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The Numerical Computation of Compressed States of Nonlinearly Elastic Anisotropic Plates

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Summary. In this paper we study the numerical computation of the compressed states of nonlinearly elastic anisotropic circular plates. The singular boundary value problem giving the compressed states depend parametrically on the applied pressure at the edge of the plate. We give a finite difference approximation of this problem and derive bounds for the global error by using the techniques of Brezzi, Rappaz and Raviart for the finite dimensional approximation of nonlinear problems. Some numerical results are given for a class of materials whose constitutive functions reflect the standard Poisson ratio effects.

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1. Introduction

In this paper we study the numerical computation of the axisymmetric compressed states of nonlinearly elastic anisotropic circular plates when subject to a uniform pressure on the edge. Most plate theories are based on the model proposed by Kàrmàn [5] which assumes a linear stress-strain relation (generalized Hooke's law). We study here the fully nonlinear model described in [1]. For this model, the compressed states for anisotropic plates have been studied in [2] by phase plane methods and in [6] by Leray-Schauder degree techniques. The isotropic case was shown in [1] to reduce to a scalar algebraic equation.

There are several aspects of this problem which are attractive from the numerical point of view.

i) In [2] and [6] it is shown that the stress at the centre of a compressed azimuthaly reinforced body is zero and that for a radially reinforced body is infinite. Thus numerical schemes that can accurately compute these solutions and corroborate these remarkable phenomena (due to anisotropy) are very desirable.

ii) The hypothesis of axisymmetry and the circular geometry lead to equations that have a polar singularity at the centre of the plate. The numerical treatment of this singularity represents another challenge. We use the *apriori* estimates on the solutions found in [2] and [6] to improve the convergence of our numerical schemes at the centre of the plate.

iii) Our equations are fully nonlinear (cf. (2.2)) and include the typical singular behaviour of nonlinear elasticity (cf. (2.8)). The numerical analysis of nonlinear problems of this sort is not standard. We use some of the techniques in [3] to deal with this problem.

The results in this paper are for materials which are azimuthaly reinforced as defined in [2] and [6] (cf. (2.6)).

The rest of the paper is organized as follows. In Sect. 2 we present the boundary value problem (BVP) for the compressed states. We discretized these equations in Sect. 3 using finite difference schemes and in Sect. 4 we do an error analysis and establish convergence for this discretization. In Sect. 5 we present some numerical results.

Notation. $\mathbb{R}^n$ denotes the n-dimensional euclidean space of n-tuples of real numbers. We denote by $\|\cdot;\mathbb{X}\|$ the norm of the normed linear space $\mathbb{X}$. If $\mathbb{X}$ and $\mathbb{Y}$ are linear normed spaces, then $L(\mathbb{X},\mathbb{Y})$ denotes the space of bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$. We let

$$C^k[0,1]=\{u:[0,1]\to\mathbb{R}\,|\,u^{(j)}\text{ is continuous in }[0,1],0\leqq j\leqq k\},$$
$$\|u;C^k[0,1]\|=\max_{0\leqq j\leqq k}\max_{0\leqq x\leqq 1}|u^{(j)}(x)|.$$

2. The Equations for the Compressed States

Consider a deformation of a *nonlinearly homogeneously elastic* circular plate of radius one which is a pure compression, i.e., no bending and no shear. Let $\rho(s)$ be the radius in the deformed configuration of the circle of radius s in the reference configuration. The strains in this problem are $(\rho(s)/s,\rho'(s))$ where $\rho'(s)=d\rho(s)/ds$. Here $\rho(s)/s$ represents the elongation of a circumferential fibre while $\rho'(s)$ accounts for the elongation of a radial fibre. The requirement that an infinitesimal volume in the reference configuration cannot be reduced to a point during the deformation leads to the inequalities (see [1])

$$\rho(s)/s,\quad \rho'(s)>0,\quad 0\leqq s\leqq 1. \tag{2.1}$$

Let $N(\rho(s)/s, \rho'(s))$ be the radial force per unit reference length exerted on the material inside the circle of radius s by that outside this circle. Let $T(\rho(s)/s, \rho'(s))$ be the force per unit reference length of a ray exerted at s by the material on one side of the ray on that of the other side. If the plate is subjected to a pressure of λ units of force per unit reference length on its edge, then in [1] and [6] it is shown that ρ satisfies the following BVP:

$$(sN(\rho(s)/s, \rho'(s)))' - T(\rho(s)/s, \rho'(s)) = 0, \qquad 0 < s < 1, \tag{2.2a}$$

$$\rho(0) = 0, \qquad -N(\rho(1), \rho'(1)) - \lambda = 0. \tag{2.2b, c}$$

Note that this BVP is equivalent to the following integro-differential equation:

$$-sN(\rho(s)/s, \rho'(s)) - \lambda - \int_s^1 T(\rho(t)/t, \rho'(t)\,dt = 0, \tag{2.3a}$$

$$\rho(0) = 0. \tag{2.3b}$$

We require that the functions $N, T\colon (0, \infty)^2 \to \mathbb{R}$ be $C^3((0, \infty)^2)$ and satisfy

$$N_v, T_\tau > 0 \qquad \forall (\tau, v) \in (0, \infty)^2, \tag{2.4a, b}$$

$$N(1, 1) = 0 = T(1, 1). \tag{2.5a, b}$$

The material of the plate is called *azimuthaly reinforced* at $\tau = 1$ if

$$N_v(1, 1) + N_\tau(1, 1) < T_v(1, 1) + T_\tau(1, 1). \tag{2.6}$$

From the results in [2] and [6] we get the following theorem concerning the existence of solutions of the BVP (2.2).

Theorem 2.1. *Let* (2.4), (2.5), *and* (2.6) *hold. Let the function N satisfy*

$$N_v(\tau, v) + N_\tau(\tau, v) > 0 \qquad \forall \tau, v > 0, \tag{2.7}$$

and

$$N(\tau, v) \to -\infty \quad as \quad \tau \to 0^+ \quad or \quad v \to 0^+ \quad and \quad \tau^2 + v^2 \leqq K^2 \tag{2.8}$$

for each fixed constant K. Hence the BVP (2.2) *has a connected set of solution pairs* (ρ, λ) *in* $C^1[0, 1] \times [0, \infty)$ *that contains* $(\rho_0 = s, 0)$, *that satisfies* (2.1) *with* $\rho'(0) = 1$, *and that is unbounded in the* λ *direction, i.e., it has a solution for each* $\lambda \geqq 0$. *If in addition N and T satisfy*

$$\begin{pmatrix} N_v & N_\tau \\ T_v & T_\tau \end{pmatrix} \quad \textit{is coercive,} \tag{2.9}$$

then the solution is unique. If $\Lambda > 0$ *is given, then there exist* $\Delta(\Lambda)$, $\delta(\Lambda) > 0$ *such that*

$$\delta(\Lambda) \leqq \rho'(s), \qquad \rho(s)/s \leqq \Delta(\Lambda) \tag{2.10}$$

for all $s \in [0, 1]$ *and* $\lambda \in [0, \Lambda]$. *Moreover if* $\rho \in C^k[0, 1]$, $k \geqq 2$, *then* $\Delta(\Lambda)$ *can be chosen so that*

$$|\rho^{(j)}(s)| \leqq \Delta(\Lambda) \quad \forall s \in [0, 1],\ 2 \leqq j \leqq k. \quad \square \tag{2.11}$$

If we strengthen condition (2.6) to

$$k(N_v(1, 1) + N_\tau(1, 1)) < T_\tau(1, 1) + k T_v(1, 1), \quad k \geqq 1, \tag{2.12}$$

then from the results in [6] it follows that $\rho \in C^k[0, 1]$ and that

$$\rho^{(j)}(0) = 0, \quad 2 \leqq j \leqq k, \tag{2.13}$$

provided $k \geqq 2$. (Condition (2.12), which clearly implies (2.6), is a measure of how "strong" is the azimuthal reinforcement of the plate at $\tau = 1$, i.e., the larger the k, the stronger is the reinforcement.)

3. The Numerical Scheme

We use finite difference schemes to obtain approximate solutions of (2.2). Let $\{s_j = jh : 0 \leqq j \leqq n\}$ be a uniform partition of $[0, 1]$ into n subintervals, $h = 1/n$. Let $u(s)$ be a function defined on $[0, 1]$. We use the notation u_i to denote an approximation of $u(s_i)$, $0 \leqq i \leqq n$. We define for any vector $\underline{u} = (u_0, u_1, \ldots, u_n)$

$$\delta u_i = (u_{i+1} - u_i)/h, \tag{3.1 a}$$

$$(u/s)_{i+1/2} = (u_i + u_{i+1})/(s_i + s_{i+1}), \tag{3.1 b}$$

$$s_{i+1/2} = (s_i + s_{i+1})/2, \tag{3.1 c}$$

for $0 \leqq i \leqq n - 1$. We discretize (2.2a) as:

$$h^{-1}[s_{i+1/2} N((\rho/s)_{i+1/2}, \delta\rho_i) - s_{i-1/2} N((\rho/s)_{i-1/2}, \delta\rho_{i-1})] \\ -[T((\rho/s)_{i+1/2}, \delta\rho_i) + T((\rho/s)_{i-1/2}, \delta\rho_{i-1})]/2 = 0, \quad 1 \leqq i \leqq n-1. \tag{3.2a}$$

The boundary conditions (2.2 b, c) are discretized as follows

$$\rho_0 = 0, \tag{3.2 b}$$

$$-s_{n-1/2} N((\rho/s)_{n-1/2}, \delta\rho_{n-1}) - \lambda - h T((\rho/s)_{n-1/2}, \delta\rho_{n-1})/2 = 0. \tag{3.2 c}$$

Remark. Equation (3.2 c) is obtained by setting $s = s_{n-1/2}$ in (2.3 a) and then using the discretizing operators (3.1 a, b) together with the endpoint rule for approximating integrals. The usual discretization of (2.2 c) is $N(\rho_n, \delta\rho_{n-1}) + \lambda = 0$. However this is $O(h)$. We will show in Sect. 4 that (3.2 c) gives an approximation $O(h)^2$ of (2.2 c).

Note that Eqs. (3.2) define a mapping $\underline{F}: \mathbb{R}^{n+1} \supset \Omega \to \mathbb{R}^n$ where

$$\Omega = \{(\rho_1, \rho_2, \ldots, \rho_n, \lambda) \in \mathbb{R}^{n+1} : \rho_i > 0, 1 \leqq i \leqq n, \rho_i < \rho_{i+1}, 1 \leqq i \leqq n-1, \lambda \geqq 0\} \tag{3.3}$$

and F_i is given by the left side of (3.2a) for $1 \leqq i \leqq n-1$, and F_n is given by the left side of (3.2c). Thus Eqs. (3.2) are equivalent to

$$\underline{F}(\underline{\rho}, \lambda) = \underline{0}, \qquad (\underline{\rho}, \lambda) \in \Omega, \tag{3.4}$$

where $\underline{\rho} = (\rho_1, \rho_2, \ldots, \rho_n)$. If the derivative of $\underline{F}$ has rank n, then the solution set of (3.4) is a one dimensional manifold M_h contained in Ω (see [7, 8]). In Sect. 4 we study the difference between this manifold and the one for the actual solutions of (2.2) which we denote by M_0.

4. The Discretization and Approximation Errors

In this section we present an error analysis for the method described in Sect. 3. To study the relation between (2.2) and (3.4) we use some of the results in [3]. These techniques are necessary because of the nonlinear nature of (2.2). Throughout this section we assume all the hypotheses in Theorem (2.1) in particular (2.9), and (2.12) (with at least $k=4$).

Let $N^0(s) = N(\rho(s)/s, \rho'(s))$, etc. We define the *discretization error* of (3.4) at $s = s_i$ by

$$DE(s_i) = (sN^0)'(s_i) - T^0(s_i) - F_i(\rho(s_{i-1}), \rho(s_i), \rho(s_{i+1})), \qquad 1 \leqq i \leqq n-1, \tag{4.1a}$$

$$DE(s_n) = -N(\rho(1), \rho'(1)) - \lambda - F_n(\rho(s_{n-1}), \rho(s_n), \lambda), \tag{4.1b}$$

where $\underline{F} = (F_1, \ldots, F_n)$. (Note that by (3.2a, c), F_i depends only on $(\rho_{i-1}, \rho_i, \rho_{i+1})$ for $1 \leqq i \leqq n-1$ and only $(\rho_{n-1}, \rho_n, \lambda)$ for $i = n$.) To study DE we shall need the following formulas which can be obtained from Taylor's Theorem by assuming that $\rho \in C^4[0, 1]$:

$$A_{i+1/2} \equiv (\rho(s_{i+1}) - \rho(s_i))/h = \rho'(s_{i+1/2}) + (h^2/24)\, \rho'''(s_{i+1/2}) + O(h^3), \tag{4.2a}$$

$$\begin{aligned} B_{i+1/2} \equiv (\rho(s_{i+1}) + \rho(s_i))/(s_{i+1} + s_i) &= \rho(s_{i+1/2})/s_{i+1/2} \\ &+ (h^2/4)\, \rho''(s_{i+1/2})/(s_{i+1} + s_i) + O(h^3). \end{aligned} \tag{4.2b}$$

If (2.12) holds with $k=4$, then $\rho \in C^4[0, 1]$ and from (2.13) we get that

$$\rho''(s_{i+1/2}) = \rho'''(\theta_i)\, s_{i+1/2}, \qquad \theta_i \in (0, s_{i+1/2}). \tag{4.3a}$$

Hence

$$|\rho''(s_{i+1/2})/(s_{i+1} + s_i)| \leqq |\rho'''(\theta_i)|. \tag{4.3b}$$

Thus the second term in (4.2b) is indeed $O(h^2)$. If we expand $N(B_{i+1/2}, A_{i+1/2})$ about $(\rho(s_{i+1/2})/s_{i+1/2}, \rho'(s_{i+1/2}))$ using Taylor's Theorem and use (4.2), then we get that

$$\begin{aligned} N(B_{i+1/2}, A_{i+1/2}) &= N^0(s_{i+1/2}) + (h^2/24)\, N_\nu^0(s_{i+1/2})\, \rho'''(s_{i+1/2}) \\ &+ (h^2/8)\, N_\tau^0(s_{i+1/2})\, \rho''(s_{i+1/2})/s_{i+1/2} + O(h^3), \qquad 0 \leqq i \leqq n-1. \end{aligned} \tag{4.4}$$

If we replace N by T in (4.4) and use the formula

$$(f(s_{i+1/2}) + f(s_{i-1/2}))/2 = f(s_i) + O(h^2), \tag{4.5}$$

which holds if $f\in C^2[0,1]$, then we get that

$$(T(B_{i+1/2},A_{i+1/2})+T(B_{i-1/2},A_{i-1/2}))/2=T^0(s_i)+O(h^2),\qquad 1\leqq i\leqq n-1. \quad (4.6)$$

It is not difficult to see now that from (4.1 a), (4.4), and (4.6) we get that

$$DE(s_i)=O(h^2),\qquad 1\leqq i\leqq n-1. \quad (4.7)$$

If we apply Taylor's Theorem to the functions ρ and $\rho(s)/s$ about $s=1$, then we get that

$$A_{n-1/2}=\rho'(1)-h\rho''(1)/2+O(h^2), \quad (4.8\,\mathrm{a})$$

$$B_{n-1/2}=\rho(1)+h(\rho(1)-\rho'(1))/2+O(h^2). \quad (4.8\,\mathrm{b})$$

If we write $s_{n-1/2}$ as $1-h/2$ in (4.1 b), expand $T(B_{n-1/2},A_{n-1/2})$ and $N(B_{n-1/2},A_{n-1/2})$ about $(\rho(1),\rho'(1))$, and use (4.8), then we get that

$$DE(s_n)=(h/2)\,[N_\tau^0(1)\,(\rho(1)-\rho'(1))-N_\nu^0(1)\,\rho''(1)+T^0(1)-N^0(1)]+O(h^2). \quad (4.9)$$

But the expression in brackets is zero if ρ satisfies (2.2 a) (expand the derivative in (2.2 a) and set $s=1$). Hence $DE(s_n)=O(h^2)$. Since by (2.11) these results are uniform in λ, we have the following result.

Lemma 4.1. *Let ρ be a solution of the* BVP (2.2). *Let* (2.12) *hold with $k=4$ (hence $\rho\in C^4[0,1]$) and $N,T\in C^3((0,\infty)^2)$. Hence if DE is defined by* (4.1) *and $\Lambda>0$ is given, then*

$$DE(s_i)=O(h^2),\qquad 1\leqq i\leqq n,\qquad \textit{uniformly for } \lambda\in[0,\Lambda].\quad \square \quad (4.10)$$

We now want to get an estimate of the global error $\max\{|\rho(s_i)-\rho_i|: 1\leqq i\leqq n\}$. First we need some notation and preliminary lemmas. Let $\mathbb{X}_h=(\mathbb{R}^n, \|\cdot;\mathbb{X}_h\|)$ and $\mathbb{Y}_h=(\mathbb{R}^n, \|\cdot;\mathbb{Y}_h\|)$ be the n-dimensional Banach spaces defined by the norms

$$\|\underset{\sim}{x};\mathbb{X}_h\|^2=h\sum_{i=0}^{n-1} s_{i+1/2}((\delta x_i)^2+(x/s)^2_{i+1/2}),\qquad \text{where we set } x_0=0, \quad (4.11)$$

$$\|\underset{\sim}{x};\mathbb{Y}_h\|^2=h\sum_{i=1}^{n-1} x_i^2+x_n^2, \quad (4.12)$$

where $\underset{\sim}{x}\in\mathbb{R}^n$. The inner product induced by $\|\cdot;\mathbb{Y}_h\|$ is

$$\langle \underset{\sim}{x},\underset{\sim}{y}\rangle_h=h\sum_{i=1}^{n-1} x_i\,y_i+x_n\,y_n,\qquad \underset{\sim}{x},\underset{\sim}{y}\in\mathbb{R}^n. \quad (4.13)$$

We shall also need the following formulas

$$\sum_{i=1}^{k} (a_i \pm a_{i-1})\, b_i = \mp a_k\, b_{k+1} \pm a_0\, b_1 \pm \sum_{i=1}^{k} (b_{i+1} \pm b_i)\, a_i. \tag{4.14}$$

We now have

Lemma 4.2. *There exists a constant $c>0$ independent of h such that*

$$\|\underset{\sim}{x}; \mathbb{Y}_h\| \leqq c\, \|\underset{\sim}{x}; \mathbb{X}_h\| \qquad \forall \underset{\sim}{x} \in \mathbb{R}^n,\ x_0 = 0. \tag{4.15}$$

Remark. Any two norms in $\mathbb{R}^n$ are equivalent. Hence the point of the lemma is to establish that c is independent of h.

Proof. From (4.14) we get that

$$x_n\, s_{n+1/2} = h \sum_{i=1}^{n} x_i + h \sum_{i=1}^{n} s_{i-1/2}\, \delta x_{i-1}. \tag{4.16a}$$

Now two applications of the Cauchy-Schwarz inequality (for the usual inner product of $\mathbb{R}^n$), together with $s_{n+1/2} \geqq 1$, and the inequality $(a+b)^2 \leqq 2(a^2+b^2)$, give that

$$|x_n|^2 \leqq 2\left[h \sum_{i=1}^{n} x_i^2 + h \sum_{i=1}^{n} s_{i-1/2} (\delta x_{i-1})^2\right]. \tag{4.16b}$$

Now from the inequalities

$$x_i^2 \leqq [(x_{i-1}+x_i)^2 + (x_i - x_{i-1})^2]/2, \qquad 1 \leqq i \leqq n, \tag{4.16c}$$

and $s_{i-1} + s_i \leqq 2$, $(h/2) \leqq s_{i-1/2}$, we get that if $h \leqq 2$ say, then

$$h \sum_{i=1}^{n} x_i^2 \leqq 2\, \|\underset{\sim}{x}; \mathbb{X}_h\|^2. \tag{4.16d}$$

Combining (4.16b, d) with the definition (4.12) we get the desired result. □

Let $D_\rho \underset{\sim}{F}$ and $D_\lambda \underset{\sim}{F}$ be the derivatives of $\underset{\sim}{F}(\underset{\sim}{\rho}, \lambda)$ with respect to $\underset{\sim}{\rho}$ and λ respectively, and $D\underset{\sim}{F} = (D_\rho \underset{\sim}{F}, D_\lambda \underset{\sim}{F})$. Let $\rho(\lambda)(\cdot)$ denote the solution of the BVP (2.2) for a given $\lambda \geqq 0$. We define

$$\Pi_h\, \rho(\lambda) = (\rho(\lambda)(s_1), \ldots, \rho(\lambda)(s_n)). \tag{4.17}$$

We now have:

Lemma 4.3. *Let $\Lambda > 0$ be given and assume* (2.9) *holds. Hence for $\lambda \in [0, \Lambda]$ and h sufficiently small, there exists a constant $c>0$ independent of h and λ such that*

$$\langle D_\rho \underset{\sim}{F}(\Pi_h\, \rho(\lambda), \lambda) \cdot \underset{\sim}{v}, \underset{\sim}{v} \rangle_h \leqq -c\, \|\underset{\sim}{v}; \mathbb{X}_h\|^2 \tag{4.18}$$

for all $\underset{\sim}{v} \in \mathbb{R}^n$.

Proof. We use the notation

$$G_{\beta}^{i+1/2} = G_{\beta}(B_{i+1/2}, A_{i+1/2}), \quad 0 \leqq i \leqq n-1, \tag{4.19a}$$

where $A_{i+1/2}, B_{i+1/2}$ are defined in (4.2), G can be any of the constitutive functions N or T, and β is either v or τ. Now from the left sides of (3.2a, c), the definition (4.13), and the identities $(4.14)_{\pm}$, we get that

$$\langle D_{\rho} \underline{F}(\Pi_h \rho(\lambda), \lambda) \cdot \underline{v}, \underline{v} \rangle_h = -h \sum_{i=0}^{n-1} s_{i+1/2} \underline{u}_{i+1/2}^t M^{i+1/2}(\lambda) \underline{u}_{i+1/2}, \tag{4.19b}$$

where

$$M^{i+1/2}(\lambda) = \begin{pmatrix} N_v^{i+1/2} & N_\tau^{i+1/2} \\ T_v^{i+1/2} & T_\tau^{i+1/2} \end{pmatrix}, \tag{4.19c}$$

$$\underline{u}_{i+1/2} = (\delta v_i, (v/s)_{i+1/2})^t. \tag{4.19d}$$

(The dependence of $M^{i+1/2}$ on λ comes through $A_{i+1/2}$, $B_{i+1/2}$ which in turn depend on $\rho(\lambda)$.) Now the expansions (4.2) and the inequalities (2.10) and (2.11) imply that for h sufficiently small and all $\lambda \in [0, \Lambda]$,

$$A_{i+1/2}, \quad B_{i+1/2} \geqq \delta(\Lambda)/2, \quad 0 \leqq i \leqq n-1. \tag{4.19e}$$

From (4.19e) and condition (2.9) we get that there exists a constant $c > 0$ independent of $\lambda \in [0, \Lambda]$ and h sufficiently small such that

$$\underline{x}^t M^{i+1/2}(\lambda)\, \underline{x} \geqq c\, \underline{x}^t \underline{x} \qquad Y \underline{x} \in \mathbb{R}^2. \tag{4.19f}$$

Hence (4.19b, f) together with the definition (4.11) yield the result. □

Lemma 4.4. *The linear operator $D_{\rho} \underline{F}(\Pi_h \rho(\lambda), \lambda)$: $\mathbb{X}_h \to \mathbb{Y}_h$ is an isomorphism and there exists a constant $\bar{c} > 0$ independent of $\lambda \in [0, \Lambda]$ and h sufficiently small such that*

$$\| D_{\rho} \underline{F}(\Pi_h \rho(\lambda), \lambda)^{-1}; L(\mathbb{Y}_h, \mathbb{X}_h) \| \leqq \bar{c}. \tag{4.20}$$

Proof. The Cauchy-Schwarz inequality for the inner product $\langle \cdot, \cdot \rangle_h$ gives that

$$\langle D_{\rho} \underline{F} \cdot \underline{v}, \underline{v} \rangle_h \geqq -\| D_{\rho} \underline{F} \cdot v; \mathbb{Y}_h \| \, \| \underline{v}; \mathbb{Y}_h \|,$$

where the arguments of $D_{\rho} \underline{F}$ are $(\Pi_h \rho(\lambda), \lambda)$. This inequality and Lemma (4.2) imply that

$$\langle D_{\rho} \underline{F} \cdot \underline{v}, \underline{v} \rangle_h \geqq -c_1 \| D\underline{\rho}\, \underline{F} \cdot \underline{v}; \mathbb{Y}_h \| \, \| \underline{v}; \mathbb{X}_h \|$$

for some constant $c_1 > 0$ independent of h and λ. This inequality and Lemma (4.3) imply that

$$\| D_{\rho} \underline{F} \cdot \underline{v}; \mathbb{Y}_h \| \geqq (c/c_1) \, \| \underline{v}; \mathbb{X}_h \|$$

where $c>0$ is as in Lemma (4.3). This shows that $D_\rho \underset{\sim}{F}(\Pi_h \rho(\lambda), \lambda)$ is $1-1$, and hence an isomorphism because $\mathbb{X}_h$ and $\mathbb{Y}_h$ are finite dimensional, and that (4.20) holds with $\bar{c}=c_1/c$. □

Lemma 4.5. *There exists a function L: $\mathbb{R}^+ \to \mathbb{R}^+$ monotonically increasing, independent of $\lambda \in [0, \Lambda]$ and h sufficiently small such that*

$$\|D_\rho \underset{\sim}{F}(\underset{\sim}{w}, \mu) - D_\rho \underset{\sim}{F}(\Pi_h \rho(\lambda), \lambda); L(\mathbb{X}_h, \mathbb{Y}_h)\| \leqq L(\varepsilon)\, h^{-3/2} \|\underset{\sim}{w} - \Pi_h \rho(\lambda); \mathbb{X}_h\| \tag{4.21 a}$$

for all $(\underset{\sim}{w}, \mu)$ such that

$$\|\underset{\sim}{w} - \Pi_h \rho(\lambda); \mathbb{X}_h\| + |\mu - \lambda| < \varepsilon h^{3/2} \tag{4.21 b}$$

where ε is sufficiently small.

Proof. From the left sides of (3.2a, c) and the definition (4.12), it is not difficult to see that for any $\underset{\sim}{v} \in \mathbb{R}^n$, $v_0 = 0$, we have that for some constant c independent of h and λ,

$$\begin{aligned}\|(D_\rho \underset{\sim}{F}(\underset{\sim}{w}, \mu) - D_\rho \underset{\sim}{F}(\Pi_h \rho(\lambda), \lambda)) \cdot \underset{\sim}{v}; \mathbb{Y}_h\|^2 &\leqq c h^{-2} \sum_{i=0}^{n-1} s_{i+1/2} \{(\delta N_\tau^{i+1/2})^2 + (\delta N_\nu^{i+1/2})^2 \\ &\quad + (\delta T_\tau^{i+1/2})^2 + (\delta T_\nu^{i+1/2})^2\} \|\underset{\sim}{v}; \mathbb{X}_h\|^2\end{aligned} \tag{4.22 a}$$

where

$$\delta N_\tau^{i+1/2} = N_\tau((w/s)_{i+1/2}, \delta w_i) - N_\tau(B_{i+1/2}, A_{i+1/2}), \quad \text{etc.,} \tag{4.22 b}$$

$0 \leqq i \leqq n-1$. From Taylor's Theorem

$$\delta N_\tau^{i+1/2} = ((w/s)_{i+1/2} - B_{i+1/2})\, \bar{N}_{\tau\tau} + (\delta w_i - A_{i+1/2})\, \bar{N}_{\tau\nu}, \tag{4.22 c}$$

where

$$\bar{N}_{\tau\tau} = \int_0^1 N_{\tau\tau}(t(w/s)_{i+1/2} + (1-t)\, B_{i+1/2}, t\,\delta w_i + (1-t)\, A_{i+1/2})\, dt, \quad \text{etc.} \tag{4.22 d}$$

Let $\Delta(\Lambda)$ and $\delta(\Delta)$ be as in (2.10), (2.11). Define

$$\Omega(\Lambda) = \{(\tau, \nu) : \delta(\Lambda) \leqq \tau, \nu \leqq \Delta(\Lambda)\}, \tag{4.23 a}$$

$$E(\Lambda) = \{(\tau, \nu) : \operatorname{dist}((\tau, \nu), \Omega(\Lambda)) \leqq \delta(\Lambda)/2\}. \tag{4.23 b}$$

From (2.10), (2.11) and (4.2) we get that for h sufficiently small and $\lambda \in [0, \Lambda]$ we have that

$$(\rho(\lambda)(s_{i+1/2}), \rho'(\lambda)(s_{i+1/2})) \in \Omega(\Lambda), \tag{4.24 a}$$

$$(B_{i+1/2}, A_{i+1/2}) \in E(\Lambda), \quad 0 \leqq i \leqq n-1. \tag{4.24 b}$$

For $\varepsilon<\delta(\Lambda)/2$ define

$$E_\varepsilon(\Lambda)=\{(\tau, v): \operatorname{dist}((\tau, v), E(\Lambda))\leqq\varepsilon\}, \tag{4.25a}$$

$$L_{11}^{(2)}(\varepsilon)=\max_{E_\varepsilon(\Lambda)}|N_{\tau\tau}(\tau, v)|, \quad \text{etc.} \tag{4.25b}$$

($L_{12}^{(1)}(\varepsilon)$ would correspond to the maximum of $|T_{\tau v}|$ over $E_\varepsilon(\Lambda)$, etc.) Now if (4.21 b) holds, since $(h^2/2)\leqq h s_{i+1/2}$, then we get that

$$(\delta w_i - A_{i+1/2})^2+((w/s)_{i+1/2}-B_{i+1/2})^2<2\varepsilon^2 h<\varepsilon^2$$

say if $h<1/2$. Thus for h sufficiently small (4.21b), (4.24b), and (4.25a) imply that

$$((w/s)_{i+1/2}, \delta w_i)\in E_\varepsilon(\Lambda), \qquad 0\leqq i\leqq n-1. \tag{4.26}$$

Hence from (4.22d) and (4.25b) we get that if (4.21b) holds, then

$$|\bar{N}_{\tau\tau}|\leqq L_{11}^{(2)}(\varepsilon), \quad \text{etc.} \tag{4.27}$$

From (4.27) and (4.22c) we get that

$$(\delta N_\tau^{i+1/2})^2\leqq 2(L_{11}^{(2)}(\varepsilon)^2+L_{12}^{(2)}(\varepsilon)^2)(((w/s)_{i+1/2}-B_{i+1/2})^2 + (\delta w_i - A_{i+1/2})^2), \quad \text{etc.} \tag{4.28}$$

Now from (4.22a), (4.28), and the definition (4.11), we get that (4.21a) holds with

$$c L(\varepsilon)^2=\sum_{i,j,k=1}^{2} L_{ij}^{(k)}(\varepsilon)^2. \quad \square$$

To state the main result establishing the convergence of the scheme (3.2) we shall need the following hypothesis. We assume that there exists a constant $\bar{K}(\Lambda)>0$ such that for any $\lambda, \mu\in[0, \Lambda]$ we have that

$$\|\rho(\lambda)(\cdot)/s-\rho(\mu)(\cdot)/s; C[0, 1]\|\leqq\bar{K}(\Lambda)|\lambda-\mu|, \tag{4.29a}$$

$$\|\rho'(\lambda)-\rho'(\mu); C[0, 1]\|\leqq\bar{K}(\Lambda)|\lambda-\mu|. \tag{4.29b}$$

(Conditions ensuring (4.29) are given in [2] and [6].) It is easy to see now that (4.29) implies that for some constant $K(\Lambda)$,

$$\|\Pi_h\rho(\lambda)-\Pi_h\rho(\mu); \mathbf{X}_h\|\leqq K(\Lambda)|\lambda-\mu| \qquad \forall\, \lambda, \mu\in[0, \Lambda]. \tag{4.30}$$

We now have the following theorem.

Theorem 4.6. *Let $\Lambda > 0$ be given. Let conditions* (2.9), (2.12) *with* $k=4$, *and* (4.30) *hold. Hence there exists a* C^1 *function* ϱ_h: $[0, \Lambda] \to \mathbb{X}_h$ *and constants* K_1, K_2 *independent of* $\lambda \in [0, \Lambda]$ *and h sufficiently small such that*

$$\underline{F}(\varrho_h(\lambda), \lambda) = \underline{0}, \qquad \lambda \in [0, \Lambda], \tag{4.31a}$$

$$\|\varrho_h(\lambda) - \Pi_h \rho(\lambda); \mathbb{X}_h\| \leqq K_1 \|\underline{F}(\Pi_h \rho(\lambda), \lambda); \mathbb{Y}_h\|, \tag{4.31b}$$

$$\|\varrho_h(\lambda) - \Pi_h \rho(\lambda); \mathbb{X}_h\| \leqq K_2 h^2. \tag{4.31c}$$

Proof. We define the spaces $\mathbb{X}, \mathbb{Y}$, and Z by

$$\mathbb{X} = \mathbb{R}, \qquad \|\cdot; \mathbb{X}\| = h^{-3/2} |.|, \tag{4.31d}$$

$$\mathbb{Y} = \mathbb{R}^n, \qquad \|\cdot; \mathbb{Y}\| = h^{-3/2} \|\cdot; \mathbb{X}_h\|, \tag{4.31e}$$

$$Z = \mathbb{R}^n, \qquad \|\cdot; Z\| = h^{-3/2} \|\cdot; \mathbb{Y}_h\|. \tag{4.31f}$$

Hence by Lemma (4.4) and the fact that $D_\lambda \underline{F} = (0, 0, \ldots, 0, 1)^t \in \mathbb{R}^n$, we get that

$$\|D_\rho \underline{F}(\Pi_h \rho(\lambda), \lambda)^{-1}; L(Z, \mathbb{Y})\| \leqq c_0$$
$$\|D_\lambda \underline{F}(\Pi_h \rho(\lambda), \lambda); L(\mathbb{X}, Z)\| \leqq c_1$$

for some constants c_0, c_1 independent of $\lambda \varepsilon [0, \Lambda]$ and h sufficiently small. Also Lemma (4.5) implies that

$$\|D\underline{F}(\underline{w}, \mu) - D\underline{F}(\Pi_h \rho(\lambda), \lambda); L(\mathbb{Y} \times \mathbb{X}, Z)\|$$
$$\leqq L(\varepsilon)\,[\|\underline{w} - \Pi_h \rho(\lambda); \mathbb{Y}\| + \|\mu - \lambda; \mathbb{X}\|].$$

Finally, from Lemma (4.1) we get that

$$\|\underline{F}(\Pi_h \rho(\lambda), \lambda); Z\| = O(h^{1/2}) \tag{4.31g}$$

uniformly in λ and h small. These results and Theorem 1 from [3] (with $f = \underline{F}$ and $y(\cdot) = \Pi_h \rho(\cdot)$) imply the existence of a constant K_1 and a C^1 function ϱ_h: $[0, \Lambda] \to \mathbb{Y}$ such that (4.31a) holds and

$$\|\varrho_h(\lambda) - \Pi_h \rho(\lambda); \mathbb{Y}\| \leqq K_1 \|\underline{F}(\Pi_h \rho(\lambda), \lambda); Z\|. \tag{4.31h}$$

If we now combine the definitions (4.31d, e, f), the estimate (4.31g) and inequality (4.31h) one gets (4.31b, c). □

Corollary 4.7. *Under the hypotheses of Theorem* (4.6), *there exists a constant K independent of* $\lambda \in [0, \Lambda]$ *and h sufficiently small such that*

$$\max\{|\rho_h(\lambda)_i - \rho(\lambda)(s_i)| : 1 \leqq i \leqq n\} \leqq K h^2. \tag{4.32}$$

Proof. From (4.31c) it follows that it is enough to show that for some constant $c > 0$ independent of h and λ,

$$\max\{|x_j| : 1 \leqq j \leqq n\} \leqq c\,\|\underline{x}; \mathbb{X}_h\| \;\forall \underline{x} \in \mathbb{R}^n, \qquad x_0 = 0. \tag{4.33a}$$

From (4.16a) with $n=j$ and two applications of the Cauchy-Schwarz inequality we get that

$$|x_j|\, s_{j+1/2} \leqq \left(s_{j+1/2} \sum_{i=1}^{j} h x_i^2\right)^{1/2} + (s_{j+1/2}/\sqrt{2}) \left(\sum_{i=1}^{j} h s_{i-1/2}\, \delta x_{i-1}^2\right)^{1/2}. \tag{4.33b}$$

Using (4.16c) one gets that

$$\sum_{i=1}^{j} h x_i^2 \leqq 2 h s_{j+1/2} \sum_{i=1}^{j} s_{i-1/2} [(x/s)_{i-1/2}^2 + \delta x_{i-1}^2]. \tag{4.33c}$$

Now combining (4.33b, c) and (4.11), we get (4.33a). □

5. Numerical Results

We present here some numerical examples for the scheme of Sect. 3. The class of materials we consider have constitutive functions

$$N(\tau, v) = G_1(\tau, v), \qquad T(\tau, v) = G_2(\tau, v), \tag{5.1a, b}$$

where

$$G_i(\tau, v) = -A_{i1}\, v^{-\alpha(i,1)} + A_{i2}\, v^{\alpha(i,2)} - A_{i3} (v\tau)^{-\alpha(i,3)} + A_{i4} (v\tau)^{\alpha(i,4)} - A_{i5}\, \tau^{-\alpha(i,5)} + A_{i6}\, \tau^{\alpha(i,6)} + B_i, \quad i=1,2, \tag{5.1c}$$

where $A_{ij}>0$, $\alpha(i,j)>1$, and the B_i's are chosen such that (2.5) holds. Clearly conditions (2.4), (2.7), and (2.8) hold. Condition (2.6) becomes

$$\alpha(1,j)\, A_{1j} + \alpha(1,3)\, A_{13} + \alpha(1,4)\, A_{14} < \alpha(2,j)\, A_{2j} + \alpha(2,3)\, A_{23} + \alpha(2,4)\, A_{24} \tag{5.2}$$

where we used the summation convention that repeated indices are summed from 1 to 6. Condition (2.9) is equivalent to

$$4 N_v\, T_\tau - (N_\tau + T_v)^2 > 0. \tag{5.3}$$

This condition can be obtained for (5.1) by making some complicated assumptions of the coefficients and exponents. Instead of doing that we note that from the proof of Lemma 4.3 the full condition (2.9) is not needed but the positive definiteness of that matrix at the discretized exact solution $\Pi_h\, \rho(\lambda)$. This could be checked numerically (up to terms $O(h^2)$) by computing the matrix in (2.9) at the computed solution $\rho_h(\lambda)$ of Theorem 4.6.

Note that now (2.2) and (5.1) form a 25 parameter problem! (These are the A_{ij}, $\alpha(i,j)$, and λ.) We now get further restrictions on these parameters. Let c_{ij}, $i,j=1, \ldots, 6$ denote the elastic parameters in the generalized Hooke's Law and let t denote the thickness of the plate. Hence from the definitions of N and T given in [1] and by requiring that the linearization of these functions about (1, 1) agree with the generalized Hooke's Law, we get that

$$N_v(1,1)=tc_{11}, \quad N_\tau(1,1)=tc_{12}, \tag{5.4a, b}$$

$$T_v(1,1)=tc_{21}, \quad T_\tau(1,1)=tc_{22}. \tag{5.5a, b}$$

Thus from (5.1), (5.4), and (5.5) we get that

$$\sum_{j=1}^{4} \alpha(i,j)\, A_{ij}=tc_{i1}, \tag{5.6a}$$

$$\sum_{j=3}^{6} \alpha(i,j)\, A_{ij}=tc_{i2}, \quad i=1,2. \tag{5.6b}$$

These equations do not specify the parameters $\alpha(i,j)$, A_{ij} uniquely. The theory that we developed in Sects. 3, 4 and the numerical examples of this section are for fixed values of the parameters $\alpha(i,j)$, A_{ij} satisfying (5.6). The full generality of (2.2), (5.1), and (5.6) shall be pursuit elsewhere.

The following coefficients c_{ij} were taken from [4]:

Material	c_{11}	c_{22}	c_{12}
Wood (Oak)	0.00127	0.00639	0.000378
Wood (Pine)	0.00124	0.0171	0.000800

where the units of the c_{ij} are 10^{13} dynes/cm^2. In both cases we took $\alpha(i,j)=2$ for all i,j and $t=0.2$ cm. The A_{ij} were chosen as

$$A_{ij}=A_i\equiv tc_{i1}/8, \quad 1\leqq j\leqq 4, \tag{5.7a}$$

$$A_{ij}=(tc_{i2}-4A_i)/4, \quad j=5,6, \tag{5.7b}$$

for $i=1,2$. (If one A_{ij} becomes less than zero, then we use the same equations with the c_{i1} and c_{i2} interchanged, the index "j" in the first equation ranging from 3 to 6, and in the second equation from 1 to 2.) Let $\varrho_n(\lambda)$ stand for the solution $\varrho_h(\lambda)$ of (4.31a) with $h=1/n$. Let $\|\cdot\|_\infty$ denote the maximum norm appearing to the left of (4.32). The computations were done for $n=2^i$, $2\leqq i\leqq 6$ and $\varrho_{64}(\lambda)$ is taken as the exact discretized solution for the purpose of estimating errors. We define the computed errors by

$$e_n(\lambda)=\|\varrho_n(\lambda)-\varrho_{64}(\lambda)\|_\infty, \tag{5.8a}$$

$$e_n^*(\lambda)=\|\varrho_n(\lambda)-\varrho_{64}(\lambda); \mathbf{X}_h\|. \tag{5.8b}$$

(Here we understand that in $\varrho_n-\varrho_{64}$ only the components of ϱ_{64} corresponding to those of ϱ_n are used in the computation.) The results were as follows (the units of λ are 10^{13} dynes):

Oak ($\lambda=10^{-4}$)

n	e_n	e_n/e_{2n}	e_n^*	e_n^*/e_{2n}^*
4	0.301200*E*-2	4.41645	0.471148*E*-2	3.99342
8	0.681996*E*-3	4.79271	0.117981*E*-2	4.14276

16	0.142299E-3	5.27014	0.284789E-3	4.77828
32	0.270009E-4	–	0.596007E-4	–

Pine ($\lambda=10^{-4}$)

n	e_n	e_n/e_{2n}	e_n^*	e_n^*/e_{2n}^*
4	0.266701E-2	4.04712	0.612105E-2	4.14702
8	0.658989E-3	4.25231	0.147601E-2	4.22327
16	0.154972E-3	4.99040	0.349495E-3	4.86671
32	0.310540E-4	–	0.718135E-4	–

The $\varrho_n(\lambda)$ were computed using a continuation method (Euler's method as predictor and Newton's method as corrector) to continue from the known solution at $\lambda=0$ to the target point at $\lambda=10^{-4}$. These results show an $O(h^2)$ convergence in the corresponding norms in these cases.

When the plate is radially reinforced at $\tau=1$, that is when the opposite of inequality (2.6) holds, then the results of Sect. 4 no longer hold because estimates like those in Lemma 4.1 are not uniform in λ due to the infinite stresses at the center of the plate. (Similar results might hold in some other norms.) An extreme case of a radially reinforced plate is the so called Taylor plate which is defined by the constitutive restrictions $T=0$ and $N_\tau=0$. (There are no circumferential effects.) In this case (2.2) reduces to

$$sN(\rho'(s))=-\lambda, \qquad \rho(0)=0. \tag{5.9}$$

In the special case where $N(v)=-A(v^{-\alpha}-1)$, then

$$\rho(s)=\int_0^s (A\tau/(A\tau+\lambda))^{1/\alpha}\,d\tau. \tag{5.10}$$

We tested our scheme with the values $A=1$, $\alpha=2$, and $\lambda=0.02$ and the results were as follows

n	e_n	e_n/e_{2n}	e_n^*	e_n^*/e_{2n}^*
4	0.121239E-1	1.63706	0.183058E-1	1.44533
8	0.740588E-2	1.87905	0.126655E-1	1.70840
16	0.394130E-2	2.17234	0.741364E-2	2.01742
32	0.181431E-2	2.44411	0.367480E-2	2.30591
64	0.742316E-3	–	0.159364E-2	–

(In the computation of the norms (5.8) in this table we used the exact solution (5.10) instead of $\varrho_{64}(\lambda)$.) Thus the scheme still converges to the exact solution (which shows the robustness of the method) but with a lower rate of convergence (approximately linear) in the norms of Sect. 4.

6. Conclusions and Remarks

We developed some finite difference schemes for the numerical approximation of the compressed states of nonlinearly elastic anisotropic circular plates. We proved that under the conditions of Theorem 4.6 we get $O(h^2)$ convergence for the strains in the norm $\|\cdot;\mathbf{X}_n\|$ and $O(h^2)$ convergence for the displacements in the maximum norm (Corollary 4.7). Here the condition (2.9) proved to be crucial. This condition prohibits certain kinds of necking instabilities that could occur if part of the plate is under sufficiently large tension in some direction.

The boundary value problem (2.2) and (5.1) form a multiparameter problem with parameters $\alpha(i,j)$, A_{ij}, and λ. Equations (5.4) and (5.5) with t, c_{ij} given and the inequality (5.2), give some conditions on the parameters $\alpha(i,j)$, A_{ij}. An additional restriction can be obtained by requiring that the material be hyperelastic ($N_\tau = T_\nu$). To study the dependence of the compressed states on the constitutive Eqs. (5.1) we would have to study (2.2), (5.1) over all admissible $\alpha(i,j)$, A_{ij} satisfying these conditions. The discretizations of these multi-parameter are usually handled by computing one dimensional submanifolds of the solution manifold (see [8]).

The set of compressed states of the plate parametrized by λ represent the trivial branch of solutions of the quasilinear system of ordinary differential equations given in [1] that describes the buckled states of the plate. As the analysis in [6] shows, the difficulties encountered in the study of the equations for the compressed states are like a microcosm of what can happen for this system of Ode's. Thus we expect that the results of this paper give some good insight into the numerical analysis of the bifurcation diagrams of the deformed states of the plate.

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