 0, \quad \theta_1(1) = 0.$$

Thus this linear boundary value problem reduces to two uncoupled second-order boundary value problems for ρ_1 and θ_1, augmented by (4.2) and (4.4), which yield expressions for the remaining unknowns.

Let us multiply (4.3) by ρ_1, integrate the resulting equation by parts over $[\epsilon, 1]$, and use (4.7) to obtain

$$\lambda g'(\rho_0(1;\lambda))\,\rho_1(1)^2 + \lim_{\epsilon\to 0}\int_\epsilon^1 \left[sN^0_\nu(\rho_1')^2 + (N^0_\tau + T^0_\nu)\,\rho_1\rho_1' + \frac{T^0_\tau \rho_1^2}{s}\right] \mathrm{d}s$$
$$= -\lim_{\epsilon\to 0}\rho_1(\epsilon)\,[\epsilon N^0_\nu(\epsilon,\lambda)\,\rho_1'(\epsilon) + N^0_\tau(\epsilon;\lambda)\,\rho_1(\epsilon)]. \tag{4.8}$$

In the cases for which $N^0(0;\lambda) > -\infty$, the functions $T^0_\tau(\cdot\,;\lambda)$, $T^0_\nu(\cdot\,;\lambda)$, $N^0_\tau(\cdot\,;\lambda)$, $N^0_\nu(\cdot\,;\lambda)$ are bounded. It then follows that if $\rho_1 \in C^1([0,1])$, then the right side of (4.8) vanishes. (This conclusion holds for much weaker restrictions on ρ_1.) Conditions (2.19), (2.28) then imply that $\rho_1 = 0$.

We get the same result when $N(0;\lambda) = -\infty$ provided that we can prove that the right side of (4.8) is $\leqslant 0$. For this purpose let us restrict our attention to (3.26) with $\Gamma = 0 = \Delta$. Then (3.27) implies that near $s = 0$, equation (4.3) has the form

$$(s^{1-\gamma}\rho_1')' - Rs^{-1-\gamma}\rho_1 + \ldots = 0, \tag{4.9a}$$

$$\gamma \equiv (\alpha_1 + \beta_1 - 1)(\chi + \phi + 2) > 0, \quad R \equiv \frac{(\alpha_1+\beta_1)\,\phi}{\chi(\chi+1)}[\chi\gamma + (\alpha_1+\beta_1)(\phi+1)] > 0. \tag{4.9b}$$

The ellipsis in (4.9a) stands for terms negligible with respect to the visible terms as $s \to 0$. The solution of (4.9a) satisfying $\rho_1(0) = 0$ has the form

$$\rho_1 = \text{const.}\, s^{\frac{1}{2}(\gamma + \surd(\gamma^2+4R))} + \ldots. \tag{4.10}$$

Thus the right side of (4.8) is

$$-\text{const.}\lim_{\epsilon\to 0}\epsilon^{\surd(\gamma^2+4R)} = 0. \tag{4.11}$$

Equation (4.5) is formally self-adjoint if and only if $A_3 = 0$. If $A_3 \neq 0$, we can replace (4.5) with the equivalent formally self-adjoint equation

$$(sM^0_\mu\phi_1')' + [(M^0_\mu A_3)' + (M^0_\sigma)' - M^0_\mu B_3^2/s]\,\phi_1 + sQ\phi_1 = 0 \tag{4.12}$$

for
$$\phi_1 = \theta_1 \exp\left[-\int s^{-1}A_3(s;\lambda)\,\mathrm{d}s\right]. \tag{4.13}$$

(There are other ways to convert (4.5) into a formally self-adjoint form, as we shall see in §7. Equation (4.12) has the virtue that the coefficient of ϕ_1'' has an especially simple form.) If we regard (4.5) as a perturbation of the formally self-adjoint problem with $A_3 = 0$, then this result indicates that real eigenvalues of the self-adjoint problem are not lost in the perturbation process by becoming complex.

In keeping with (3.21) we assume that

$$\alpha_1(\lambda)+\beta_1(\lambda) > 1, \quad \alpha_3(\lambda)+\beta_3(\lambda) > 1. \tag{4.14a, b}$$

Conditions (4.14*a*, *b*) are equivalent to

$$T^0_\tau(0;\lambda)+T^0_\nu(0;\lambda) > N^0_\tau(0;\lambda)+N^0_\nu(0;\lambda), \tag{4.14c}$$

$$\Sigma^0_\sigma(0;\lambda)+\Sigma^0_\mu(0;\lambda) > M^0_\sigma(0;\lambda)+M^0_\mu(0;\lambda). \tag{4.14d}$$

If constraint (2.6) is adopted with $\omega(s,z)$ having the special but reasonable form $\zeta(s)z$, then the methods of Antman (1978, §10, especially equations (10.52) and (10.53)) show that there is a positive-valued function q such that

$$\begin{aligned}(\Sigma^0_\sigma(s;\lambda),\Sigma^0_\mu(s;\lambda),M^0_\sigma(s;\lambda),M^0_\mu(s;\lambda))\\ = q(\tau_0(s;\lambda),\nu_0(s;\lambda))\,(T^0_\tau(s;\lambda),T^0_\nu(s;\lambda),N^0_\tau(s;\lambda),N^0_\nu(s;\lambda)).\end{aligned} \tag{4.15}$$

(If ζ is prescribed, then q would just be a function of s. We adopt this more general form of q to free our analysis from too slavish an adherence to (2.6). See the discussion of this question in §10.)

When (4.15) holds, (4.14*c*) and (4.14*d*) are equivalent. It is important to note that (4.15) shows that it is unwarranted to make the attractive assumption that the terms on the left side of (4.15) are constants.

Suppose that $N^0(0;\lambda) > -\infty$ and that (4.14) holds. To determine whether (4.12) has a well-behaved set of eigenvalues, we note that (3.22) implies that $(M^0_\mu A_3)'$ and $(M^0_\sigma)'$ behave like $s^{\alpha_1+\beta_1-2}$ for s small. Note that $\alpha_1+\beta_1-2 > -1$ by (4.14). Equation (3.22) also implies that there are positive-valued functions M^+_μ, E, P, R *of* λ such that

$$M_\mu \leqslant M^+_\mu, \quad -(M^0_\mu A_3)'-(M^0_\sigma)'+M^0_\mu B^2_3/s \leqslant M^+_\mu E^2/s, \tag{4.16}$$

$$Q \geqslant \begin{cases} M^+_\mu P^2 > 0 & \text{if} \quad N^0(0;\lambda) < 0, \\ M^+_\mu R^2 s^{\alpha_1+\beta_1} & \text{if} \quad N^0(0;\lambda) = 0. \end{cases}$$

Then the Sturmian comparison theory asserts that the solutions of (4.12) oscillate more rapidly than those of

$$(s\phi')' - \frac{E^2\phi}{s} + \begin{Bmatrix} sP^2 \\ s^{1+\alpha_1+\beta_1}R^2 \end{Bmatrix}\phi = 0 \quad \text{for} \quad N^0(0;\lambda)\begin{Bmatrix} < 0 \\ = 0 \end{Bmatrix}. \tag{4.17}$$

The solutions of (4.17) that are regular at $s = 0$ are the Bessel functions

$$J_{E(\lambda)}(P(\lambda)\,s) \quad \text{when} \quad N^0(0;\lambda) < 0, \tag{4.18a}$$

$$J_{\delta(\lambda)\,E(\lambda)}(\delta(\lambda)\,R(\lambda)\,s^{1/\delta(\lambda)}) \quad \text{when} \quad N^0(0;\lambda) = 0, \tag{4.18b}$$

where $\delta(\lambda) \equiv 2/(\alpha_1(\lambda)+\beta_1(\lambda)+2)$. These functions are oscillatory for $s \in (0,\infty)$. If E, P, R, δ are such that these functions oscillate more rapidly as λ is increased, then the same is true of (4.12), and the Sturmian theory can be used in a standard way to ensure that (4.12) has real eigenvalues in various range of the positive λ-axis.

We take the view that this fact shows that given the constitutive functions N^0, M^0_μ, etc. one could in principle compute the eigenvalues for (4.12). It is not our aim here to determine detailed information about the disposition of the eigenvalues. We note, however, that such an effort can be complicated not only by the possible discontinuous dependence of $N^0, M^0_\mu, \ldots$ on λ, but also by some more subtle aspects of the material response: an important factor causing (4.18) to oscillate more rapidly with an increase in λ is the dependence of P or R, and thus of Q, on λ. From (3.4*b*), (4.5*b*), and the remarks at the end of §3, we note that $N^0(1;\lambda)^2 \to \infty$ as $\lambda \to \infty$ and we might expect the same for $N^0(s;\lambda)^2$ for $s \in (0,1]$. If H^0_η behaves appropriately, then we could expect $Q(s;\lambda) \to \infty$ as $\lambda \to \infty$. Now P behaves like Q/M^+_μ and (4.15) suggests that M^+_μ behaves like N^0_ν. Thus P behaves like $(N^0)^2/N^0_\nu$, provided H^0_η is innocuous. For a large class of functions $(\tau, \nu) \mapsto N(\boldsymbol{w})$, one can show that $(N^0)^2/N^0_\nu \to \infty$ as $\lambda \to \infty$. In this case we could expect (4.12) to have a countable infinity of unbounded eigenvalues. On the other hand, if the material is unshearable so that $H^0_\eta = \infty$ (a formal result, which can be rigorously justified), then $Q = -N^0\rho_0'$. Condition (2.34*a*) and the results of §3 then imply that Q/N^0_ν is bounded as $\lambda \to \infty$. Thus we might expect (from a comparison of (4.12) with an equation having coefficients satisfying inequalities opposite to those of (4.16)) that the eigenvalues of (4.12) are confined to a bounded interval of the λ-axis. These issues arise in the buckling of nonlinearly elastic structures with a rich enough repertoire of permissible deformations. A careful presentation of the heuristic argument of this paragraph applied to a simpler buckling problem is given by Antman & Rosenfeld (1978). Some of these notions will now be illustrated.

Rather than conducting an analogous development for the case that $N^0(0;\lambda) = -\infty$, which would rely on ideas like those used in (4.12)–(4.14), we simply discuss the Taylor plate with $g = 1$ (cf. (2.18)) and with q of (4.15) depending only on ν. Then (3.31) and (3.32) reduce (4.5) to

$$\left[\frac{sq(\nu^\#(-\lambda s^{-1}))}{\nu^\#_n(-\lambda s^{-1})}\theta_1'\right]' + \lambda[(H^0_\eta)^{-1}\lambda s^{-1} + \nu^\#(-\lambda s^{-1})]\,\theta_1 = 0, \tag{4.19}$$

which is subject to
$$\theta_1(0) = 0 = \theta_1(1). \tag{4.20}$$

For simplicity, we restrict our attention to Taylor plates for which

$$\nu^\#(n) = (1-Kn)^{-k} \quad \text{for} \quad n \leqslant 0, \tag{4.21}$$

$$q(\nu) = L\nu^{-l} \quad \text{for} \quad \nu \leqslant 1, \tag{4.22}$$

where $K, L, k > 0; l \geqslant 0$. Then (4.19) becomes

$$[s^{-k(1+l)}(K\lambda+s)^{1+k(1+l)}\theta_1']' + \lambda C[(H^0_\eta)^{-1}\lambda s^{-1} + s^k(K\lambda+s)^k]\,\theta_1 = 0, \tag{4.23}$$

where $C \equiv KkL^{-1}$.

To analyse the spectral properties of (4.23) we introduce the Prüfer transformation

$$s^{-k(1+l)}(K\lambda+s)^{1+k(1+l)}\theta_1' = r\cos\omega, \quad \theta_1 = r\sin\omega. \tag{4.24}$$

Then (4.23) and (4.20) imply that ω satisfies

$$\omega' = \frac{s^{-k(1+l)}\cos^2\omega}{(K\lambda+s)^{1+k(1+l)}} + \lambda C[(H^0_\eta)^{-1}\lambda s^{-1} + s^k(K\lambda+s)^{-k}]\sin^2\omega, \tag{4.25}$$

$$\omega(0) = 0, \quad \omega(1) = (n+1)\,\pi, \tag{4.26}$$

where $n+1$ is a positive integer. Problem (4.23) and (4.20) has a non-trivial solution θ_1 with exactly n zeros on $(0,1)$, each of which is simple, if and only if there is a value $\lambda > 0$ for which (4.25) and (4.26) has a solution. To exploit these observations, we observe that

$$F(\omega; a, b) \equiv \int_0^\omega \frac{d\zeta}{a\cos^2\zeta + b\sin^2\zeta} = \frac{n\pi}{2(ab)^{\frac{1}{2}}} + \frac{1}{(ab)^{\frac{1}{2}}}\arctan\left(\left(\frac{b}{a}\right)^{\frac{1}{2}}\tan(\omega - n\pi)\right) \quad (4.27)$$

for $\frac{1}{2}(2n-1)\pi < \omega \leqslant \frac{1}{2}(2n+1)\pi$, $n \geqslant 0, 1, 2, \ldots$.

Let us first study the case in which the Taylor plate is unshearable, so that $H^0_\eta = \infty$. Thus (4.25) implies that

$$\omega' \leqslant (K\lambda + s)^{-1}(\cos^2\omega + \lambda C\sin^2\omega). \quad (4.28)$$

Thus (4.27) yields

$$F(\omega(1); 1, \lambda C) \leqslant \ln((K\lambda+1)/K\lambda). \quad (4.29)$$

In particular, if $\omega(1) = (n+1)\pi$ (cf. (4.26)), then (4.27) reduces (4.29) to

$$[\tfrac{1}{2}(n+1)\pi]^2 \leqslant \lambda C \ln(1 + (1/K\lambda)). \quad (4.30)$$

By l'Hôpital's rule, the right side of (4.30) approaches 0 as $\lambda \to 0$ and approaches C/K as $\lambda \to \infty$. It thus follows from (4.30) that if n exceeds some threshold $\bar{n}$ (depending on the fixed parameters C, K), then there is no positive value of λ for which (4.30) is satisfied. If we assume that $\bar{n}$ is an integer, then there are consequently no eigenfunctions having more than $\bar{n}$ interior zeros. In particular, if C is small enough, then $\bar{n}$ could be taken $= -1$, in which case there are no eigenvalues at all for (4.23). It also follows from (4.30) that if (4.23) does have an eigenfunction with exactly n interior zeros, then the corresponding eigenvalues cannot accumulate at $\lambda = 0$. (We have offered no reason to preclude their accumulation at any point of $(0, \infty]$. Antman & Rosenfeld (1978) show how such accumulation points can be assigned in a simpler problem by a judicious choice of constitutive functions.)

Now we turn our attention to shearable Taylor plates for which H^0_η is a positive constant. (The treatment of problems for which H^0_η depends on ν follows the same lines.) From (4.25) we get

$$\omega' \geqslant s^{k(1+l)}\left[\frac{\cos^2\omega}{(K\lambda+1)^{1+k(1+l)}} + C(H^0_\eta)^{-1}\lambda^2\sin^2\omega\right], \quad (4.31)$$

whence we obtain

$$F(\omega(1); (K\lambda+1)^{1+k(1+l)}, C(H^0_\eta)^{-1}\lambda^2) \geqslant [1 + k(1+l)]^{-1}. \quad (4.32)$$

It then follows from (4.27) that if $k(1+l) < 1$, then $\omega(1) \to \infty$ as $\lambda \to \infty$. In this case (4.23), (4.20) has a sequence $\{\lambda_n\}$ of eigenvalues going to ∞ with corresponding eigenfunctions having exactly n interior zeros. (There may be other eigenvalues and eigenfunctions besides these.)

5. Formulation of equivalent governing nonlinear integral equations when $N^0(0;\lambda) > -\infty$

The equations of our boundary value problem are singular at $s = 0$. The nature of the singularity depends on the behaviour of M^0_μ, etc. at $s = 0$. Here we treat problems for which $N^0(0;\lambda) > -\infty$. In particular, we restrict our attention here to the case in which (4.14) holds (cf. theorem 2), the corresponding equalities having been treated by Antman (1978). The treatment of the Taylor plate, typical of problems with $N^0(0;\lambda) = -\infty$, is given in §7.

In view of Theorems 1 and 2, the functions $M^0_\mu(\cdot\,;\lambda)\ldots$ are Hölder continuous on $[0, 1]$ for fixed λ; they may depend discontinuously on λ. We obtain our integral equations by constructing inverses of differential operators modelled on those of (4.3)–(4.7).

We substitute (2.26) into (2.22), (2.23) and then carry out the differentiations on the left sides of the resulting equations. Condition (2.27a) enables us to use Cramer's rule to solve these equations for $s\nu', s\eta', s\mu' \equiv s\theta''$. We then use (2.8) and (2.9) to force these equations into a mould suggested by (4.3)–(4.7):

$$\begin{aligned}L_1(\rho-\rho_0) &\equiv [s(\rho-\rho_0)']' - 2\alpha_1(\lambda)(\rho-\rho_0)' + [\alpha_1(\lambda)^2-\beta_1(\lambda)^2]s^{-1}(\rho-\rho_0)\\ &= (\Delta_1\cos\theta-\Delta_2\sin\theta)/\Delta + (N^0-T^0+N^0_\tau(\rho_0'-s^{-1}\rho_0))/N^0_\nu\\ &\quad +[1-2\alpha_1(\lambda)](\rho-\rho_0)' - [\alpha_1(\lambda)^2-\beta_1(\lambda)^2]s^{-1}(\rho-\rho_0) - s\zeta'\theta' \equiv f_1,\end{aligned} \tag{5.1}$$

$$L_2\zeta \equiv (s\zeta')' = (\Delta_1\sin\theta - \Delta_2\cos\theta)/\Delta + \zeta' + s\rho'\theta' \equiv f_2, \tag{5.2}$$

$$\begin{aligned}L_3\theta &\equiv (s\theta')' - 2\alpha_3(\lambda)\theta' + [\alpha_3(\lambda)^2-\beta_3(\lambda)^2]s^{-1}\theta\\ &= -sQ(s;\lambda)\theta + \Delta_3/\Delta + [1-2\alpha_3(\lambda)^2]\theta' + [\alpha_3(\lambda)^2-\beta_3(\lambda)^2]s^{-1}\theta + sQ(s;\lambda)\theta\\ &\equiv -sQ(s;\lambda)\theta + f_3\end{aligned} \tag{5.3}$$

where $\alpha_1, \beta_1, \alpha_3, \beta_3$ are defined in (3.20), (4.1),

$$\Delta \equiv \det\frac{\partial(N,H,M)}{\partial(\nu,\eta,\mu)}, \tag{5.4}$$

Δ_i is the determinant obtained from Δ by replacing its ith column with

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$$

given by

$$\left.\begin{aligned}a_1 &= -sN_\tau\tau' - sN_\sigma\sigma' - N + sH\mu + T\cos\theta,\\ a_2 &= -sH_\tau\tau' - sH_\sigma\sigma' - H - sN\mu - T\sin\theta,\\ a_3 &= -sM_\tau\tau' - sM_\sigma\sigma' - M + \Sigma\cos\theta + s(\eta N - \nu H),\end{aligned}\right\} \tag{5.5}$$

$$\rho(0) - \rho_0(0;\lambda) = 0, \tag{5.6}$$

$$\begin{aligned}N^0_\nu(1;\lambda)[\rho'(1)-\rho_0'(1;\lambda)] + N^0_\tau(1;\lambda)[\rho(1)-\rho_0(1;\lambda)]\\ = -N(\boldsymbol{w}(1)) - \lambda g(\rho(1)) + N^0_\nu(1;\lambda)[\rho'(1)-\rho_0'(1;\lambda)]\\ + N^0_\tau(1;\lambda)[\rho(1)-\rho_0(1;\lambda)] \equiv b,\end{aligned} \tag{5.7}$$

$$\zeta(0) = 0, \quad \zeta'(0) = 0, \quad \zeta'(1) = 0, \tag{5.8}$$

$$\theta(0) = 0, \quad \theta(1) = 0. \tag{5.9}$$

(In (5.7) $\rho_0'(1;\lambda)$ is defined to be $(\partial/\partial s)\,\rho_0(1;\lambda)$. We adhere to this convection in the sequel.) Everywhere in f_1, f_2, f_3, b where ν and η appear, we replace them with

$$\nu = \rho' \cos\theta + \zeta' \sin\theta, \quad \eta = -\rho' \sin\theta + \zeta' \cos\theta, \tag{5.10}$$

which come from (2.8), (2.9). Thus we have reduced our boundary value problem to (5.1)–(5.9), a sixth-order system for $\rho - \rho_0, \zeta, \theta$.

We now convert this system into an equivalent system of integral equations. Green's function for L_1 subject to (5.6) and to the vanishing of the left side of (5.7) is given by

$$K_1(s,t;\lambda) \equiv \frac{1}{2\beta_1}\begin{cases}(\gamma s^{\alpha_1+\beta_1} - s^{\alpha_1-\beta_1})\, t^{\beta_1-\alpha_1} & \text{for} \quad t \leqslant s,\\ (\gamma t^{\beta_1-\alpha_1} - t^{-\alpha_1-\beta_1})\, s^{\alpha_1+\beta_1} & \text{for} \quad t \geqslant s,\end{cases} \tag{5.11a}$$

$$\gamma(\lambda) \equiv \frac{N_\tau^0(1;\lambda) + N_\nu^0(1;\lambda)\,(\alpha_1 - \beta_1)}{N_\tau^0(1;\lambda) + N_\nu^0(1;\lambda)\,(\alpha_1 + \beta_1)}. \tag{5.11b}$$

Green's function for L_3 subject to (5.9) is

$$K_3(s,t;\lambda) \equiv \frac{1}{2\beta_1}\begin{cases}(s^{\alpha_3+\beta_3} - s^{\alpha_3-\beta_3})\, t^{\beta_3-\alpha_3} & \text{for} \quad t \leqslant s,\\ (t^{\beta_3-\alpha_3} - t^{-\alpha_3-\beta_3})\, s^{\alpha_3+\beta_3} & \text{for} \quad t \geqslant s.\end{cases} \tag{5.12}$$

(In the important case that $N^0(0;\lambda) = 0$, it follows from Theorem 2 that α_3, β_3, and therefore K_3, are independent of λ.)

Let $\epsilon(\lambda)$ be a positive number to be chosen below. We define functions $(u_1, u_2, u_3) \equiv \boldsymbol{u}$ by

$$s^\epsilon u_1 \equiv L_1(\rho - \rho_0), \quad s^\epsilon u_2 \equiv L_2\,\zeta, \quad s^\epsilon u_3 \equiv L_3\,\theta. \tag{5.13a–c}$$

Note that (5.8), (5.13*b*) imply that

$$\int_0^1 s^\epsilon u_2(s)\,\mathrm{d}s = 0. \tag{5.14}$$

Assuming that $\boldsymbol{u}$ is continuous and satisfies (5.14), we can convert (5.13), (5.6)–(5.9) into an equivalent system that gives ζ and θ explicitly in terms of $\boldsymbol{u}$ and that relates $\rho - \rho_0$ to $\boldsymbol{u}$:

$$\rho - \rho_0 = G_1\,u_1 + E(\lambda)^{-1}\, b s^{\alpha_1(\lambda)+\beta_1(\lambda)}, \quad \zeta = G_2\,u_2, \quad \theta = G_3\,u_3, \tag{5.15a–c}$$

where

$$(G_1\,u_1)\,(s;\lambda) \equiv \int_0^1 K_1(s,t;\lambda)\, t^{\epsilon(\lambda)} u_1(t)\,\mathrm{d}t, \tag{5.16a}$$

$$(G_2\,u_2)\,(s;\lambda) \equiv \int_0^1 \frac{1}{\xi}\int_0^\xi t^{\epsilon(\lambda)} u_2(t)\,\mathrm{d}t\,\mathrm{d}\xi, \tag{5.16b}$$

$$(G_3\,u_3)\,(s;\lambda) \equiv \int_0^1 K_3(s,t;\lambda)\, t^{\epsilon(\lambda)} u_3(t)\,\mathrm{d}t, \tag{5.16c}$$

$$E(\lambda) \equiv N_\nu^0(1;\lambda)\,[\alpha_1(\lambda) + \beta_1(\lambda)] + N_\tau^0(1;\lambda). \tag{5.16d}$$

We now wish to solve (5.15a) for $\rho-\rho_0$ in terms of $\boldsymbol{u}$. We evaluate (5.15a) and its s-derivative at $s=1$ and then replace b from (5.7) to get the following system for $\rho(1), \rho'(1)$:

$$\begin{aligned} P_0(\rho(1),\rho'(1),\theta'(1);\lambda) &\equiv N_\nu^0(1;\lambda)(\alpha_1+\beta_1)[\rho(1)-\rho_0(1;\lambda)] \\ &\quad -N_\nu^0(1;\lambda)[\rho'(1)-\rho_0'(1;\lambda)]+\lambda g(\rho(1))+N(\rho(1),\rho'(1),0,0,\theta'(1)) \\ &= E(\lambda)(G_1u_1)(1;\lambda), \end{aligned} \tag{5.17a}$$

$$\begin{aligned} P_1(\rho(1),\rho'(1),\theta'(1);\lambda) &\equiv -N_\tau^0(1;\lambda)[\rho(1)-\rho_0(1;\lambda)] \\ &\quad -(\alpha_1+\beta_1)^{-1}N_\tau^0(1;\lambda)[\rho'(1)-\rho_0'(1;\lambda)]+\lambda g(\rho(1))+N(\rho(1),\rho'(1),0,0,\theta'(1)) \\ &= E(\lambda)(G_1u_1)'(1;\lambda), \end{aligned} \tag{5.17b}$$

$$\begin{aligned} &\partial(P_0,P_1)/\partial(\rho(1),\rho'(1)), \\ &= \begin{pmatrix} N_\nu^0(1;\lambda)(\alpha_1+\beta_1)+N_\tau(\boldsymbol{w}(1))+\lambda g'(\rho(1)) & N_\nu(\boldsymbol{w}(1))-N_\nu^0(1;\lambda) \\ N_\tau(\boldsymbol{w}(1))-N_\tau^0(1;\lambda)+\lambda g'(\rho(1)) & (\alpha_1+\beta_1)^{-1}N_\tau^0(1;\lambda)+N_\nu(\boldsymbol{w}(1)) \end{pmatrix}. \end{aligned} \tag{5.18}$$

Conditions (2.30) and (4.11) ensure that (5.18) is positive-definite. We assume this conclusion. Conditions (2.34) ensure that

$$\begin{aligned} &\{[P_0(\rho(1),\rho'(1),\theta'(1);\lambda)-P_0(1,1,\theta'(1);\lambda)][\rho(1)-1] \\ &\quad +[P_1(\rho(1),\rho'(1),\theta'(1);\lambda)-P_1(1,1,\theta'(1);\lambda)][\rho'(1)-1]\} \\ &\quad \times\{[\rho(1)-1]^2+[\rho'(1)-1]^2\}^{-\frac{1}{2}}\to\infty, \end{aligned} \tag{5.19}$$

as $\rho(1)\to 0$ or ∞ for fixed $\rho'(1)>\omega(Z)\,|\theta'(1)|$ and as $\rho'(1)\to\omega(Z)\,|\theta'(1)|$ or ∞ for fixed $\rho(1)>0$. The positive-definiteness of (5.18) and the coercivity condition (5.19) support a global implicit function theorem (based on the Brouwer degree theory) that ensures that (5.17) has a unique solution for $\rho(1)$ and $\rho'(1)$ and thus for $\rho(1)-\rho_0(1;\lambda)$ and $\rho'(1)-\rho_0'(1;\lambda)$ as three times continuously differentiable functions of $\theta'(1)$, $E(\lambda)(G_1u_1)(1;\lambda)$, $E(\lambda)(G_1u_1)'(1;\lambda)$ and as a function of λ. Let us replace this $\theta'(1)$ with $(G_3u_3)'(1;\lambda)$ from (5.15c). Then b can be expressed as a function of $(G_1u_1)(1;\lambda)$, $(G_1u_1)'(1;\lambda)$, $(G_3u_3)'(1;\lambda)$ and λ. We substitute this representation into (5.15a) to express $\rho-\rho_0$ entirely in terms of $\boldsymbol{u}$. Let us now replace ρ,ζ,θ and their derivatives wherever they appear in f_1, f_2, f_3 with their representations obtained thus from (5.15), denoting the resulting expressions by $F_1[\boldsymbol{u};\lambda]$, $F_2[\boldsymbol{u};\lambda]$, $F_3[\boldsymbol{u};\lambda]$. Then (5.1)–(5.9) is equivalent to the system of integral equations

$$u_i(s)=s^{-\epsilon(\lambda)}F_i[\boldsymbol{u};\lambda](s), \quad i=1,2, \tag{5.20a, b}$$

$$u_3(s)+s^{1-\epsilon(\lambda)}Q(s;\lambda)(G_3u_3)(s;\lambda)=s^{-\epsilon(\lambda)}F_3[\boldsymbol{u};\lambda](s). \tag{5.20c}$$

We seek classical solutions $\boldsymbol{u}$ of (5.20). They generate classical solutions of the boundary value problem via (5.15).

6. Compactness of the integral operators when $N^0(0;\lambda) > -\infty$

In this section we prove that the operators appearing on the right-hand side of (5.20) are compact from $C^0([0,1])$ to itself when $N^0(0;\lambda) > -\infty$ (and (4.14) holds). We choose $\epsilon(\lambda)$ to satisfy

$$0 < \epsilon(\lambda) < \min\{\chi(\lambda), 2[\alpha_1(\lambda)+\beta_1(\lambda)-1], 2[\alpha_3(\lambda)+\beta_3(\lambda)-1]\}, \tag{6.1}$$

where
$$\chi(\lambda) \equiv \begin{cases} \alpha_3(\lambda)+\beta_3(\lambda)+1 & \text{if } N^0(0;\lambda) < 0 \\ \alpha_1(\lambda)+\beta_1(\lambda)+\alpha_3(\lambda)+\beta_3(\lambda)+1 & \text{if } N^0(0;\lambda) = 0 \end{cases}.$$

(We can then use (3.22) to show that $s \mapsto s^{-\epsilon(\lambda)} L_1(\rho_0(\cdot,\lambda)-\rho_0(0,\lambda))(s;\lambda)$ is continuous. In view of (5.15*a*), this fact suggests that it is reasonable to seek continuous solutions $\boldsymbol{u}$ of (5.20).)

We denote the components $(\tau_0, \nu_0, 0, 0, 0)$ of $\boldsymbol{w}_0$ by $(w_1^0, \ldots, w_5^0)$. The indices i, j range from 1 to 5. When repeated twice they are summed over their range. The Mean Value Theorem implies that

$$\begin{aligned} N(\boldsymbol{w}) &= N(\boldsymbol{w}_0) + N_i(\boldsymbol{w}_0, \boldsymbol{w}-\boldsymbol{w}_0)(w_i - w_i^0) \\ &= N(\boldsymbol{w}_0) + N_{w_i}(\boldsymbol{w}_0)(w_i - w_i^0) + N_{ij}(\boldsymbol{w}_0, \boldsymbol{w}-\boldsymbol{w}_0)(w_i - w_i^0)(w_j - w_j^0), \end{aligned} \tag{6.2}$$

where
$$N_i(\boldsymbol{w}_0, \boldsymbol{w}-\boldsymbol{w}_0) \equiv \int_0^1 N_{w_i}(\boldsymbol{w}_0 + t(\boldsymbol{w}-\boldsymbol{w}_0))\, \mathrm{d}t,$$
$$N_{ij}(\boldsymbol{w}_0, \boldsymbol{w}-\boldsymbol{w}_0) \equiv \int_0^1 (1-t)\, N_{w_i w_j}(\boldsymbol{w}_0 + t(\boldsymbol{w}-\boldsymbol{w}_0))\, \mathrm{d}t.$$

The same kind of representations hold for other constitutive functions. Thus, for example, we obtain from (5.5) that

$$\begin{aligned} a_1 = &-[N_\tau^0 + N_{\tau_j}(\boldsymbol{w}_0, \boldsymbol{w}-\boldsymbol{w}_0)(w_j - w_j^0)](\rho' - \rho/s) \\ &- N_{\sigma j}(\boldsymbol{w}_0, \boldsymbol{w}-\boldsymbol{w}_0)(w_j - w_j^0)(\theta' \cos\theta - \sin(\theta/s)) - N^0 - N_\tau^0 s^{-1}(\rho-\rho_0) \\ &- N_\nu^0(\rho' \cos\theta + \zeta' \sin\theta - \rho_0') - N_{ij}(\boldsymbol{w}_0, \boldsymbol{w}-\boldsymbol{w}_0)(w_i - w_i^0)(w_j - w_j^0) \\ &+ [H_\eta^0(-\rho' \sin\theta + \zeta' \cos\theta) + H_{ij}(\boldsymbol{w}_0, \boldsymbol{w}-\boldsymbol{w}_0)(w_i - w_i^0)(w_j - w_j^0)]\, s\theta' \\ &+ [T^0 + T_\tau^0 s^{-1}(\rho-\rho_0) + T_\nu^0(\rho' \cos\theta + \zeta' \sin\theta - \rho_0') + T_{ij}(\boldsymbol{w}_0, \boldsymbol{w})(w_i - w_i^0)(w_j - w_j^0)] \\ &\times [1 + (\cos\theta - 1)] \end{aligned} \tag{6.3a}$$

$$= -[N^0 - T^0 + N_\tau^0(\rho_0' - s^{-1}\rho_0)] + (T_\nu^0 - N_\tau^0 - N_\nu^0)(\rho' - \rho_0') + T_\tau^0 s^{-1}(\rho - \rho_0) + \ldots, \tag{6.3b}$$

where the ellipsis stands for a sum of products of continuous functions of $\boldsymbol{w}-\boldsymbol{w}_0$, θ, $s\theta'$ with expressions of the form

$$\left.\begin{array}{c} (w_i - w_i^0)(w_j - w_j^0), \quad (w_j - w_j^0)(\rho' - \rho/s), \\ (w_j - w_j^0)[\theta' - \theta/s + \theta'(\cos\theta - 1) + (\sin(\theta) - \theta)/s], \\ (\rho' - \rho_0')(\cos\theta - 1) + \rho_0'(\cos\theta - 1), \quad \zeta'\theta, \quad s(\rho' - \rho_0')\theta\theta' + s\rho_0'\theta\theta', \quad s\zeta'\theta'. \end{array}\right\} \tag{6.4}$$

Note that

$$\left.\begin{aligned} &w_1-w_1^0 = s^{-1}(\rho-\rho_0), \quad w_2-w_2^0 = (\rho'-\rho_0')\cos\theta-\rho_0'(1-\cos\theta)+\zeta'\sin\theta, \\ &w_3-w_3^0 = -(\rho'-\rho_0')\sin\theta-\rho_0'\sin\theta+\zeta'\cos\theta, \quad w_4-w_4^0 = s^{-1}\theta(\sin(\theta)/\theta), \\ &w_5-w_5^0 = \theta'. \end{aligned}\right\} \tag{6.5}$$

Now a_1 occurs in (5.1) as the coefficient of

$$\varDelta^{-1}\left[\frac{\partial(H,M)}{\partial(\eta,\mu)}\cos\theta+\frac{\partial(H,M)}{\partial(\nu,\mu)}\sin\theta\right] = (N_\nu^0)^{-1}+\dots, \tag{6.6}$$

where the ellipsis stands for a sum of products of continuous functions of $\boldsymbol{w}-\boldsymbol{w}_0$ and θ with $\boldsymbol{w}-\boldsymbol{w}_0$ and θ. Thus we find that there is a considerable cancellation within f_1 so that it consists merely of the terms accounted for in (6.4) together with

$$\begin{aligned} (N_\nu^0)^{-1}(\rho'-\rho_0')\,[T_\nu^0-T_\nu^0(0;\lambda)-N_\tau^0+N_\tau^0(0;\lambda)-N_\nu^0+N_\nu^0(0;\lambda)] \\ +(N_\nu^0)^{-1}s^{-1}(\rho-\rho_0)\,[T_\tau^0-T_\tau^0(0;\lambda)]. \end{aligned} \tag{6.7}$$

It is not surprising that f_1 and likewise f_2 and f_3 have this structure because the terms accounting for the linearization and the expansion of coeffcients about $s=0$ have been incorporated into the operators L_1, L_2, L_3. (It is nevertheless necessary for our analysis to carry out the preceding exercise in order to determine the precise role of the variable s in f_1, f_2, f_3.)

THEOREM 3. *The operators taking* $\boldsymbol{u}$ *into the functions* $s\mapsto s^{-\epsilon(\lambda)}F_i[\boldsymbol{u};\lambda](s)$ *are compact from* $C^0([0,1])$ *into itself.*

Sketch of the proof. Because many of the terms of F_i have the form (6.4) we can split up the effect of the singular term $s^{-\epsilon(\lambda)}$ between each factor of the products of (6.4). Thus, for example, a typical term of $s^{-\epsilon(\lambda)}F_i[\boldsymbol{u};\lambda](s)$ is

$$[s^{-\frac{1}{2}\epsilon}(G_3\boldsymbol{u}_3)'(s;\lambda)]^2\,q(\boldsymbol{u};\lambda)(s), \tag{6.8}$$

where q is a composition of a continuous function of $\boldsymbol{w}-\boldsymbol{w}_0$, θ, $s\theta'$ with the representations for these variables in terms of $\boldsymbol{u}$ induced by (5.15). The term (6.8) corresponds to $(\theta')^2$ appearing in (6.4). Now (5.12), (5.16*c*) imply that

$$\begin{aligned} s^{-\frac{1}{2}\epsilon(\lambda)}(G_3u_3)'(s;\lambda) = {}& [(\alpha_3+\beta_3)\,s^{\alpha_3+\beta_3-1-\frac{1}{2}\epsilon(\lambda)}-(\alpha_3-\beta_3)\,s^{\alpha_3-\beta_3-1-\frac{1}{2}\epsilon(\lambda)}]\int_0^s t^{\beta_3-\alpha_3+\epsilon(\lambda)}u_3(t)\,\mathrm{d}t \\ &+(\alpha_3+\beta_3)\,s^{\alpha_3+\beta_3-1-\frac{1}{2}\epsilon(\lambda)}\int_s^1[t^{\beta_3-\alpha_3+\epsilon(\lambda)}-t^{-\alpha_3-\beta_3+\epsilon(\lambda)}]\,u_3(t)\,\mathrm{d}t. \end{aligned} \tag{6.9}$$

For u_3 confined to a bounded subset of $C^0([0,1])$, it is easily checked that (6.1) ensures that (6.9) is uniformly bounded and equicontinuous. By the Ascoli–Arzelà Theorem it follows that (6.9) and therefore (6.8) generate compact and continuous operators from $C^0([0,1])$ to itself. Indeed, the same argument shows that all the terms accounted for in (6.4) can be treated likewise.

To handle the contribution

$$\{s^{-1-\frac{1}{2}\epsilon(\lambda)}(\rho-\rho_0)\}\{s^{-\frac{1}{2}\epsilon(\lambda)}[T_\tau^0-T_\tau^0(0;\lambda)]\,(N_\nu^0)^{-1}\} \tag{6.10}$$

of a typical term of (6.7) to the right side of (5.20), we observe that

$$\begin{aligned} T^0_\tau(s;\lambda)-T^0_\tau(0;\lambda) &= T_\tau(\boldsymbol{w}_0(s;\lambda))-T_\tau(\boldsymbol{w}_0(0;\lambda)) \\ &= T_{\tau j}(\boldsymbol{w}_0(0;\lambda), \boldsymbol{w}_0(s;\lambda)-\boldsymbol{w}_0(0;\lambda))\,[w^0_j(s;\lambda)-w^0_j(0;\lambda)], \end{aligned} \tag{6.11}$$

$$\left.\begin{aligned} w^0_1(s;\lambda)-w^0_1(0;\lambda) &= A(\lambda)\,s^{\alpha_1+\beta_1-1}, \\ w^0_2(s;\lambda)-w^0_2(0;\lambda) &= (\alpha_1+\beta_1)\,A(\lambda)\,s^{\alpha_1+\beta_1-1} \end{aligned}\right\} \tag{6.12}$$

by (3.22). It follows from (6.12) that the second term in braces in (6.10) is a continuous function of s. It follows by the argument centred on (6.9) that the first term in braces in (6.10) generates a compact and continuous mapping of $\boldsymbol{u}$ from $C^0([0,1])$ to $C^0([0,1])$. The same kind of argument shows that the terms of (6.7) and related terms generate compact and continuous operators from $C^0([0,1])$ to itself.

The development we have just carried out also shows that the linear mapping from $C^0([0,1])$ into itself taking u_3 into

$$s \mapsto s^{1-\epsilon(\lambda)}Q(s;\lambda)\,(G_3 u_3)\,(s;\lambda)$$

is compact and continuous (because $\epsilon < \chi$ in (6.1)) and that

$$F_i[\boldsymbol{u};\lambda] = o(\|\boldsymbol{u};C^0([0,1]\|),$$

as $\boldsymbol{u} \to \boldsymbol{0}$. These observations together with Theorem 3 form the essential hypotheses of the global bifurcation Theorem of Rabinowitz (1971) and of associated continuation theorems of Leray–Schauder type (Rabinowitz 1973; Alexander & Yorke 1976). In applying these results to our problem, we need only account for the possiblity that the operators of (5.20) need not depend continously on λ.

To state our result we introduce

$$\Lambda^h = \{\lambda \geqslant 0 : \bar{n}(\lambda) \in \mathcal{N}^h\}, \text{ etc.,} \tag{6.13}$$

(see (3.16) and (3.18)). The continuity of $\bar{n}$, a consequence of the implicit function theorem in virtue of (2.19) and (3.18), ensures that Λ^h and Λ^v are open subsets of $(0,\infty)$. We thus have

THEOREM 4. *Let $[\lambda^-,\lambda^+)$ contain a number χ such that (χ,λ^+) is a component open interval of Λ^h and such that $(\lambda^-,\chi)\cap\Lambda^h = \varnothing$. ($N^0$ and τ_0 are continuous for $\lambda\in(\lambda^-,\lambda^+)$, see figure 1.) If κ is an eigenvalue of the linear problem (4.5) subject to $\theta_1(0) = 0 = \theta_1(1)$ (or equivalently of $u_3(s)+s^{1-\epsilon(\lambda)}Q(s;\lambda)\,(G_3u_3)\,(s;\lambda) = 0$) having odd algebraic multiplicity, and if $\kappa\in(\lambda^-,\lambda^+)$, then bifurcating from the point $(\lambda,\boldsymbol{u}) = (\kappa,\boldsymbol{0})$ in (λ^-,λ^+) is a maximal connected family $\mathscr{C}(\kappa)$ of non-trivial solution pairs of (5.20) having at least one of the following properties:* (i) *$\mathscr{C}(\kappa)$ is unbounded in $(\lambda^-,\lambda^+)\times C^0([0,1])$,* (ii) *the closure of $\mathscr{C}(\kappa)$ contains a point of the form $(\kappa^*,\boldsymbol{0})$ where κ^* is another eigenvalue of the linearized problem with $\kappa^*\in(\lambda^-,\lambda^+)$,* (iii) *the closure of $\mathscr{C}(\kappa)$ contains a point of the form $(\lambda^-,\boldsymbol{u})$ or $(\lambda^+,\boldsymbol{u})$. If $\mathscr{C}$ is any maximal connected set of solution pairs in $(\lambda^-,\lambda^+)\times C^0([0,1])$ that does not contain a point of the form $(\kappa,\boldsymbol{0})$ where κ is an eigenvalue of the linearized problem, then $\mathscr{C}$ has at least one of properties* (i), (iii), *or* (iv): *$\mathscr{C}$ contains a loop in the sense that there is an essential mapping of $\mathscr{C}$ onto a circle in $(\lambda^-,\lambda^+)\times C^0([0,1])$.*

7. Analysis of the Taylor plate

We now study the global bifurcation of solutions to the equations for the Taylor plate (cf. the last paragraph of §2). Our results are representative of those for problems for which $N^0(0;\lambda) = -\infty$. By restricting our attention to Taylor plates, we can carry out the formulation and analysis of the governing integral equations in a way that is both far more efficient and closer to the underlying mechanics than that of §§5 and 6. Either that approach or the present one can be adapted, with some effort, to handle general problems for which $N^0(0;\lambda) = -\infty$. (One approach is to embed the constitutive equations for Taylor plate into a family of constitutive equations for which $N^0(0;\lambda) = -\infty$ and then use global multi-parameter bifurcation theory; see Negrón-Marrero (1985).)

We retain (2.27a) and the relevant specializations (2.31)–(2.36). Then the mapping $(\nu,\eta,\mu)\mapsto(N(\nu,\eta,\mu),H(\nu,\eta,\mu),M(\nu,\eta,\mu))$ has an inverse $(\nu^{\#},\eta^{\#},\mu^{\#})$ with

$$\omega(Z)\,|\mu^{\#}| \leqslant \nu^{\#}, \tag{7.1}$$

$$\nu^{\#}(n,h,m)-\omega(Z)\,|\mu^{\#}(n,h,m)|\to 0 \tag{7.2}$$

as $n\to-\infty$ or as $|m|\to\infty$,

$$\nu^{\#} \text{ is even in } h \text{ and } m, \tag{7.3a}$$

$$\nu^{\#} \text{ is odd in } h \text{ and even in } m, \tag{7.3b}$$

$$\mu^{\#} \text{ is even in } h \text{ and odd in } m. \tag{7.3c}$$

There is some ambiguity in (7.2) in that it is not apparent where on the curve $\nu = \omega|\mu|$ in (ν,μ)-space the strains ν and μ end up as $n\to-\infty$ or $|m|\to\infty$ in some prescribed way. When (n,h,m) approaches an extreme along a curve, the actual values of ν,η,μ are defined as the limits (provided that they exist). We shall illustrate this fact below.

Let us replace (2.20c) with the integral of (2.23) from 0 to s. Using $\nu^{\#},\eta^{\#},\mu^{\#}$ we convert the governing equations for the Taylor plate, embodied by (2.8a), (2.9), (2.12), (2.14), (2.16), (2.20), (2.23) and (2.26), to the following system of integral equations:

$$\nu(s) = \nu^{\#}(n(s),h(s),m(s)), \tag{7.4}$$

$$\eta(s) = \eta^{\#}(n(s),h(s),m(s)), \tag{7.5}$$

$$\mu(s) = \mu^{\#}(n(s),h(s),m(s)), \tag{7.6}$$

where

$$sn(s) \equiv -\lambda g(\rho(1))\cos\theta(s), \tag{7.7}$$

$$sh(s) \equiv \lambda g(\rho(1))\sin\theta(s), \tag{7.8}$$

$$sm(s) \equiv a-\lambda g(\rho(1))\int_0^s [\nu(t)\sin\theta(t)+\eta(t)\cos\theta(t)]\,\mathrm{d}t. \tag{7.9}$$

Here the θ and $\rho(1)$ appearing on the right-hand sides of (7.7)–(7.9) are defined by

$$\theta(s) = \int_0^s \mu(t)\,\mathrm{d}t, \tag{7.10}$$

$$\rho(1) = \int_0^1 [\nu(t)\cos\theta(t) - \eta(t)\sin\theta(t)]\,\mathrm{d}t, \tag{7.11}$$

and $$a \equiv \lim_{s\to 0} sm(s) = \hat{M}(1) + \lambda g(\rho(1)) \int_0^1 [\nu(t)\sin\theta(t) + \eta(t)\cos\theta(t)]\,\mathrm{d}t \tag{7.12}$$

is defined to be the functional of ν, η, μ satisfying

$$\int_0^1 \mu^{\#}(n(t), h(t), m(t))\,\mathrm{d}t = 0. \tag{7.13}$$

Our monotonicity and coercivity conditions imply that (7.13) can be uniquely solved for a. (Note that (2.8) and (2.9) imply that the integrand in (7.12) is just $\zeta'(t)$.)

Under these definitions, (7.4)–(7.6) constitute our integral equations for the Taylor plate. The right-hand sides define mappings of

$$\left\{\nu, \eta, \mu \in C^0([0,1]) : \nu(s) > \omega(Z)\,|\mu(s)|, \int_0^1 \mu(t)\,\mathrm{d}t = 0\right\}. \tag{7.14}$$

Under mild constitutive restrictions to be imposed, we shall show that these mappings are compact self-mappings.

Example. If we adopt (2.6) with $\omega(z) = z$, if we take a three-dimensional stored-energy function Φ to have the form

$$\Phi(\nu, \eta, \mu, z) = \alpha^{-1}A(\nu - z\mu)^{-\alpha} + A(\nu - z\mu) + B\eta^2, \tag{7.15}$$

where $A > 0, B > 0, \alpha > 1$, and if we define the stored energy function for a Taylor plate by

$$W(\nu, \eta, \mu) = \int_{-Z}^{Z} \Phi(\nu, \eta, \mu, z)\,\mathrm{d}z, \tag{7.16}$$

then (2.38) yields

$$\begin{aligned} A^{-1}N(\nu, \eta, \mu) &= \int_{-Z}^{Z} [1 - (\nu - z\mu)^{-\alpha-1}]\,\mathrm{d}z \\ &= \begin{cases} 2Z + (\alpha\mu)^{-1}[(\nu + Z\mu)^{-\alpha} - (\nu - Z\mu)^{-\alpha}] & \text{for } \mu \neq 0, \\ 2Z[1 - \nu^{-\alpha-1}] & \text{for } \mu = 0, \end{cases} \end{aligned} \tag{7.17a}$$

$$B^{-1}H(\nu, \eta, \mu) = 4Z\eta, \tag{7.17b}$$

$$\begin{aligned} A^{-1}M(\nu, \eta, \mu) &= \int_{-Z}^{Z} (\nu - z\mu)^{-\alpha-1} z\,\mathrm{d}z = (\alpha\mu)^{-1} \int_{-Z}^{Z} z \frac{\partial}{\partial z}(\nu - z\mu)^{-\alpha}\,\mathrm{d}z \\ &= \begin{cases} [\alpha(\alpha-1)\mu^2]^{-1}[(\nu + Z\mu)^{-\alpha}(\nu + \alpha Z\mu) - (\nu - Z\mu)^{-\alpha}(\nu - \alpha Z\mu)] & \text{for } \mu \neq 0, \\ 0 \quad \text{for } \mu = 0. \end{cases} \end{aligned} \tag{7.17c}$$

Note that the form of (7.15) is appropriate for a Taylor plate under compression. Representations (7.17 a, c) depend continuously on μ for $\nu - Z|\mu| > 0$. Constitutive equation (7.17 a) reduces to (4.21) when $\mu = 0$ if $k^{-1} = \alpha+1$, $K = (2Z)^{-1}$. We also find that $q(\nu)$ of (4.22) reduces to $\frac{1}{3}Z^2$ (cf. (4.15)).

The inverse $(\nu^{\#}, \mu^{\#})$ of (7.17 a, b) cannot in general be expressed in closed form. Nevertheless, it is easy to verify that (7.1)–(7.3) hold. We can use (7.17) to illustrate the comment following (7.3) in the process of showing that $\nu^{\#}(n(s), h(s), m(s))$ and $\mu^{\#}(n(s), h(s), m(s))$ have well defined limits as $s \searrow 0$. Let us divide (7.17 c) by (7.17 a) and then replace $N(\nu, \eta, \mu)$ and $M(\nu, \eta, \mu)$ in the resulting equation with n and m of (7.7) and (7.9). Letting $s \to 0$ and using the fact that $\theta(s) \to 0$ as $s \to 0$, we get

$$\frac{(\alpha-1)\,a\mu}{\lambda g(\rho(1))} = \frac{(\nu+Z\mu)^{\alpha}(\nu-\alpha Z\mu)-(\nu-Z\mu)^{\alpha}(\nu+\alpha Z\mu)}{2\alpha Z\mu(\nu^2-Z^2\mu^2)^{\alpha}+(\nu-Z\mu)^{\alpha}-(\nu+Z\mu)^{\alpha}}, \tag{7.18}$$

where ν and μ are evaluated at 0. If $\mu(0) = 0$, then (7.2) implies that $\nu(0) = 0$ and vice versa. Let us suppose otherwise, that $\nu(0) > 0$ and $Z\mu(0) = \pm\nu(0)$. The substitution of these values into (7.18) implies that

$$A = \lambda g(\rho(1))\, Z \operatorname{sign} \mu(0). \tag{7.19}$$

Note that it is unlikely that the solution a of (7.13) would satisfy this condition. (Indeed, for $|\theta|$ small enough, it cannot.) Let us show that (7.19) is inconsistent with the full system (7.17 a, c), which we rewrite as

$$2AZ(\nu-Z\mu)^{-\alpha} = (\alpha-1)\,\mu M-(\nu+\alpha Z\mu)\,(N-2AZ), \tag{7.20a}$$

$$2AZ(\nu+Z\mu)^{-\alpha} = (\alpha-1)\,\mu M-(\nu-\alpha Z\mu)\,(N-2AZ). \tag{7.20b}$$

Suppose that $Z\mu(0) \to n(0) > 0$. We set

$$\epsilon(s) = \nu(s) - Z\mu(s). \tag{7.21}$$

Replacing N and M in (7.20) with n and m of (7.7) and (7.9) and using (7.19), we deduce from (7.20 a) that

$$\epsilon(s) = [AZs/\lambda g\alpha\nu(0)]^{1/\alpha} + \ldots, \tag{7.22}$$

where the ellipsis stands for terms negligible with respect to the visible term as $s \to 0$. Because $\theta(s) = \mu(0)\,s + \ldots$, we now obtain from (7.20 b) that

$$2AZ(2\nu-\epsilon)^{-\alpha} = \lambda g\, s^{-1}\epsilon(s) = \lambda g\left[\frac{AZ}{\lambda g\alpha\nu(0)}\right]^{1/\alpha} s^{(1/\alpha)-1} + \ldots. \tag{7.23}$$

As $\nu(0)$ is presumed positive, equation (7.23) yields a contradiction in the limit as $s \to 0$. Thus we deduce that if (7.17) holds, then

$$\nu^{\#}(n(s), h(s), m(s)), \quad \eta^{\#}(n(s), h(s), m(s)), \quad \mu^{\#}(n(s), h(s), m(s)) \to 0 \tag{7.24}$$

as $s \to 0$.

This conclusion is compatible with the results of §§3 and 4 for Taylor plates. (Note that the singularities in the equations for Taylor plates are such that a need not be 0 even though $\mu(0) = 0$.) Indeed, by a more complicated version of the analysis centred on (7.22) and (7.23) we may deduce from (7.20) that there are constants C_1 and C_2 depending on λ and the solution such that

$$\nu^{\#}(n(s), h(s), m(s)) = C_1 s^{1/(\alpha+1)} + \ldots, \mu^{\#}(n(s), h(s), m(s)) = C_2 s^{1/(\alpha+1)} + \ldots. \quad (7.25)$$

Let us assume that the constitutive functions $\nu^{\#}, \eta^{\#}, \mu^{\#}$, in general, enjoy the properties of this example. Specifically, we require that (7.24) hold and that (7.25) has the following generalization. The moduli of continuity of $s \mapsto \nu^{\#}(n(s), h(s), m(s))$, $\eta^{\#}(n(s), h(s), m(s))$, $\mu^{\#}(n(s), h(s), m(s))$ have bounded C^0-norms when the functions ν, η, μ occurring in the definitions of n, h, m are confined to a bounded subset of (7.14).

Under these conditions, we can invoke the Arzelà–Ascoli theorem to show that the right-hand sides of (7.4)–(7.6) define a compact self-mapping of (7.14). We thus obtain an analogue of Theorem 4.

If we add to the restrictions of (7.14) the requirements that $|\theta(s)| \leqslant \frac{1}{2}\pi$ and $\nu(s) \leqslant 1$, then we can use the methods of §10 to show that the right-hand sides of (7.4)–(7.6) still define a self-mapping of the modified (7.14). The only significance of this observation is that it justifies (7.15)–(7.17), which are meaningful only where the plate is under compression. This issue is not crucial, however, because the essential consequences of (7.15)–(7.17) are (7.24) and (7.25), which are statements about what happens near $s = 0$, and the centre of the plate is always under compression.

8. Preservation of nodal properties on branches

In this section we show that each connected set of solution pairs not containing a trivial solution has a distinctive nodal pattern for θ and that branches bifurcating from the trivial branch at simple eigenvalues inherit their nodal patterns from the corresponding eigenfunctions.

Let us outline the basic ideas, due to Crandall & Rabinowitz (1970, 1971), used to study the preservation of nodal properties. Let $\mathscr{C}$ be a connected set of pairs (λ, θ) in $[0, \infty) \times C^1([0, 1])$. Let

$$\mathscr{S}_k \equiv \{\theta \in C^1([0, 1]) : \theta(0) = 0 = \theta(1), \theta \text{ has exactly } k \text{ zeros on } (0, 1), \text{ each of which is simple}\}. \quad (8.1)$$

$\mathscr{S}_k$ is open. Its boundary

$$\partial \mathscr{S}_k \subset \{\theta \in C^1([0, 1]) : \theta \text{ as a double zero}\}. \quad (8.2)$$

Let $(\lambda^{\#}, \theta^{\#}) \in \mathscr{C}$ with $\theta^{\#} \in \mathscr{S}_k$. Then for every $(\lambda, \theta) \in \mathscr{C}$, the function $\theta \in \mathscr{S}_k$ unless there is a $(\lambda^{\flat}, \theta^{\flat}) \in \mathscr{C}$ with $\theta^{\flat}$ having a double zero. (If $\mathscr{C}$ were a curve, then this result would say that the only place on $\mathscr{C}$ that θ could change its nodal pattern is where it has a double zero.) If $\theta^{\flat}$ is the solution of a regular second-order ordinary differential equation admitting the solution $\theta = 0$, then the standard uniqueness theorem for initial value problems would imply that $\theta^{\flat} = 0$.

Our problem, however, is not regular. Moreover (4.11), (5.12), (5.15c), (5.16c) and (6.1) imply that $\theta(0) = 0 = \theta'(0)$, yet there are non-trivial solutions. We must therefore refine our approach. We now treat the case in which $N^0(0;\lambda) > -\infty$ and (4.14) holds. We first show that (5.3) behaves very much like a second-order ordinary differential equation for θ alone. We then transform this equation to one for which we can readily obtain a useful uniqueness theorem. Thus we must study Δ_3/Δ. For this purpose we get some preliminary results.

The positivity of H_η (from (2.27 a)) and inequality (2.35) imply that the function $\eta \mapsto H(\boldsymbol{w})$ has an inverse $h \mapsto \tilde{\eta}(\tau, \nu, h, \sigma, \mu)$. As $\tilde{\eta}(\tau, \nu, 0, \sigma, \mu) = 0$ by (2.31), we have

$$\tilde{\eta}(\tau, \nu, H(\boldsymbol{w}), \sigma, \mu) = \left[\int_0^1 \tilde{\eta}_H(\tau, \nu, tH(\boldsymbol{w}), \sigma, \mu)\,\mathrm{d}t\right] H(\boldsymbol{w}) \equiv \varphi(\boldsymbol{w})\, H(\boldsymbol{w}). \tag{8.3}$$

As $\theta(s)/s$, $\theta'(s) \to 0$ as $s \to 0$, we can use (5.15) to show that $N(\boldsymbol{w}(s)) \to N^0(0;\lambda)$ as $s \to 0$. Thus (2.20 a) implies that

$$\Gamma(s;\lambda) \equiv s^{-1}\left[\lambda g(\rho(1)) + \int_s^1 T(\boldsymbol{w}(t))\,\mathrm{d}t\right] \to N^0(0;\lambda) \quad \text{as} \quad s \to 0. \tag{8.4}$$

(Here and below we regard $\boldsymbol{w}, \theta$, etc., as corresponding to a solution ensured by theorem 4.) It follows from (2.20 b) and (8.3) that

$$H(\boldsymbol{w}(s)) = \Gamma(s;\lambda)\sin\theta(s), \quad \eta(s) = \varphi(\boldsymbol{w}(s))\,\Gamma(s;\lambda)\sin\theta(s). \tag{8.5}$$

Let us now study Δ_3. Using analogues of (6.2) we find that a_3 of (5.5) has the form

$$\begin{aligned} a_3 = &-(M_{\tau 4}\sin\theta/s + M_{\tau 5}\,\theta')\,(\rho' - \rho/s) - M_\sigma(\cos\theta\theta' - \sin\theta/s) - M_4\sin\theta/s - M_5\,\theta' \\ &+ (\Sigma_4\sin\theta/s + \Sigma_5\,\theta')\cos\theta + (\varphi N - \nu)\,\Gamma\sin\theta, \end{aligned} \tag{8.6}$$

that a_2 has a similar form, and that the cofactor of a_1 in Δ_3 equals

$$H_{\nu 3}\,\eta(M_{\eta 4}\,\sigma + M_{\eta 5}\,\theta') - H_\eta(M_{\nu 4}\,\sigma + M_{\nu 5}\,\theta'). \tag{8.7}$$

By treating Δ_3/Δ just as we treated Δ_1/Δ in the first part of §6, we deduce that f_3 consists of terms like those of (6.4) and (6.7). But the argument centred on (8.6) or (8.7) shows that each term in f_3 actually contains either $(\sin\theta)/s = [(\sin\theta)/\theta]\,(\theta/s)$ or θ'. By using (5.11), (5.12), (5.15), (5.16) and (6.1) we find that the coefficients of θ/s and θ' in f_3 can be written as continuous functions of s times $s^{\frac{1}{2}\epsilon(\lambda)}$. Thus (5.3) has the form

$$L_3\,\theta = -sQ(s;\lambda)\,\theta + s^{\frac{1}{2}\epsilon(\lambda)}[X(s)\,\theta/s + Y(s)\,\theta'], \tag{8.8}$$

where X and Y are continuous functions (depending on a solution of the boundary value problem, see theorem 4).

To handle the fact that $\theta(0) = 0 = \theta'(0)$, we note that near $s = 0$ the regular non-trivial solution of (4.5 a) has the form $c(s)\,s^{\alpha_3+\beta_3}$ where c is continuous on $[0,1]$ and $c(0) \neq 0$. Accordingly we might be led to set

$$\theta(s) = s^{\alpha_3(\lambda)+\beta_3(\lambda)}\chi(s), \tag{8.9}$$

and find that (8.8) is equivalent to

$$(s^{2\beta_3+1}\chi')' = -s^{2\beta_3+2}Q\chi/s + s^{2\beta_3+\frac{1}{2}\epsilon}\{[X + (\alpha_3+\beta_3)\,Y]\,\chi/s + Y\chi'\}. \tag{8.10}$$

If we substitute (8.9) into (5.15*c*), we readily obtain representations for χ and χ', which show that χ and $s^{\alpha_3+\beta_3-\epsilon}\chi'$ are continuous. Thus χ' is continuous if $\epsilon \geqslant \alpha_3+\beta_3$. This inequality is compatible with (6.1) if

$$\alpha_3+\beta_3 \geqslant 2, \quad 2\alpha_1-\alpha_3+2\beta_1-\beta_3 \geqslant 2,$$

the second inequality being equivalent to the first when (4.12) holds. Because we merely require that $\alpha_3+\beta_3 > 1$ and because the failure of χ' to be continuous would prove troublesome, we replace (8.9) with

$$\theta(s) = s^{\alpha_3(\lambda)+\beta_3(\lambda)-1}\psi(s), \tag{8.11}$$

so that (5.15*c*) and (8.8) are equivalent to the less elegant relations:

$$\psi = (s-s^{-2\beta_3+1})\int_0^s t^{\beta_3-\alpha_3+\epsilon}u_3(t)\,\mathrm{d}t + s\int_s^1 (t^{\beta_3-\alpha_3+\epsilon}-t^{-\alpha_3-\beta_3+\epsilon})\,u_3(t)\,\mathrm{d}t, \tag{8.12}$$

$$(s^{2\beta_3-1}\psi')' + (1-2\beta_3)\,s^{2\beta_3-2}\psi/s = -s^{2\beta_3-1}Q\psi + s^{2\beta_3-2+\frac{1}{2}\epsilon}[Z\psi/s+Y\psi'], \tag{8.13}$$

where $Z \equiv X+(\alpha_3+\beta_3-1)\,Y$.

As u_3 is continuous, we deduce from (8.12) that $\psi\in C^1([0,1])\cap C^2((0,1])$ and furthermore that $|\psi''|, |(\psi/s)'| \leqslant \text{const.}\, s^{\epsilon-\alpha_3-\beta_3}$. Thus ψ'', $(\psi/s)'$ are integrable and ψ', (ψ/s) are absolutely continuous when $\epsilon \geqslant \alpha_3+\beta_3-1$. This inequality is compatible with (6.1) when $\alpha_3+\beta_3 \geqslant 1$ (which we assume) and

$$2\alpha_1-\alpha_3+2\beta_1-\beta_3 \geqslant 1. \tag{8.14}$$

In light of the comments of the paragraph containing (8.11), we henceforth assume (8.14).

THEOREM 5. *Let θ correspond to a solution of the boundary value problem and let ψ be defined by* (8.11). *If there is an $a\in[0,1]$ such that $\psi(a) = 0 = \psi'(a)$, then $\psi = 0$.*

Proof. If $a\in(0,1]$, then the result follows from the standard uniqueness theorems for regular initial value problems. Thus let $a = 0$. By using variation of constants we find that (8.13) subject to the initial conditions

$$\psi(0) = 0 = \psi'(0) \tag{8.15}$$

is equivalent to

$$2\beta_3\,\psi(s) = \int_0^s (st^{1-2\beta_3}-s^{1-2\beta_3}t)\,t^{2\beta_3}\left\{-\frac{Q(t)\,\psi(t)}{t}+t^{-2+\frac{1}{2}\epsilon}\left[\frac{Z(t)\,\psi(t)}{t}+Y(t)\,\psi'(t)\right]\right\}\mathrm{d}t, \tag{8.16}$$

whence we compute that

$$|[\psi(s)/s]'|+|\psi''(s)| \leqslant C(s+s^{\frac{1}{2}\epsilon-1}) \sup_{0\leqslant t\leqslant s} [|\psi(t)/t|+|\psi'(t)|], \tag{8.17}$$

where C is a constant. (The meaning of C can vary from appearance to appearance.) Let

$$\boldsymbol{v}(s) \equiv (\psi(s)/s, \psi'(s)), \quad v(s) \equiv |\boldsymbol{v}(s)| \equiv |\psi(s)/s|+|\psi'(s)|. \tag{8.18}$$

As ψ/s and ψ' are absolutely continuous, it follows that $|\boldsymbol{v}|' \leqslant |\boldsymbol{v}'|$ so that (8.17) yields

$$v'(s) \leqslant Cs^{\kappa-1}y(s), \quad \kappa \equiv 1+\min(1, \tfrac{1}{2}\epsilon-1) > 0, \quad y(s) \equiv \sup_{0\leqslant t\leqslant s} v(t). \tag{8.19}$$

As (8.15) implies that $v(0) = 0$, we can integrate (8.19) to obtain

$$v(s) \leqslant Cs^{\kappa}y(s) \leqslant CS^{\kappa}y(S) \quad \text{for} \quad 0 \leqslant s \leqslant S, \tag{8.20}$$

whence

$$y(S) \leqslant CS^{\kappa}y(S) \quad \text{for} \quad S\in[0,1]. \tag{8.21}$$

Now choose $S > 0$ so small that $CS^{\kappa} < 1$. Then (8.22) implies that $y(S) = 0$ so that $\psi = 0$ on $[0, S]$ and thus on all of $[0, 1]$.

Remark. The corresponding proof of Antman (1978) is faulty. Straightforward modifications of the proof just given correct it. Alternatively, the analogue of (8.8), which is more specific than the equation used by Antman, can be attacked directly by a uniqueness theorem of Kamke type.

Let $\bar{\lambda}$ be an eigenvalue of the linearized eigenvalue problem and let $\bar{\boldsymbol{u}}$ be the corresponding eigenfunction (in the equivalent formulation of the problem as an integral equation). In particular, the eigenfunction θ_1 is related to $\bar{u}_3$ by (5.15*c*). Note that $\bar{u}_3$ satisfies the linearization of (5.20*c*), but that this linearization is not obtained from (5.20*c*) by discarding $s^{-\epsilon}F_3$: there are linear terms hidden in F_3 as (5.3) shows.

The work of Crandall & Rabinowitz (1970, 1971) can be applied to our problem to yield

Lemma 1. *Let $\bar{\lambda}$ be an eigenvalue of the linearized problem with (the generic property of having) algebraic multiplicity* 1. *Let $\bar{\boldsymbol{u}}$ be the corresponding normalized eigenfunction and let $\boldsymbol{P}(\tilde{\lambda})$ project $C^0([0,1])$ onto span $(\bar{\boldsymbol{u}})$. Then there is a number $a > 0$ and there are continuous functions*

$$\kappa: [-a, a] \to \mathbb{R}, \quad \boldsymbol{v}: [-a, a] \to [\boldsymbol{I}-\boldsymbol{P}(\bar{\lambda})]\, C^0([0,1])$$

with $\kappa(0) = 0$, $\boldsymbol{v}(0) = \boldsymbol{0}$ such that $(\bar{\lambda}+\kappa(t), t[\bar{\boldsymbol{u}}+\boldsymbol{v}(t)])$ is a solution pair of (5.20). *Moreover, there is a neighbourhood $\mathscr{E}$ of $(\bar{\lambda}, \boldsymbol{0})$ such that if $(\lambda, \boldsymbol{u})$ is a solution pair of* (5.20) *lying in $\mathscr{E}$, then either $\boldsymbol{u} = \boldsymbol{0}$, or else there is a $t\in[-a, a]$ such that $(\lambda, \boldsymbol{u}) = (\bar{\lambda}+\kappa(t), t[\boldsymbol{u}+\boldsymbol{v}(t)])$.*

Let ψ_1 correspond to the eigenfunction θ_1 through (8.11) or equivalently correspond to $\bar{u}_3$ by (8.12). From (8.12) and lemma 1 we conclude that there is a continuously differentiable function $[-a, a]\times[0,1] \ni (t, s) \mapsto \chi(t, s) \in \mathbb{R}$ such that on the non-trivial solution branch bifurcating from $(\bar{\lambda}, \boldsymbol{0})$

$$(\lambda, \psi) = (\bar{\lambda}+\kappa(t),\, t[\psi_1(\cdot)+\chi(t, \cdot)]). \tag{8.22}$$

As $\psi \in C^1([0, 1])$, we readily deduce (cf. Crandall & Rabinowitz 1970, 1971) the following.

Theorem 6. *Let (λ^-, λ^+) be an interval with the properties described in Theorem* 4. *Let $\bar{\lambda}\in(\lambda^-, \lambda^+)$ and let $\bar{\lambda}$ be an eigenvalue of the linearized problem with algebraic multiplicity* 1. *Then $\psi = s^{1-\alpha_3-\beta_3}\theta$ on the non-trivial branch bifurcating from $(\bar{\lambda}, \boldsymbol{0})$ inherits its nodal pattern from ψ_1 and preserves that pattern along the branch as long as $\lambda\in(\lambda^-, \lambda^+)$.*

These nodal patterns are thus preserved globally when the trivial problem and thus the full problem have solutions depending continuously on λ. When the solution of the trivial problem has a discontinuity at $\lambda = \xi \in (0, \infty)$, then the bifurcation diagram may have the form shown in figure 2. In the next section we shall show that the branch $\mathscr{b}$ can have the same nodal pattern as $\mathscr{a}$, despite their separation.

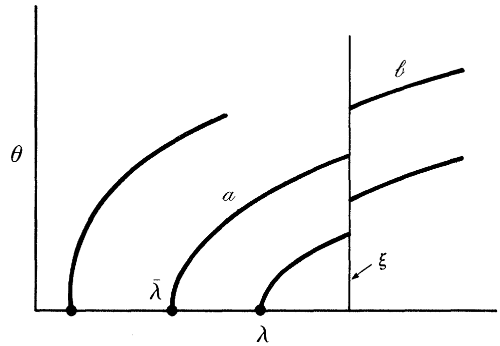

FIGURE 2. Typical bifurcation diagram when the problem has a discontinuity at $\lambda = \xi$. The ordinate corresponds to some convenient functional of θ. The branch $\mathscr{a}$ has a nodal pattern that it inherits from the eigenfunction of the linearized problem corresponding to $\bar{\lambda}$.

We now study the preservation of nodal structure for Taylor plates, which are representative of plates for which $N^0(0;\lambda) = -\infty$. We set

$$J(s) = sm(s). \tag{8.23}$$

Then the problem (7.4)–(7.10) is equivalent to

$$\theta'(s) = \mu^{\#}(-s^{-1}\lambda g(\rho(1))\cos\theta(s), s^{-1}\lambda g(\rho(1))\sin\theta(s), s^{-1}J(s)), \tag{8.24}$$

$$J'(s) = -\lambda g(\rho(1))\,[\nu^{\#}\sin\theta(s) + \eta^{\#}\cos\theta(s)], \tag{8.25}$$

together with the boundary conditions $\theta(0) = 0 = \theta(1)$. In (8.25) the arguments of $\nu^{\#}$ and $\eta^{\#}$ are those of $\mu^{\#}$ in (8.24).

In studying the nodal properties of θ we again face the difficulty that $\theta(0) = 0 = \theta'(0)$ as a consequence of (7.25). Rather than introduce a new variable by an analogue of (8.11) (which would be an effective procedure), we pursue an alternative approach, which is far more efficient for Taylor plates and which more directly exploits the constitutive properties: we study the nodal properties of J.

From (7.3b, c) and Taylor's Theorem, we obtain

$$\eta^{\#}(n,h,m) = \left[\int_0^1 \eta_h^{\#}(n,th,m)\,\mathrm{d}t\right]h \equiv \overline{\eta_h}(n,h,m)\,h, \tag{8.26a}$$

$$\mu^{\#}(n,h,m) = \left[\int_0^1 \mu_m^{\#}(m,h,tm)\,\mathrm{d}t\right]m \equiv \overline{\mu_m}(n,h,m)\,m, \tag{8.26b}$$

with $\overline{\eta_h}$ and $\overline{\mu_m}$ positive by (2.27). We study nodal properties only for

$$|\theta(s)| \leqslant \tfrac{1}{2}\pi. \tag{8.27}$$

Then (8.25), (8.26a) imply that $J'(s) = 0$ if and only if $\theta(s) = 0$. Thus J can change its nodal properties only where θ and J have simultaneous zeros (provided (8.27) holds). If θ and J have a simultaneous zero at $s_0 > 0$, then $\theta = 0 = J$ by standard uniqueness theory. Thus to get an analogue of theorem 6, we need only prove an analogue of theorem 5, namely that the only solution of (8.24), (8.25) satisfying

$$\theta(0) = 0 = J(0) \tag{8.28}$$

is the trivial solution.

Suppose that (7.17) holds. Then by the same kind of analysis of (7.20) that yielded (7.25) we find that

$$\mu_m^{\#}(n(s), h(s), m(s)) = Cs^{\frac{\alpha+2}{\alpha+1}} + \ldots, \tag{8.29}$$

and that $\overline{\mu_m}$ has the same character. (A similar relation holds for $\eta_h^{\#}$ and $\bar{\eta}_h$.) Thus we deduce from (8.24)–(8.26) that there is a positive number C (depending on the solution and λ) such that

$$|\theta|' \leqslant C|J|, \quad |J|' \leqslant C|\theta|. \tag{8.30a, b}$$

By standard methods of differential inequalities, we find that the only functions satisfying (8.28), (8.30) are trivial. The full force of (8.29) was not needed to produce this desired conclusion. It would have sufficed if the right side of (8.29) were replaced with Cs^{γ} with $\gamma > 0$. (In this case (8.30b) would be replaced with $|J|' \leqslant Cs^{\gamma-1}|\theta|$.)

Thus *for a large class of Taylor plates, the only place where the nodal properties of J can change on a connected branch in* (8.27) *is at the trivial solution. Moreover, J inherits its nodal properties from eigenfunctions of the linearized problem corresponding to eigenvalues of algebraic multiplicity* 1.

9. Preservation of nodal properties across gaps

We now focus our attention on figure 2. We wish to obtain conditions ensuring that the branch $\mathscr{b}$ has the same nodal properties as branch $\mathscr{a}$, despite their disconnectedness. Our basic idea is to embed our constitutive functions into a family continuously parametrized by a real variable q with our actual functions corresponding to $q = 1$ and with a set of constitutive functions for isotropic materials corresponding to $q = 0$. We thus obtain a two-parameter bifurcation problem to which we could conceivably apply global multiparameter continuation and bifurcation theory (see, Alexander & Yorke 1976; Alexander & Antman 1981; Fitzpatrick *et al.* 1983; Ize *et al.* 1986). We should hope that the disconnected branches $\mathscr{a}$, $\mathscr{b}$ of figure 2 could be connected to a solution branch for the isotropic plate, which is known to be connected (cf. Antman 1978), by a 'sheet' of solution pairs $(\boldsymbol{u}, \lambda, q)$ of our two-parameter problem. In this case $\mathscr{a}$ and $\mathscr{b}$ would have the same nodal properties (as the branch for the isotropic plate) by virtue of the analysis of §8.

Let us note that because we study only axisymmetric solutions, the isotropy conditions for plates imply that (Antman 1978):

$$\left.\begin{aligned} T(\tau,\nu,0,\sigma,\mu) &= N(\nu,\tau,0,\mu,\sigma),\\ \Sigma(\tau,\nu,0,\sigma,\mu) &= M(\nu,\tau,0,\mu,\sigma),\\ H(\tau,\nu,0,\sigma,\mu) &= H(\nu,\tau,0,\mu,\sigma).\end{aligned}\right\} \tag{9.1}$$

We adopt (9.1) as our definition of isotropy.

We now show that there are subtle difficulties to be overcome in transforming these intuitive notions into hard mathematics, but that the essential ideas survive the transformation. We first study the construction of the one-parameter family of constitutive functions. The mathematically obvious choice of such a family is

$$T(\boldsymbol{w},q) = qT(\boldsymbol{w}) + (1-q)\,\bar{T}(\boldsymbol{w}), \text{etc.}, \tag{9.2}$$

where $\bar{T}$, etc. are constitutive functions for an isotropic plate. The isoclines for the phase portrait for the equations for the trivial equilibrium state of our anisotropic plate are given by $\tau = h(n), \tau = v(n)$. We denote the corresponding isoclines for (9.2) by

$$\tau = h(n;q), \quad \tau = v(n;q) \quad \text{with} \quad h(n;1) = h(n), \quad v(n;1) = v(n). \tag{9.3}$$

For illustrative purposes, we suppose that the roles of $h(n)$ and $v(n)$ in figure 1 are switched. Then the direction of the arrows on the curve $\tau = f(n)$ is reversed. In this case the trivial solutions jumps as λ crosses the value $-a/g(f(a))$ where the curve $n = -\lambda g(\tau)$ passes through $(f(a), a)$. This value of λ corresponds to ξ of figure 2. We refer to the resulting phase portrait as the reversal of figure 1. It has the same critical points as figure 1.

Let us choose $\bar{N}, \bar{T}$ so that the critical points of the (reversal of) phase portrait figure 1 correspond to trivial states for the isotropic plate. As (3.12), (3.13) and (9.1) imply, this choice is effected by requiring these critical points to lie on the curve $n = \bar{N}(\tau,\tau)$ of (the reversal of) figure 1. Then we can expect (9.2) to have the isoclines shown in figure 3. By following the arguments of §3, we find that the range of the trivial solutions $\tau_0(\cdot\,;\lambda,q), N^0(\cdot\,;\lambda,q)$ for any fixed value of λ does not change at all for $q \in (0,1]$. Consequently, as $q \to 0$, the functions $\tau_0(\cdot\,;\lambda,q)$ and $N^0(\cdot\,;\lambda,q)$ cannot approach the constant values they must have for an isotropic material. Thus the governing equations for a plate of material (9.2) with $q > 0$ depend discontinuously on λ; moreover, the size of the discontinuities does not go to zero as $q \to 0$. We can therefore expect that the bifurcation diagram in (θ,λ,q)-space for the family of materials (9.1) has the character of that of figure 4. However nice the sheets of solutions are, we have no grounds to expect them to touch a non-trivial solution branch for the isotropic plate and thus to expect a and $\mathscr{b}$ of figure 2 to have the same nodal properties.

Thus we cannot achieve our goals with (9.2). Instead of (9.2) we must require that as $q \to 0$ the ranges of $\tau_0(\cdot\,;\lambda,q), N^0(\cdot\,;\lambda,q)$ reduce to single values. Let us again choose $\bar{N}, \bar{T}$ so that the singular points in the (reversal of) phase portrait figure 1 for the trivial solutions of the anisotropic plate correspond to trivial equilibrium states for the isotropic plate. A glance at the equations for the latter shows that

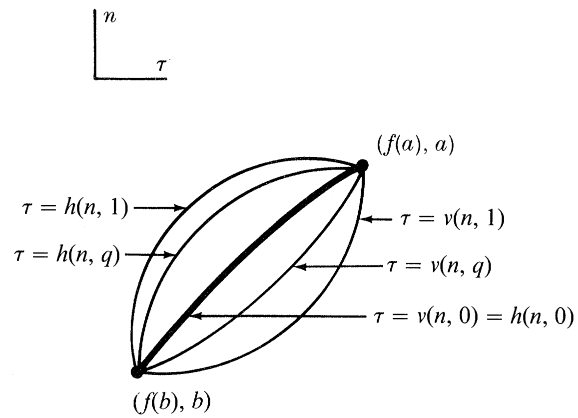

FIGURE 3. The family of isoclines corresponding to (9.3) when the reversal of figure 1 holds. Here $q \in (0,1)$. This figure corresponds to that part of the reversal of figure 1 for $b \leqslant n \leqslant a$. In this figure we have assumed that $\bar{N}, \bar{T}$ are such that $(f(a), a)$ and $(f(b), b)$ correspond to trivial states for the isotropic plate.

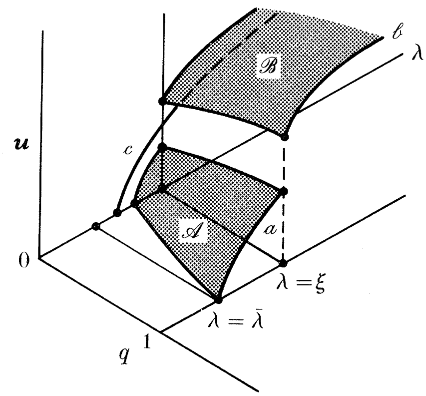

FIGURE 4. Bifurcation diagram for the material of (9.2). $\boldsymbol{u}$ represents any convenient property of the solution $\boldsymbol{u}$, e.g. $\max|\theta(s)|$. The branches $a, \mathscr{b}$ on the plane $q=1$ correspond to those of figure 2. τ is the branch for an isotropic plate. a and b are the edges of the sheet $\mathscr{A}, \mathscr{B}$. For the naïve homotopy (9.2), we cannot expect $\mathscr{A}, \mathscr{B}, c$ to touch one another.

it is an easy matter to construct $\bar{N}, \bar{T}$ thus. We wish to construct the family $(\boldsymbol{w}, q) \mapsto T(\boldsymbol{w}; q)$, etc. so that the corresponding isoclines have the behaviour illustrated in figure 5.

We first show how to construct the embedding appropriate to trivial solutions, i.e. we construct continuous functions $q \mapsto \tilde{T}(\tau, \psi; q), \tilde{N}(\tau, \psi; q)$ so that

$$\tilde{T}(\tau, \nu; 0) = \bar{T}(\tau, \nu, 0, 0, 0), \quad \tilde{T}(\tau, \nu; 1) = T(\tau, \nu, 0, 0, 0), \text{etc.} \tag{9.4}$$

We restrict our attention to solutions satisfying the bounds

$$\tau(s) \geqslant \epsilon > 0, \quad \nu(s) \geqslant \epsilon > 0 \quad \forall s \in [0,1], \tag{9.5}$$

where ϵ is prescribed, so we need only concern ourselves with constitutive functions defined on the corresponding region of (τ, ν)-space. This restriction

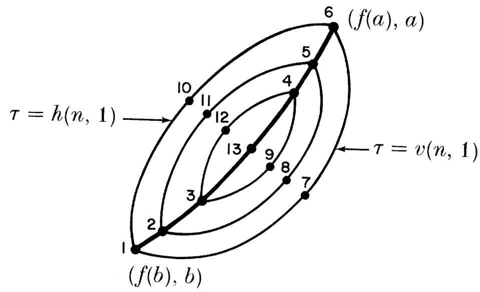

FIGURE 5. The desired nesting of isoclines when the reversal of figure 1 holds for $q = 1$. If $0 < q_1 < q_2 < 1$, then $\tau = v(n) = v(n, 1)$ corresponds to path 1, 7, 6; $\tau = v(n; q_2)$ to path 1, 2, 8, 5, 6; $\tau = v(n; q_1)$ to path 1, 3, 9, 4, 6; and $\tau = v(n, 0) = h(n, 0)$ to path 1, 13, 6. Similarly $\tau = h(n, q_2)$ corresponds to path 1, 2, 11, 5, 6. (The common limit $\tau = h(n, 0) = v(n, 0)$ as $q \to 0$ need not be related to $\tau = f(n)$.)

means that we do not fully treat cases in which the curve $n = -\lambda g(\tau)$ intersects $\tau = f(n)$ at a value of n in an unbounded component of $\mathcal{N}^v$ (cf. figure 1).

To fix ideas let us set $F(\tau, \nu; q) = \tilde{T}(\tau, \nu; q) - \bar{T}(\tau, \nu)$. By our requirement that the critical points of (the reversal of) figure 1 correspond to trivial equilibrium states of the isotropic plate we know that $F(c, c; 1) = 0$ when $\nu = c = \tau$ defines such a state. We must contrive a function F such that the set $\mathscr{E}_q \equiv \{(\tau, \nu): \epsilon \leqslant \tau, \nu \leqslant 1, F(\tau, \nu; q) = 0\}$ has the properties that $[\epsilon, 1] \times [\epsilon, 1] = \mathscr{E}_0 \supset \mathscr{E}_{q_1} \supset \mathscr{E}_{q_2}$ if $0 \leqslant q_1 \leqslant q_2$, and that the area of $\mathscr{E}_q$ (is defined) and is a continuous, decreasing function of q. If $\tilde{N}$ is defined analogously, then figure 5 is valid.

In figure 6, we illustrate the graph of $F(\cdot, \cdot; 1)$ and a typical section of it. We define a $q \mapsto F(\cdot, \cdot; q)$, which flattens this graph as $q \mapsto 0$, by showing how to flatten this section. The steps are illustrated in figure 7. Formally, for $k \geqslant 1$, we set

$$F(k^{-1}\nu, \nu; q) = F(k^{-1}\nu, \nu; 1) - 2(1-q)\left[F(k^{-1}b, b, 1) + \frac{F(k^{-1}a, a, 1) - F(k^{-1}b, b, 1)}{a-b}(\nu - b)\right], \quad (9.6)$$

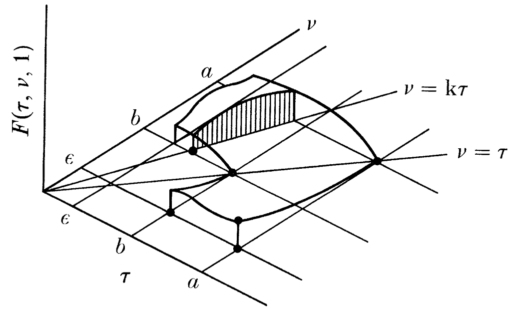

FIGURE 6. The graph of $F(\cdot, \cdot; 1)$ over the L-shaped region $\{(\tau, \nu): \epsilon \leqslant \tau \leqslant b, a \leqslant \nu \leqslant b\} \cup \{(\tau, \nu): a \leqslant \tau \leqslant b, \epsilon \leqslant \nu \leqslant b\}$. The area under the section of this graph with the typical plane $\nu = k\tau$ containing the line $\tau = 0 = \nu$ is shaded. We have represented this graph as lying *above* the (τ, ν)-plane only for illustrative convenience.

when $\frac{1}{2} \leqslant q \leqslant 1$,

$$F(k^{-1}\psi, \nu; q) = 0 \quad \text{for} \quad b \leqslant \nu \leqslant \beta(q), \quad \alpha(q) \leqslant \nu \leqslant a, \quad \text{when} \quad 0 \leqslant q \leqslant \tfrac{1}{2},$$

$$F(k^{-1}\nu, \nu; q) = F(k^{-1}\nu, \nu; \tfrac{1}{2}) - (1-2q)\Big[F(k^{-1}\beta(q), \beta(q), \tfrac{1}{2}) + \frac{F(k^{-1}\alpha(q), \alpha(q), \frac{1}{2}) - F(k^{-1}\beta(q), \beta(q), \frac{1}{2})}{\alpha(q)-\beta(q)}(\nu - \beta(q))\Big]$$

for $\beta(q) \leqslant \nu \leqslant \alpha(q)$ when $0 \leqslant q \leqslant \frac{1}{2}$,

$$\alpha(q) = a - (\tfrac{1}{2} - q)(a-b) \quad \text{for} \quad 0 \leqslant q \leqslant \tfrac{1}{2},$$

$$\beta(q) = b + (\tfrac{1}{2} - q)(a-b) \quad \text{for} \quad 0 \leqslant q \leqslant \tfrac{1}{2}.$$

The adjustments for $k \leqslant 1$ are immediate. To avoid any trouble with the fact that the function in figure 7*c* is not differentiable, we finally smooth out the rough spots locally by a standard mollification process. In this way we ensure that F is continuous and that $F(\cdot, \cdot; q)$ inherits the smoothness of T and $\bar{T}$. We construct $\tilde{N}$ by an identical process.

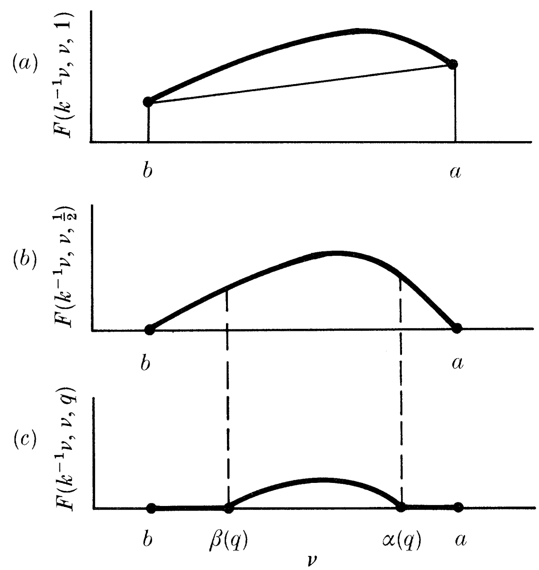

FIGURE 7. Flattening of the shaded section of figure 6 in a way that produces figure 5.

We now replace (9.2) with

$$\left.\begin{aligned} T(\mathbf{w}; q) &= \tilde{T}(\tau, \nu; q) + q[T(\mathbf{w}) - T(\tau, \nu, 0, 0, 0)] + (1-q)[\bar{T}(\mathbf{w}) - \bar{T}(\tau, \nu, 0, 0, 0)], \\ H(\mathbf{w}; q) &= qH(\mathbf{w}) + (1-q)\bar{H}(\mathbf{w}), \ldots, \end{aligned}\right\} \quad (9.7)$$

with $N(\mathbf{w}; q)$ defined as $T(\mathbf{w}; q)$ and with $\Sigma(\mathbf{w}; q)$ and $M(\mathbf{w}; q)$ defined as $H(\mathbf{w}; q)$. The embedding (9.7) produces figure 5. This figure implies that if the trivial solution for the anisotropic plate has a jump at a critical value of λ, then the use of (9.7) halves the size of the jump at $q = 1$, and then as $q \to 0$ separates the values of λ at which the jumps occur while diminishing the size of the two resulting jumps. It is intuitively reasonable to attribute the same kind of behaviour of discontinuities to the bifurcation diagram. In this case, we should obtain something like figure 8 in place of the unsatisfactory figure 4. When figure 8 is valid, we find that the branches a and b have the same nodal properties because they can be connected by branches e, c, h, which contain no trivial solutions.

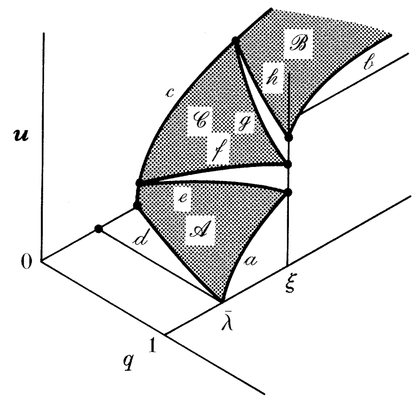

FIGURE 8. Bifurcation diagram for material of (9.7). $\mathscr{A}, \mathscr{B}, \mathscr{C}$ are solution sheets with f lying above e and with h lying above g. c is a branch for an isotropic plate.

There are, however, certain reservations to be overcome before we can embrace figure 8 as an accurate depiction of the bifurcation process: (i) We have no assurance that the branches e and f ever touch c as shown. Branch e, for example, could end on the eigencurve d of the linearization of the two-parameter problem about the trivial solution. In this case a cannot be connected to b so as to ensure that they have the same nodal properties. (ii) A careful reading of the references on global continuation and bifurcation theory cited in the first paragraph of this section shows that there is no warrant to presume that the solution sheets $\mathscr{A}, \mathscr{B}$, $\mathscr{C}$ are surfaces; they are merely guaranteed to be connected sets (not necessarily maximal) each point of which has Lebesgue dimension 2. Under modest restrictions on a and b (of a kind to be described below) we can construct such sheets $\mathscr{A}$ and $\mathscr{B}$ containing a and b, but we have no assurance that these sheets also contain c. These sheets could become unbounded in $\boldsymbol{u}$ at a positive value of q for each fixed λ or they could be folded over, never touching the plane $q = 0$, but each intersecting the plane $q = 1$ on disconnected branches.

We could show that such possibilities cannot occur by producing very sharp estimates for solutions. Some very useful estimates are obtained by Antman & Negrón-Marrero (1989), but they are inadequate for the preclusion of behaviour qualitatively different from that of figure 8. Moreover, the complexity and generality of our problem would make the derivation of the requisite estimates a formidable task.

We finesse these difficulties by the simple expedient of constructing the sheets

$\mathscr{A}, \mathscr{B}, \mathscr{C}$, not as continuations of a and b, but as continuations of c. We accordingly sacrifice the assurance that $\mathscr{A}$ contains a and $\mathscr{B}$ contains b. Nevertheless, for each $q_0 > 0$, the intersection of the sheets $\mathscr{A}, \mathscr{B}, \mathscr{C}$ with the plane $q = q_0$ is the bifurcation diagram for an anisotropic plate. Antman & Negrón-Marrero (1989) furnish estimates ensuring that solutions $\boldsymbol{u}$ cannot blow up in certain large regions of solution-parameter space. The global continuation theory of Alexander & Yorke (1976) then implies that in these regions the sheets $\mathscr{A}, \mathscr{B}, \mathscr{C}$ reach the plane $q = 1$.

That the sheets $\mathscr{A}, \mathscr{B}, \mathscr{C}$ can be continued from c (except at the intersections of c with e and g is shown by an argument based on the implicit function theorem (cf. Alexander & Yorke (1976)). The essential condition for this argument to be valid is that the Fréchet derivative of (5.20) with respect to $\boldsymbol{u}$ on c be non-singular. (This derivative would be singular at points of c at which secondary bifurcation occurs.) The verification of this non-singularity is not a triviality, but it only requires an analysis of the equations for an isotropic plate. Our basic construction is still valid as long as the points on c where the Fréchet derivative is singular are discrete. In this case we might obtain more sheets than if this derivative is regular.

Let us note that there could well be bifurcations like that of figure 2 with no non-trivial solution branches to the left of the line $\lambda = \xi$, or, more generally, there could be a number of non-trivial branches generated on this line and lying to its right that do not bifurcate from the trivial branch and that have no corresponding branches to its left. Such phenomena can be studied by the homotopy methods we have developed in this section.

10. Comments

For the sake of brevity we have eschewed a careful treatment of non-degenerate problems with $N^0(0;\lambda) = -\infty$ in favour of a careful treatment of the Taylor plate. An inkling of the complexity engendered by studying non-degenerate problems can be gained by examining the paragraphs containing (3.24)–(3.28) (also cf. Antman & Negrón-Merrero 1987, §9). By adapting the methods we have developed for problems with $N^0(0;\lambda) > -\infty$ and for Taylor plates, we could readily handle non-degenerate problems with $N^0(0;\lambda) = -\infty$ at the cost of conducting rather lengthy and dull analyses.

If $N^0(0;\lambda) = -\infty$, our solutions obviously exhibit extremely singular behaviour at $s = 0$. This response is manifested at the slightest pressure. The nature of solutions depends critically on the shear response. Many studies of nonlinear problems for elastic structures use constitutive laws giving certain stresses or resultants as linear functions of certain strains under the assumption (often unjustified) that such linear laws are typical of any reasonable law. But there is no way that a linear law can be presumed capable of describing what happens at the centre of the plate when $N^0(0;\lambda) = -\infty$. (In fact, if the constitutive law were linear and $N^0(0;\lambda) = -\infty$, then the orientation of the material would be reversed near the centre of the plate.) These remarks indicate why a central role of nonlinear elasticity is to explain the behaviour at singularities.

Whether or not $N^0(0;\lambda) = -\infty$, the solutions of our problems for anisotropic plates satisfy $\theta(0) = 0 = \theta'(0)$ (cf. §4). In studying nodal properties of solutions in §4, we had to make special scalings (cf. (8.11) and (8.23)) to compensate for the fact that a non-trivial θ can have a double zero at $s = 0$. The same kind of scalings introduced in §5 supported the compactness theorems of §6. In contrast, in Antman's (1978) corresponding treatment of isotropic plates a non-trivial solution cannot have a double zero at the origin. This fact is also reflected in his formulation of integral equations and in the ensuing proof of compactness, which requires an exploitation of the isotropy in a way that is neither needed nor possible for anisotropic plates. In this sense, the theory of isotropic plates seems to represent a singular extreme of that for anisotropic plates and cannot be reckoned as easier to analyse than the latter theory. A desire to understand the nature of difficulties that arise in the study of geometrically exact theories of isotropic plates was one of the motivations leading to the research described above.

In (2.6) we could have taken ω to have the form $\omega(s, \phi, z) = \delta(s, \phi)\, g(z)$, say, where g is given and δ is an unknown function measuring thickness variations (cf. Antman & Carbone 1977). We would then add δ to (2.2) and continue to use the modified (2.6) only for motivation. In this case there would be two more strain variables appearing in (2.13), namely δ and δ'. Under these conditions, the character of the governing equations would be considerably altered. In particular, the simple and elegant analysis supporting §3 would have to be replaced with a difficult technical treatment in which physical interpretation would be submerged. (The appropriate techniques would be an extension of those developed by Negrón-Marrero (1985).) The treatment of §4, which relies on the results of §3 would necessarily become significantly more complicated. The treatments of §§5 and 6 would remain essentially the same. It now appears conceivable that the techniques of §8 could be carried over to this more general problem. (This question will be analysed elsewhere in a more transparent setting.) Even if δ were constrained to depend on the other strain variables of (2.13) (so as to model incompressible media or to cause an appropriate stress to vanish somewhere), the same kinds of complications would ensue.

Were δ constrained to depend on the unknown strains τ and ν, then condition (4.12) would still hold. If δ were a strain variable, a generalization of (4.12) would follow from the generalized (2.6).

We have adopted the policy of prescribing δ so as to keep our treatment as transparent as possible, while permitting q in (4.12) to be fairly general, so as to capture the main effects of the presence of transverse strains.

Our construction in §9, characterized by figure 8, actually affords a mechanism by which solution pairs on the branch $\mathscr{B}$ can be reached by continuation from accessible solutions. Our result could therefore lead to effective numerical procedures for determining $\mathscr{B}$. We also remark that the possession of a distinctive nodal pattern by a branch can be used to check numerical constructions of points on the branch. If a continuation procedure leads to a solution not having the appropriate nodal properties, that solution must be deemed to be spurious and the numerical procedure deemed to have ceased to be reliable.

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