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Axisymmetric Deformations of Buckling and Barrelling Type for Cylinders Under Lateral Compression – The Linear Problem

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Abstract. We consider the nonlinear boundary value problem of specifying the displacement of the lateral surface of a cylindrical body subject to zero normal stresses on the top and the bottom and sliding conditions (i.e. no tangential components of the surface traction) at the lateral surface. We restrict our analysis to study the existence of axisymmetric deformations assuming that the material of the body is homogeneous, isotropic and hyperelastic. We study the linearization of the nonlinear equations about a trivial solution and show that smooth solutions of the linear problem must be separable. We classify the nontrivial axisymmetric solutions of the linearized problem in two types that we call buckling and barrelling like solutions. We characterize the eigenvalues for both solutions types as well as those displacements of the lateral surface at which the complementing condition for the linearized equations fails to be satisfied. For a class of Blatz–Ko type materials we give a complete characterization of the existence, multiplicity and disposition of the corresponding eigenvalues. We show, for such material, that the eigenvalues of buckling and barrelling types are simple, and that they form monotone sequences (decreasing for the former and increasing for the latter) both of which converge to a value at which the complementing condition fails. Moreover, it is shown that the cylinder looses stability first to buckling rather than to barrelling.

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1. Introduction

The boundary value problem for the deformations of columns under lateral compression has been studied among others by Guo [9], Antman [1, 2], Negrón-Marrero [12], Negrón-Marrero and Antman [14] for thin plates; and for cylinders by Guo [10] (incompressible materials), Sensenig [16] (for a compressible material with a quadratic stored energy function), and Negrón-Marrero [13] (for compressible Hadamard–Green type materials). This problem however has not brought as much attention as that of uniaxial compression. This apparent lack of interest in the lateral compression problem, we believe, is due mostly to the difficulty in realizing such boundary conditions in the laboratory. However, with this paper (and a forthcoming companion paper) we want to convey the idea that the lateral compression problem for nonlinearly elastic cylinders lends itself to interesting analytical studies that can help to understand some of the experimentally observed behavior of nonlinearly elastic columns in uniaxial compression.

The problem of columns under uniaxial compression has been studied extensively both experimentally and theoretically. We mention here those works most relevant to the present paper and refer to [7] for a thorough review of the literature up to 1989. Simpson and Spector studied in [17] and [18] the problem of a homogeneous isotropic hyperelastic right circular cylinder subject to uniaxial compression. First, they proved the existence of a homogeneous deformation (with diagonal deformation gradient), from now on referred to as *the trivial solution*, and then studied the corresponding linearized problem around that trivial solution. They showed that all axisymmetric solutions of the linearized problem can be obtained by separation of variables, and deduced a necessary and sufficient condition that characterized the values of the compression ratio for which the linearized problem would have nontrivial axisymmetric (barrelling)* solutions. For those values of the compression ratio the trivial solution would become linearly unstable in the sense that it ceases to be a weak local minimizer of the energy with respect to barrelling perturbations. Being unable to establish whether that condition is satisfied for some value of the compression ratio for a completely general homogeneous isotropic hyperelastic material they considered in [18] a strongly elliptic Hadamard–Green type material for which they showed the existence of compression ratios at which the corresponding linearized problem has nontrivial barrelling solutions. In [17] they further specialized the material constitutive law and considered a Blatz–Ko type material for which they were able to give a complete characterization of the compression ratios for which the linearized axisymmetric problem has nontrivial solutions. However, in neither paper, [17] or [18], is there an analysis of buckling type solutions. In fact, the study of buckled states for nonlinearly elastic cylinders confronts technical difficulties that so far have been insurmountable.

Davies [7] analyzed both buckling and barrelling for 2d columns and showed that for sufficiently large compression the trivial solution becomes unstable. The type of the associated instability, buckling or barrelling, depends on the sign of a certain parameter related to the material behavior and compression ratio. Davies showed that for various realistic materials such as Neo–Hookean, Mooney–Rivlin, and Hadamard–Green materials, the sign of this parameter is such that the trivial solution always loses stability first to buckling. She also proved that the values of the compression ratio for which there exist nontrivial buckling solutions of the linearized problem are bounded below by a value of the compression ratio

* In the engineering literature the term *buckling* refers to either *skew-symmetric buckling* or *symmetric buckling*. In this paper we use the terminology from elasticity theory in which *buckling* stands for skew-symmetric buckling and *barrelling* stands for symmetric buckling.

at which the Complementing Condition, cf. [19] and [3], first fails, call it λ_c, and that the corresponding values of the compression ratio for which there exist nontrivial barrelling solutions for the linearized problem are bounded above by the same value λ_c. In another paper, [8], Davies studied the same problem treated by Simpson and Spector in [18] and [17] and, among other things, her results suggest a relationship between the instability results in [17] and [18] and the failure of the Complementing Condition. In particular it is shown that the column loses stability to barrelling at a compression ratio which is bounded above or below a certain value at which the complementing condition fails. Which type of bound we get, again depends on the sign of a certain material dependent parameter.

Since it has not been possible to study the existence of buckling type solutions for genuinely three dimensional columns under uniaxial compression, it has not been possible to consider simultaneously buckling and barrelling instabilities for three dimensional columns. Davies compared the value of the compression ratio at which the first buckling mode occurs in a rectangular 2d column, with the compression ratio at which the first barrelling mode occurs for a cylindrical 3d column. As Davies herself recognizes, that comparison is arbitrary and there is no reason to expect a relationship between instabilities for rectangular two dimensional columns and cylindrical three dimensional ones.

In this paper we consider the nonlinear boundary value problem of specifying the displacement of the lateral surface of a cylindrical body subject to zero normal stresses on the top and the bottom and sliding conditions (i.e. no tangential components of the surface traction) at the lateral surface. This problem was considered by Guo [9] in which the existence of a trivial homogeneous solution is established. (See Section 3.) The linearized equations about the homogeneous solution with respect to arbitrary deformations (not necessarily axisymmetric) were obtained. These equations were further specialized to thin plates and a characterization of the lateral displacements (critical loads) for which stability is lost was given. (The term stability loss here refers to a value of the lateral displacement for which the time independent linearized equation has a nontrivial solution.) In [10] Guo considers the linearized equations for incompressible materials and axisymmetric deformations. He obtained numerically for specific materials, including the Mooney material, that the critical loads of buckling and barrelling type are simple, that they form monotone sequences (decreasing for the former and increasing for the latter) both of which converge to a value which he called the *condensation curve*. In this paper we generalize the results in Guo [10] to axisymmetric deformations assuming that the material of the body is homogeneous, compressible, isotropic and hyperelastic. Furthermore for Blatz–Ko type materials, we do a rigorous treatment of the numerical results in [10] concerning the condensation curve, in particular we show that such a critical load corresponds to a value at which the complementing condition fails. Unlike the problem of uniaxial axisymmetric compression for 3d columns in which it is impossible to study buckling type solutions, in the problem of lateral compression for axisymmetric deformations, we can treat both types of solutions in a three dimensional setting.

In Section 2 we give some basic definitions and present the equations of three-dimensional elasticity for homogeneous isotropic hyperelastic materials. In Section 3 we show that our nonlinear boundary value problem admits a trivial homogeneous solution representing a pure expansion and/or compression of the cylinder. In Section 4 we study the linearization of our boundary value problem about the trivial homogeneous solution. First, we give conditions for the elasticity tensor at the homogeneous solution to be strongly elliptic. We then show that all smooth (C^2) solutions of the linear problem, without considering the boundary conditions at the top and the bottom of the cylinder, can be obtained by separation of variables and can be represented by a Fourier–Bessel series that converges uniformly. The coefficients of these series representations are then characterized as solutions of a family of linear boundary value problems, that incorporate the boundary conditions at the top and the bottom of the cylinder, whose solutions are given in terms of hyperbolic functions. The condition for these coefficient functions to be not all zero gives the characteristic equations defining the compression ratios at which the linearized problem has nontrivial axisymmetric solutions. Those compression ratios are given as solutions of two equations. One of the equations represents solutions that are asymmetric with respect to the middle plane of the cylindrical reference configuration. For this reason we call them *buckling* type solutions. The other equation represents solutions that are symmetric with respect to that middle plane, and which we call *barrelling* type solutions. This classification into buckling and barrelling type solutions is not arbitrary. It corresponds to the observation that if we look at the axisymmetric solutions transversally they resemble the buckling and barrelling type solutions of a planar rectangular bar. Thus we can make direct comparisons between buckling and barrelling modes in a three-dimensional setting. That this is a genuinely three-dimensional problem is a consequence of the fact that the first Piola–Kirchhoff stress tensor for the lateral compression problem has five nonzero components in its dyadic representation. (See [13], equation (6.19).)

In Section 5 we give a characterization of the values of the lateral compression ratio for which the linearized problem fails to satisfy the complementing condition. This is an algebraic condition, also known as the Lopatinsky–Shapiro condition, between the coefficients of the leading parts of the differential and boundary operators of a given elliptic boundary value problem (see [3, 19, 22], and, for an elementary exposition [15]). The complementing condition figures prominently in the approach of Healey and Simpson in [11] to the problem of global continuation of solutions in three-dimensional nonlinear elasticity.

The existence and properties of the solutions (eigenvalues) of the characteristic equations (cf. (4.55), (4.56)) is in general a very difficult task. In [13] the existence of solutions of these equations is established for a family of Hadamard–Green type materials but there are no results on the disposition and multiplicity of these eigenvalues. In Section 6 we study these equations for Blatz–Ko type materials (cf. (6.1)) which are used to model certain rubbery materials [6]. In this case we give a complete characterization of the existence, multiplicity and disposition of the corresponding eigenvalues. We show that the eigenvalues of buckling and barrelling type are simple, that they form monotone sequences (decreasing for the former and increasing for the latter) both of which converge to a value at which the complementing condition fails. As a corollary it follows that the cylinder always loses stability first to buckling rather than to barrelling. This does not agree with the experimental observations at least for bars under uniaxial compression where barrelling occurs first for bars which are relatively thicker than longer, while buckling occurs first in bars which are relatively longer than thicker.* However as pointed out in [7], all analytical studies, including now our three-dimensional results, support the idea that the appearance of barrelling first might be due to end effects, like friction at the ends of the bar, not taken into account by the analytical models.

2. Problem Formulation

We consider a body which in its reference configuration occupies the region $\overline{\Omega}$, where

$$\Omega = \left\{(x_1, x_2, x_2) \in \mathbb{R}^3\colon\ x_1^2 + x_2^2 < 1,\ -h < x_3 < h\right\}.$$

We write $\partial\Omega = \partial\Omega_B \cup \partial\Omega_S \cup \partial\Omega_T$, where

$$\begin{aligned}
\partial\Omega_B &= \left\{(x_1, x_2, x_3)\colon\ x_1^2 + x_2^2 \leqslant 1,\ x_3 = -h\right\},\\
\partial\Omega_T &= \left\{(x_1, x_2, x_3)\colon\ x_1^2 + x_2^2 \leqslant 1,\ x_3 = h\right\},\\
\partial\Omega_S &= \left\{(x_1, x_2, x_3)\colon\ x_1^2 + x_2^2 = 1,\ -h \leqslant x_3 \leqslant h\right\}.
\end{aligned}$$

For any given deformation tensor $\mathbf{F}$ with $\det(\mathbf{F}) > 0$, we let $\mathbf{S}(\mathbf{F})$ be the (First) Piola–Kirchhoff stress tensor. Our boundary value problem for any given deformation $\mathbf{f}\colon\ \Omega \to \mathbb{R}^3$, with $\det \nabla\mathbf{f} > 0$, $\mathbf{f} = (f_1, f_2, f_3)$, is now given by

$$\operatorname{div}\mathbf{S}(\nabla\mathbf{f}) = \mathbf{0} \quad \text{in } \Omega, \tag{2.1}$$

$$f_1^2 + f_2^2 = \lambda^2 \quad \text{on } \partial\Omega_S, \tag{2.2}$$

$$(\mathbf{S}(\nabla\mathbf{f}) \cdot \mathbf{n}) \cdot \mathbf{t} = 0 \quad \text{on } \partial\Omega_S, \qquad \mathbf{t} \cdot \mathbf{n} = 0, \tag{2.3}$$

$$\mathbf{S}(\nabla\mathbf{f}) \cdot \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega_B \cup \partial\Omega_T, \tag{2.4}$$

where $\mathbf{n}$ is the outward unit normal field of $\partial\Omega$, and $\lambda \in (0, 1]$. We assume that the material of the cylinder is *isotropic* and *hyperelastic*, i.e., there exists a *stored energy* function $\hat{\sigma}(\mathbf{F})$ of the form:

$$\hat{\sigma}(\mathbf{F}) = \sigma\left(\frac{1}{2}\mathbf{F} \cdot \mathbf{F}, \frac{1}{4}\mathbf{F}\mathbf{F}^{\mathrm{t}} \cdot \mathbf{F}\mathbf{F}^{\mathrm{t}}, \det\mathbf{F}\right), \tag{2.5}$$

* For details see [4, 5], and the discussion in [7] (pp. 147–149).

where $\mathbf{F} \cdot \mathbf{H} = \mathrm{trace}(\mathbf{F}\mathbf{H}^t)$ denotes the inner product of the tensors $\mathbf{F}$ and $\mathbf{H}$. We now have that

$$\mathbf{S}(\mathbf{F}) = \frac{\mathrm{d}\hat{\sigma}(\mathbf{F})}{\mathrm{d}\mathbf{F}} = \sigma_{,1}\mathbf{F} + \sigma_{,2}\mathbf{F}\mathbf{F}^t\mathbf{F} + (\det \mathbf{F})\sigma_{,3}\mathbf{F}^{-t}. \tag{2.6}$$

The *elasticity tensor* $\mathcal{A}(\mathbf{F})$ is defined by

$$\mathcal{A}(\mathbf{F}) = \frac{\mathrm{d}\mathbf{S}(\mathbf{F})}{\mathrm{d}\mathbf{F}}.$$

We say that the elasticity tensor is *strongly-elliptic* (at $\mathbf{F}$) provided that

$$\mathbf{ab} \cdot \mathcal{A}(\mathbf{F})[\mathbf{ab}] > 0, \quad \forall\, \mathbf{a}, \mathbf{b} \neq \mathbf{0}.$$

We assume that $\hat{\sigma}$ satisfies the growth conditions:

$$\lim_{\det \mathbf{F} \to 0^+} \hat{\sigma}(\mathbf{F}) = \infty, \qquad \lim_{\|\mathbf{F}\| \to \infty} \hat{\sigma}(\mathbf{F}) = \infty. \tag{2.7}$$

3. The Homogeneous Solution

We consider now a (homogeneous) deformation of the form $\mathbf{f}_\lambda = (\lambda x_1, \lambda x_2, \omega x_3)$. Note that

$$\nabla \mathbf{f}_\lambda = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \omega \end{pmatrix}. \tag{3.1}$$

Thus

$$\begin{aligned} \mathbf{S}(\nabla \mathbf{f}_\lambda) \;=\; \mathrm{diag}\big\{&\lambda\sigma_{,1} + \lambda^3\sigma_{,2} + \lambda\omega\sigma_{,3}, \\ &\lambda\sigma_{,1} + \lambda^3\sigma_{,2} + \lambda\omega\sigma_{,3}, \omega\sigma_{,1} + \omega^3\sigma_{,2} + \lambda^2\sigma_{,3}\big\}. \end{aligned} \tag{3.2}$$

The unit normal vector field on $\partial\Omega_S$ is $\mathbf{n} = \mathbf{e}_1 = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}$, and its tangent vectors are $\mathbf{t} = \alpha\mathbf{e}_2 + \beta\mathbf{k}$, where $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 \neq 0$, and $\mathbf{e}_2 = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}$. It follows from (3.2) that

$$\mathbf{S}(\nabla \mathbf{f}_\lambda) \cdot \mathbf{n} = (\lambda\sigma_{,1} + \lambda^3\sigma_{,2} + \lambda\omega\sigma_{,3})\mathbf{e}_1,$$

which implies that

$$(\mathbf{S}(\nabla \mathbf{f}) \cdot \mathbf{n}) \cdot \mathbf{t} = 0.$$

On the other hand, we have that $\mathbf{n} = \pm\mathbf{k}$ on $\partial\Omega_B \cup \partial\Omega_T$. Thus it follows from (3.2) that $\mathbf{S}(\nabla \mathbf{f}) \cdot \mathbf{n} = \mathbf{0}$ if and only if

$$\omega\sigma_{,1} + \omega^3\sigma_{,2} + \lambda^2\sigma_{,3} = 0. \tag{3.3}$$

For (3.1) we get that

$$\frac{1}{2}\mathbf{F}\cdot\mathbf{F}=\frac{1}{2}\big(2\lambda^2+\omega^2\big),\qquad \frac{1}{4}\mathbf{F}\mathbf{F}^{\mathrm{t}}\cdot\mathbf{F}\mathbf{F}^{\mathrm{t}}=\frac{1}{4}\big(2\lambda^4+\omega^4\big),\qquad \det\mathbf{F}=\lambda^2\omega.$$

Define for any given λ,

$$g(\omega)=\sigma\left(\frac{1}{2}\big(2\lambda^2+\omega^2\big),\frac{1}{4}\big(2\lambda^4+\omega^4\big),\lambda^2\omega\right).$$

Then, it follows from (2.7) that $g(\omega)\to\infty$ as $\omega\to 0^+$ or $\omega\to\infty$. Hence, there exists $\hat{\omega}(\lambda)$ such that $g'(\hat{\omega}(\lambda))=0$, i.e. such that (3.3) holds. Therefore,

$$\mathbf{f}_\lambda=\big(\lambda x_1,\lambda x_2,\hat{\omega}(\lambda)x_3\big) \tag{3.4}$$

is a solution of the given boundary value problem. This result is essentially due to Guo [9].

4. The Linearized Equations

We now consider the linearization of the boundary value problem of Section 3 about the trivial solution (3.4). This linearization is given, in terms of the displacement field $\mathbf{u}$, by:

$$\operatorname{div}\mathcal{A}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{u}]=\mathbf{0}\quad\text{in }\Omega, \tag{4.1}$$

$$u_1^2+u_2^2=0\quad\text{on }\partial\Omega_S, \tag{4.2}$$

$$\big(\mathcal{A}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{u}]\cdot\mathbf{n}\big)\cdot\mathbf{t}=0\quad\text{on }\partial\Omega_S,\qquad \mathbf{t}\cdot\mathbf{n}=0, \tag{4.3}$$

$$\mathcal{A}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{u}]\cdot\mathbf{n}=\mathbf{0}\quad\text{on }\partial\Omega_B\cup\partial\Omega_T, \tag{4.4}$$

where

$$\begin{aligned}\mathcal{A}(\mathbf{F})[\mathbf{H}] \;=\;& \sigma_{,1}\mathbf{H}+\sigma_{,2}(\mathbf{H}\mathbf{F}^{\mathrm{t}}\mathbf{F}+\mathbf{F}\mathbf{H}^{\mathrm{t}}\mathbf{F}+\mathbf{F}\mathbf{F}^{\mathrm{t}}\mathbf{H})\\ &+(\det\mathbf{F})\sigma_{,3}\big((\mathbf{F}^{-\mathrm{t}}\cdot\mathbf{H})\mathbf{I}-\mathbf{F}^{-\mathrm{t}}\mathbf{H}^{\mathrm{t}}\big)\mathbf{F}^{-\mathrm{t}}+\sum_{i,j=1}^{3}(\mathbf{G}^i\cdot\mathbf{H})\sigma_{,ij}\mathbf{G}^j,\end{aligned} \tag{4.5}$$

and

$$\mathbf{G}^1=\mathbf{F},\qquad \mathbf{G}^2=\mathbf{F}\mathbf{F}^{\mathrm{t}}\mathbf{F},\qquad \mathbf{G}^3=(\det\mathbf{F})\mathbf{F}^{-\mathrm{t}}. \tag{4.6}$$

We define for $i=1,2,3$,

$$\begin{aligned} t_i &= \sigma_{,3}+\big(\lambda_i^2\sigma_{,1}+\lambda_i^4\sigma_{,2}\big)/(\lambda_1\lambda_2\lambda_3),\\ \beta_i &= \sigma_{,1}+\sigma_{,2}\big(\lambda_1^2+\lambda_2^2+\lambda_3^2-\lambda_i^2\big),\\ \tau_i &= \frac{\lambda_1\lambda_2\lambda_3}{\lambda_i}\frac{\partial t_i}{\partial\lambda_i},\end{aligned} \tag{4.7}$$

where the arguments of $\sigma_{,3}$, etc. are $(\lambda_1, \lambda_2, \lambda_3)$. Here the t_i's are the principal stresses (of the Cauchy stress tensor), and the λ_i's are the principal stretches. It follows for the deformation (3.4), writing ω instead of $\hat{\omega}(\lambda)$, that equations (4.7) reduce to

$$t_1 = t_2 = \sigma_{,3} + \frac{\sigma_{,1} + \lambda^2\sigma_{,2}}{\omega}, \tag{4.8}$$

$$t_3 = \sigma_{,3} + \omega\frac{\sigma_{,1} + \omega^2\sigma_{,2}}{\lambda^2} = 0 \quad \text{(cf. (3.3))}, \tag{4.9}$$

$$\beta_1 = \beta_2 = \sigma_{,1} + \sigma_{,2}\big(\lambda^2 + \omega^2\big), \qquad \beta_3 = \sigma_{,1} + 2\lambda^2\sigma_{,2}, \tag{4.10}$$

$$\begin{aligned} \tau_1 = \tau_2 \;=\; & \sigma_{,1} + 3\lambda^2\sigma_{,2} + \sigma_{,11}\lambda^2 + \sigma_{,12}\lambda^4 + \sigma_{,13}\lambda^2\omega \\ & + \sigma_{,21}\lambda^4 + \sigma_{,22}\lambda^6 + \sigma_{,23}\lambda^4\omega + \lambda\omega\big(\sigma_{,31}\lambda + \sigma_{,32}\lambda^3 + \sigma_{,33}\lambda\omega\big), \end{aligned} \tag{4.11}$$

$$\begin{aligned} \tau_3 \;=\; & \sigma_{,1} + 3\omega^2\sigma_{,2} + \sigma_{,11}\omega^2 + \sigma_{,12}\omega^4 + \sigma_{,13}\lambda^2\omega \\ & + \sigma_{,21}\omega^4 + \sigma_{,22}\omega^6 + \sigma_{,23}\lambda^2\omega^3 + \lambda^2\big(\sigma_{,31}\omega + \sigma_{,32}\omega^3 + \sigma_{,33}\lambda^2\big), \end{aligned} \tag{4.12}$$

where the arguments of $\sigma_{,3}$, etc., are $(\lambda, \lambda, \omega)$. Let $\mathbf{B} = \mathcal{A}(\nabla\mathbf{f}_\lambda)[\mathbf{H}]$. Then combining (3.1), and (4.5)–(4.12), we get that the components of $\mathbf{B}$ are

$$\begin{aligned} B_{11} &= \tau_1 H_{11} + (\omega t_1 + \tau_1 - 2\beta_3)H_{22} + XH_{33}, \\ B_{22} &= \tau_1 H_{22} + (\omega t_1 + \tau_1 - 2\beta_3)H_{11} + XH_{33}, \\ B_{33} &= \tau_3 H_{33} + X(H_{11} + H_{22}), \\ B_{12} &= \beta_3 H_{12} + (\beta_3 - \omega t_1)H_{21}, \\ B_{13} &= \beta_1 H_{13} + (\omega/\lambda)\beta_1 H_{31}, \\ B_{21} &= \beta_3 H_{21} + (\beta_3 - \omega t_1)H_{12}, \\ B_{23} &= \beta_1 H_{23} + (\omega/\lambda)\beta_1 H_{32}, \\ B_{31} &= \beta_1 H_{31} + (\omega/\lambda)\beta_1 H_{13}, \\ B_{32} &= \beta_1 H_{32} + (\omega/\lambda)\beta_1 H_{23}, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} X \;=\; & \lambda\sigma_{,3} + \omega\big(\sigma_{,11}\lambda + \sigma_{,12}\lambda^3 + \sigma_{,13}\lambda\omega\big) \\ & + \omega^3\big(\sigma_{,21}\lambda + \sigma_{,22}\lambda^3 + \sigma_{,23}\lambda\omega\big) + \lambda^2\big(\sigma_{,31}\lambda + \sigma_{,32}\lambda^3 + \sigma_{,33}\lambda\omega\big), \end{aligned} \tag{4.14}$$

and the H_{ij}'s are the components of $\mathbf{H}$.

We now consider an *axisymmetric deformation*, i.e. one of the form

$$\mathbf{u} = \begin{pmatrix} \phi(r,z)x_1 \\ \phi(r,z)x_2 \\ \ell(r,z) \end{pmatrix}, \tag{4.15}$$

where $r^2 = x_1^2 + x_2^2$ and $z = x_3$. It follows that

$$\nabla\mathbf{u} = \begin{pmatrix} \phi + \frac{x_1^2}{r}\phi_r & \frac{x_1 x_2}{r}\phi_r & x_1\phi_z \\ \frac{x_1 x_2}{r}\phi_r & \phi + \frac{x_2^2}{r}\phi_r & x_2\phi_z \\ \frac{x_1}{r}\ell_r & \frac{x_2}{r}\ell_r & \ell_z \end{pmatrix}. \tag{4.16}$$

If we set $\mathbf{H} = \nabla\mathbf{u}$ in the expression for $\mathbf{B}$, using (4.13), then we can write (4.1) as

$$\big(\operatorname{div}\mathcal{A}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{u}]\big)_i = x_i\left(\tau_1\left(\phi_{rr} + \frac{3}{r}\phi_r\right) + \beta_1\phi_{zz} + \frac{\mathcal{N}}{r}\ell_{rz}\right) = 0,$$
$$i = 1, 2,$$
$$\big(\operatorname{div}\mathcal{A}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{u}]\big)_3 = \beta_1\left(\ell_{rr} + \frac{1}{r}\ell_r\right) + \mathcal{N}(2\phi_z + r\phi_{rz}) + \tau_3\ell_{zz} = 0,$$

where

$$\mathcal{N} = X + (\omega/\lambda)\beta_1, \tag{4.17}$$

which further reduce to

$$\tau_1\left(\phi_{rr} + \frac{3}{r}\phi_r\right) + \beta_1\phi_{zz} + \frac{\mathcal{N}}{r}\ell_{rz} = 0, \tag{4.18}$$

$$\beta_1(r\ell_r)_r + \mathcal{N}(r^2\phi_z)_r + \tau_3 r\ell_{zz} = 0. \tag{4.19}$$

The boundary conditions (4.2)–(4.4) reduce to

$$\phi(1, z) = 0, \quad -h \leqslant z \leqslant h, \tag{4.20}$$

$$\beta_1\ell_r + (\omega/\lambda)\beta_1 r\phi_z = 0, \quad -h \leqslant z \leqslant h,\ r = 1, \tag{4.21}$$

$$\beta_1\phi_z + (\omega/\lambda)\beta_1\frac{\ell_r}{r} = 0, \quad 0 \leqslant r \leqslant 1,\ z = \pm h, \tag{4.22}$$

$$\tau_3\ell_z + (X/r)(r^2\phi)_r = 0, \quad 0 \leqslant r \leqslant 1,\ z = \pm h. \tag{4.23}$$

At $r = 0$ we further require that

$$\phi(0, z) = 0, \quad -h \leqslant z \leqslant h, \tag{4.24}$$

$$\lim_{r\to 0}\big(r\beta_1\ell_r + (\omega/\lambda)\beta_1 r^2\phi_z\big) = 0, \quad -h \leqslant z \leqslant h. \tag{4.25}$$

Let us define

$$v(r, z) = r\phi(r, z). \tag{4.26}$$

Hence, (4.18)–(4.25) can be written as

$$\tau_1\big(-(rv_r)_r + v/r\big) - \beta_1 r v_{zz} = r\mathcal{N}\ell_{rz}, \tag{4.27}$$

$$-\beta_1(r\ell_r)_r - \tau_3 r\ell_{zz} = \mathcal{N}(rv_z)_r, \tag{4.28}$$

$$v(0,z)=0=v(1,z), \qquad -h\leqslant z\leqslant h, \tag{4.29}$$
$$\beta_1\ell_r+(\omega/\lambda)\beta_1 v_z=0, \qquad -h\leqslant z\leqslant h,\ r=1, \tag{4.30}$$
$$\lim_{r\to 0} r\big(\beta_1\ell_r+(\omega/\lambda)\beta_1 v_z\big)=0, \qquad -h\leqslant z\leqslant h, \tag{4.31}$$
$$\beta_1 v_z+(\omega/\lambda)\beta_1\ell_r=0, \qquad 0\leqslant r\leqslant 1,\ z=\pm h, \tag{4.32}$$
$$\tau_3\ell_z+(X/r)(rv)_r=0, \qquad 0\leqslant r\leqslant 1,\ z=\pm h. \tag{4.33}$$

The proof of the following result is almost identical to that of Theorem A.1 in [18], p. 122, with the corresponding change in notation, and thus we omit it here.

LEMMA 4.1. *Necessary and sufficient conditions for the elasticity tensor $\mathcal{A}(\nabla\mathbf{f}_\lambda)$ to be strongly-elliptic are*

$$\beta_1>0, \quad \beta_3>0, \quad \tau_1>0, \quad \tau_3>0, \tag{4.34}$$
$$\sqrt{\tau_1\tau_3}+\beta_1>|\mathcal{N}|, \tag{4.35}$$

where $\mathcal{N}$ is given by (4.17).

Note that (4.29) implies that

$$v_z(0,z)=0=v_z(1,z), \quad \text{for all } z. \tag{4.36}$$

Assuming that $\mathcal{A}(\nabla\mathbf{f}_\lambda)$ is strongly elliptic, Lemma 4.1 implies that $\beta_1>0$. It follows from (4.30), (4.31), and (4.36) that

$$\lim_{r\to 0+} r\ell_r(r,z)=0=\ell_r(1,z), \quad \text{for all } z. \tag{4.37}$$

LEMMA 4.2. *Let $v,\ell\in C^2([0,1]\times[-h,h])$ and satisfy* (4.29) *and* (4.37)*. Let (k_n) be the sequence of positive zeros of the Bessel function J_1. Then v,ℓ can be represented by the following Fourier–Bessel series:*

$$v(r,z)=\sum_{n=1}^{\infty} v_n(z)J_1(k_nr), \qquad \ell(r,z)=\sum_{n=1}^{\infty}\ell_n(z)J_0(k_nr), \tag{4.38}$$

and both of these series converge uniformly on $[0,1]\times[-h,h]$.

Proof. The functions $(J_1(k_nr))$ have the following orthogonality properties

$$\int_0^1 rJ_1(k_nr)J_1(k_mr)\,\mathrm{d}r=\begin{cases}0, & m\neq n,\\ \frac{1}{2}J_2^2(k_n), & m=n.\end{cases} \tag{4.39}$$

Now combining the identities

$$\int xJ_0(x)\,\mathrm{d}x=xJ_1(x), \qquad J_0'(x)=-J_1(x),$$

with (4.39) we get that

$$\int_0^1 r J_0(k_n r) J_0(k_m r)\,\mathrm{d}r = \begin{cases} 0, & m \neq n, \\ \frac{1}{2} J_2^2(k_n), & m = n. \end{cases} \tag{4.40}$$

That $v, \ell \in C^2([0,1]\times[-h,h])$ have representations like (4.38) is a standard result in Fourier–Bessel series (see [21]) and (4.39) and (4.40) imply that

$$v_n(z) = \frac{2}{J_2^2(k_n)} \int_0^1 r v(r,z) J_1(k_n r)\,\mathrm{d}r, \tag{4.41}$$

$$\ell_n(z) = \frac{2}{J_2^2(k_n)} \int_0^1 r \ell(r,z) J_0(k_n r)\,\mathrm{d}r. \tag{4.42}$$

Now integrating by parts twice and using (4.29), (4.37), and that $J_1(k_n) = 0$ for all n, we get the following equivalent expressions for (4.41) and (4.42):

$$v_n(z) = -\frac{2}{k_n^2 J_2^2(k_n)} \int_0^1 \big[(r v_r)_r - v/r\big] J_1(k_n r)\,\mathrm{d}r,$$

$$\ell_n(z) = -\frac{2}{k_n^2 J_2^2(k_n)} \int_0^1 (r\ell_r)_r J_0(k_n r)\,\mathrm{d}r.$$

It follows now using (4.29a) that for some constant K depending on v, w we have

$$|v_n(z)| \leqslant \frac{K}{k_n^2 J_2^2(k_n)} \int_0^1 |J_1(k_n r)|\,\mathrm{d}r, \tag{4.43}$$

$$|\ell_n(z)| \leqslant \frac{K}{k_n^2 J_2^2(k_n)} \int_0^1 |J_0(k_n r)|\,\mathrm{d}r. \tag{4.44}$$

To get useful bounds for the right hand sides of these inequalities we shall use the following estimates:

$$\big|J_0(x)\big| \leqslant 1, \qquad \Big|J_1(x)\Big| \leqslant K_1 x, \quad \text{for all } x,$$

$$\big|J_\nu(x)\big| \leqslant \frac{M_\nu}{\sqrt{x}}, \quad \text{for all } x,$$

for some constants K_1, M_ν. We now have in (4.43) that

$$\begin{aligned}
\int_0^1 \big|J_1(k_n r)\big|\,\mathrm{d}r &= \int_0^{k_1/k_n} \big|J_1(k_n r)\big|\,\mathrm{d}r + \int_{k_1/k_n}^1 \big|J_1(k_n r)\big|\,\mathrm{d}r \\
&\leqslant \int_0^{k_1/k_n} K_1 k_n r\,\mathrm{d}r + \frac{1}{k_n} \int_{k_1}^{k_n} \big|J_1(u)\big|\,\mathrm{d}u \\
&\leqslant \frac{K_1 k_1^2}{2k_n} + \frac{M_1}{k_n} \int_{k_1}^{k_n} \frac{\mathrm{d}u}{\sqrt{u}} \\
&\leqslant \frac{K_1 k_1^2}{2k_n} + \frac{2M_1}{k_n}\big(\sqrt{k_n} - \sqrt{k_1}\big) \leqslant \frac{C}{\sqrt{k_n}}.
\end{aligned}$$

Using this result in (4.43) together with the fact that $J_2^2(k_n)$ is asymptotic to $1/k_n$, we get that

$$\left|v_n(z)\right| \leqslant \frac{C'}{k_n^{3/2}}, \quad \text{for all } n, \tag{4.45}$$

for some constant C' depending on v, ℓ only. Similarly we can show that (4.44) has a bound like (4.45). These bounds together with the Weierstrass Comparison Test show that the series (4.38) are uniformly convergent on $[0, 1] \times [-h, h]$. □

LEMMA 4.3. *Let* $v, \ell \in C^2([0, 1] \times [-h, h])$ *be a solution of the boundary value problem* (4.27)–(4.33). *Then* v *and* ℓ *have the series representations* (4.38) *where* v_n, ℓ_n *satisfy*

$$\tau_1 k_n^2 v_n(z) - \beta_1 v_n''(z) \;=\; -\mathcal{N} k_n \ell_n'(z), \tag{4.46}$$
$$\beta_1 k_n^2 \ell_n(z) - \tau_3 \ell_n''(z) \;=\; \mathcal{N} k_n v_n'(z), \tag{4.47}$$
$$v_n'(\pm h) - (\omega/\lambda) k_n \ell_n(\pm h) \;=\; 0, \tag{4.48}$$
$$k_n X v_n(\pm h) + \tau_3 \ell_n'(\pm h) \;=\; 0. \tag{4.49}$$

Proof. According to Lemma 4.2, v and ℓ have the series representations (4.38) converging uniformly on $[0, 1] \times [-h, h]$. We now multiply equation (4.27) by $J_1(k_n r)$ and integrate from 0 to 1 to get that

$$\int_0^1 \big(\tau_1\big[-(r v_r)_r + v/r\big] - r\beta_1 v_{zz}\big) J_1(k_n s)\mathrm{d}r = \int_0^1 \mathcal{N} r \ell_{rz} J_1(k_n r)\mathrm{d}r. \tag{4.50}$$

But

$$\begin{aligned}
&\int_0^1 \big[-(r v_r)_r + v/r\big] J_1(k_n r)\mathrm{d}r \\
&\quad = -r v_r J_1(k_n r)\big|_{r=0}^{r=1} + \int_0^1 \Big(r\big(J_1(k_n r)\big)_r v_r + (v/r) J_1(k_n r)\Big)\,\mathrm{d}r \\
&\quad = r\big(J_1(k_n r)\big)_r v\big|_{r=0}^{r=1} + \int_0^1 \Big(-\big(r(J_1(k_n r))_r\big)_r + J_1(k_n r)/r\Big) v\,\mathrm{d}r \\
&\quad = \int_0^1 r k_n^2 J_1(k_n r) v\,\mathrm{d}r \\
&\quad = \frac{1}{2} J_2^2(k_n) k_n^2 v_n(z),
\end{aligned} \tag{4.51}$$

where we used that

$$-\big(r(J_1(k_n r))_r\big)_r + J_1(k_n r)/r = r k_n^2 J_1(k_n r).$$

Also

$$\int_0^1 r v_{zz} J_1(k_n r)\,\mathrm{d}r = \frac{1}{2} J_2^2(k_n) v_n''(z), \tag{4.52}$$

$$\begin{aligned}\int_0^1 r \ell_{rz} J_1(k_n r)\,\mathrm{d}r &= r\ell_z J_1(k_n r)\Big|_{r=0}^{r=1} - k_n \int_0^1 r\ell_z J_0(k_n r)\,\mathrm{d}r \\ &= -\frac{1}{2} J_2^2(k_n) k_n \ell_n'(z).\end{aligned} \tag{4.53}$$

Combining (4.50)–(4.53) we get (4.46). A similar argument, now multiplying (4.28) by $J_0(k_n r)$, gives (4.47). Also multiplying (4.32) by $r J_1(k_n r)$, and (4.33) by $r J_0(k_n r)$ and integrating from 0 to 1, we get the boundary conditions (4.48), (4.49). □

The next result, including the case $\hat{\mu}_1 = \hat{\mu}_2$, is shown in [13].

LEMMA 4.4. *Let the roots of*

$$\beta_1 \tau_3 \mu^4 - \left(\tau_1\tau_3 + \beta_1^2 - \mathcal{N}^2\right)\mu^2 + \tau_1\beta_1 = 0, \tag{4.54}$$

be given by $\pm\hat{\mu}_1, \pm\hat{\mu}_2$ *where* $\hat{\mu}_1 \neq \hat{\mu}_2$. *Then the boundary value problem* (4.46)–(4.49) *has a nontrivial solution if and only if* λ *satisfies one of the following two equations:*

$$\begin{aligned} f_{\mathrm{BAR}}(\lambda, k_n) \equiv\ & \left(\hat{\mu}_1 - (\omega/\lambda)P_1\right)(X + \tau_3\hat{\mu}_2 P_2)\tanh(k_n\hat{\mu}_1 h) \\ & - \left(\hat{\mu}_2 - (\omega/\lambda)P_2\right)(X + \tau_3\hat{\mu}_1 P_1)\tanh(k_n\hat{\mu}_2 h) = 0, \end{aligned} \tag{4.55}$$

$$\begin{aligned} f_{\mathrm{BUC}}(\lambda, k_n) \equiv\ & \left(\hat{\mu}_1 - (\omega/\lambda)P_1\right)(X + \tau_3\hat{\mu}_2 P_2)\tanh(k_n\hat{\mu}_2 h) \\ & - \left(\hat{\mu}_2 - (\omega/\lambda)P_2\right)(X + \tau_3\hat{\mu}_1 P_1)\tanh(k_n\hat{\mu}_1 h) = 0, \end{aligned} \tag{4.56}$$

where

$$P_i = \frac{\mathcal{N}\hat{\mu}_i}{\beta_1 - \tau_3\hat{\mu}_i^2}, \quad i = 1, 2. \tag{4.57}$$

Moreover the solution pair of (4.46)–(4.49) *corresponding to* (4.55) *is given by*

$$\begin{aligned} v_n(z) &= (X + \tau_3\hat{\mu}_2 P_2)\cosh(k_n\hat{\mu}_2 h)\cosh(k_n\hat{\mu}_1 z) \\ &\quad - (X + \tau_3\hat{\mu}_1 P_1)\cosh(k_n\hat{\mu}_1 h)\cosh(k_n\hat{\mu}_2 z), \\ \ell_n(z) &= (X + \tau_3\hat{\mu}_2 P_2)P_1\cosh(k_n\hat{\mu}_2 h)\sinh(k_n\hat{\mu}_1 z) \\ &\quad - (X + \tau_3\hat{\mu}_1 P_1)P_2\cosh(k_n\hat{\mu}_1 h)\sinh(k_n\hat{\mu}_2 z), \end{aligned} \tag{4.58}$$

while the pair corresponding to (4.56) *is given by*

$$\begin{aligned} v_n(z) &= (X + \tau_3\hat{\mu}_2 P_2)\sinh(k_n\hat{\mu}_2 h)\sinh(k_n\hat{\mu}_1 z) \\ &\quad - (X + \tau_3\hat{\mu}_1 P_1)\sinh(k_n\hat{\mu}_1 h)\sinh(k_n\hat{\mu}_2 z), \\ \ell_n(z) &= (X + \tau_3\hat{\mu}_2 P_2)P_1\sinh(k_n\hat{\mu}_2 h)\cosh(k_n\hat{\mu}_1 z) \\ &\quad - (X + \tau_3\hat{\mu}_1 P_1)P_2\sinh(k_n\hat{\mu}_1 h)\cosh(k_n\hat{\mu}_2 z). \end{aligned} \tag{4.59}$$

A first order approximation to $\mathbf{f}$ in (2.1)–(2.4) is given by $\hat{\mathbf{f}} = \mathbf{f}_\lambda + \mathbf{u}$ where $\mathbf{f}_\lambda$ and $\mathbf{u}$ are given by (3.4) and (4.15), respectively. Note that $\hat{\mathbf{f}}$ is symmetric with respect to the middle plane of Ω when (4.58) is used, and asymmetric when (4.59) is used. There are numerically generated pictures in [13] that illustrate these symmetry properties.*

5. The Complementing Condition

In this section we characterize the values of λ for which the *reduced* boundary value problem (4.27)–(4.33) fails to satisfy the complementing condition. We say that the complementing condition holds if the only exponentially bounded solution of a certain auxiliary boundary value problem on a half space, is the zero solution. Thompson in [20] made the observations that in the context of linearized elasticity the complementing condition is equivalent to the condition that all Rayleigh waves travel with nonzero velocity (see also [19]).

For the boundary value problem (4.27)–(4.33), the leading parts of the differential operators and boundary conditions** at $z = \pm h$ are given by:

$$\begin{aligned} -\tau_1 v_{rr} - \beta_1 v_{zz} &= \mathcal{N}\ell_{rz}, \\ -\beta_1 \ell_{rr} - \tau_3 \ell_{zz} &= \mathcal{N} v_{rz}, \quad \text{and} \end{aligned} \tag{5.1}$$

$$\begin{aligned} v_z + (\omega/\lambda)\ell_r &= 0, \\ \tau_3 \ell_z + X v_r &= 0, \end{aligned} \tag{5.2}$$

respectively.

We look for solutions of the above equations of the form

$$v(r,z) = w_1(z)\,\mathrm{e}^{\mathrm{i}\xi r}, \qquad \ell(r,z) = w_2(z)\,\mathrm{e}^{\mathrm{i}\xi r}, \tag{5.3}$$

where $z > 0$, $r, \xi \in \mathbb{R}$ and w_1, w_2 are bounded functions. If we substitute (5.3) into (5.1)–(5.2), we get that w_1, w_2 must be solutions of

$$\begin{aligned} \tau_1 \xi^2 w_1 - \beta_1 w_1'' &= \mathcal{N}\mathrm{i}\,\xi w_2', \\ \beta_1 \xi^2 w_2 - \tau_3 w_2'' &= \mathcal{N}\mathrm{i}\,\xi w_1', \end{aligned} \tag{5.4}$$

$$\begin{aligned} w_1'(0) + (\omega/\lambda)\mathrm{i}\,\xi w_2(0) &= 0, \\ \tau_3 w_2'(0) + X\mathrm{i}\xi w_1(0) &= 0. \end{aligned} \tag{5.5}$$

An easy computation shows now that $w_1(z) = A_1\,\mathrm{e}^{\rho z}$ and $w_2(z) = A_2\,\mathrm{e}^{\rho z}$ are solutions of (5.4) if and only if $\rho = \pm\xi\hat{\mu}_1, \pm\xi\hat{\mu}_2$ where $\hat{\mu}_1, \hat{\mu}_2$ are the roots of (4.54).

* In that paper the author refers to buckling and barrelling type solutions as symmetry breaking and symmetry preserving solutions respectively.

** We only check the boundary conditions at the top and bottom of the cylinder. A similar argument shows that the boundary condition at the lateral surface, which is not of Dirichlet type, complements the differential operator.

Moreover the general solution of (5.4) satisfying the condition of boundedness, is given by

$$w_1(z) = A\mathrm{e}^{\xi\hat{\mu}_1 z} + B\mathrm{e}^{\xi\hat{\mu}_2 z}, \qquad w_2(z) = A\mathrm{i}P_1\mathrm{e}^{\xi\hat{\mu}_1 z} + B\mathrm{i}P_2\mathrm{e}^{\xi\hat{\mu}_2 z},$$

where P_1, P_2 are given by (4.57) and where we assumed that $\mathrm{Re}(\hat{\mu}_j) < 0$, $j = 1, 2$. The boundary conditions (5.5) are now equivalent to

$$\begin{pmatrix} \hat{\mu}_1 - (\omega/\lambda)P_1 & \hat{\mu}_2 - (\omega/\lambda)P_2 \\ \mathrm{i}(\tau_3\hat{\mu}_1 P_1 + X) & \mathrm{i}(\tau_3\hat{\mu}_2 P_2 + X) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.6}$$

This system and consequently (5.1)–(5.2) can have nontrivial solutions if and only if the determinant of the coefficient matrix in (5.6) vanishes. Hence, we have proved the following proposition.

LEMMA 5.1. *The complementing condition fails for the boundary value problem* (4.27)–(4.33) *if and only if*

$$\big(\hat{\mu}_1 - (\omega/\lambda)P_1\big)(\tau_3\hat{\mu}_2 P_2 + X) - \big(\hat{\mu}_2 - (\omega/\lambda)P_2\big)(\tau_3\hat{\mu}_1 P_1 + X) = 0. \tag{5.7}$$

6. Blatz–Ko Type Materials

We now study the existence and properties of solutions of equations (4.55) and (4.56) for the case of a Blatz–Ko type material, that is one with stored energy function σ (cf. (2.5)) given by

$$\sigma(a, b, c) = a + (1/m)c^{-m}, \tag{6.1}$$

for some positive constant m. In this case (3.3) reduces to

$$\omega - \lambda^2\big(\lambda^2\omega\big)^{-(m+1)} = 0,$$

which has solution

$$\hat{\omega}(\lambda) = \lambda^{-2m/(m+2)}.$$

We let

$$\eta \equiv \frac{\hat{\omega}(\lambda)}{\lambda} = \lambda^{-(3m+2)/(m+2)}. \tag{6.2}$$

It follows now from (4.10)–(4.12), (4.14), and (4.17) that

$$\tau_1 = 1 + (m+1)\eta^2, \qquad \tau_3 = m + 2, \qquad \beta_1 = 1, \tag{6.3}$$

$$X = m\eta, \qquad \mathcal{N} = (m+1)\eta. \tag{6.4}$$

It is easy to check now that the solutions of (4.54) are given by

$$\hat{\mu}_1^2 = 1, \qquad e^2 \equiv \hat{\mu}_2^2 = \frac{1 + (m+1)\eta^2}{m+2}, \tag{6.5}$$

and that (4.55) is given by

$$-\left(1+\eta^2\right)^2 \tanh(k_n h) + 4e\eta^2 \tanh(k_n e h) = 0. \tag{6.6}$$

If we let $\rho = k_n h$ and divide (6.6) by $4e\eta^2$, then we get that (6.6) is equivalent to

$$\tanh(\rho e) - \frac{(1+\eta^2)^2}{4e\eta^2} \tanh(\rho) = 0. \tag{6.7}$$

Using (6.5b) we can eliminate η^2 from (6.7) to get the following equation in terms of ρ, e:

$$f_{\mathrm{BAR}}(\rho, e) \equiv \tanh(\rho e) - \frac{((m+2)e^2+m)^2}{4(m+1)e((m+2)e^2-1)} \tanh(\rho) = 0, \tag{6.8}$$

where f_{BAR}: $[0,\infty) \times [1,\infty) \to \mathbb{R}$. To simplify the notation from now we will suppress the subscript BAR. It follows now that

$$\frac{\partial f}{\partial e} = \rho\, \mathrm{sech}^2(\rho e) - \frac{p_1(e)\tanh(\rho)}{4(m+1)e^2[(m+2)e^2-1]^2}, \tag{6.9}$$

$$\frac{\partial^2 f}{\partial e^2} = -2\rho^2 \tanh(\rho e)\, \mathrm{sech}^2(\rho e) - \frac{p_2(e)\tanh(\rho)}{2(m+1)e^3[(m+2)e^2-1]^3}, \tag{6.10}$$

where

$$\begin{aligned} p_1(e) &= \big((m+2)e^2+m\big)\big[(m+2)^2e^4 - 3(m+1)(m+2)e^2 + m\big], \\ p_2(e) &= (2m+1)(m+2)^3e^6 + 3(2m^2+2m+1)(m+2)^2e^4 \\ &\quad - 3m^2(m+2)e^2 + m^2. \end{aligned}$$

Since $p_1(1) = -4(m+1)^3 < 0$ it follows from (6.8) and (6.9) that

$$f(\rho,1) = 0, \qquad \frac{\partial f}{\partial e}(\rho,1) > 0, \qquad \lim_{e\to\infty} f(\rho,e) = -\infty. \tag{6.11}$$

Thus equation (6.8) has a solution $e > 1$ for any given ρ. A lengthy but otherwise elementary computation shows that

$$p_2(1) > 0, \qquad p_2'(1) > 0, \qquad p_2''(1) > 0, \qquad p_2'''(e) > 0,$$

which shows that $p_2(e) > 0$ and thus from (6.10) that

$$\frac{\partial^2 f}{\partial e^2}(\rho, e) < 0 \quad \text{for all } e > 1. \tag{6.12}$$

Hence (6.8) has a unique solution $\hat{e}(\rho)$ where $\hat{e} : (0,\infty) \to (1,\infty)$. An argument based on the Implicit Function Theorem (see [17]) shows that $\hat{e}$ is a continuously differentiable function of ρ. Note that from (6.5b) it follows that there is a 1–1 correspondence between e and η. Furthermore, there is a 1–1 correspondence between $\eta \in [1,\infty)$ and $\lambda \in (0,1]$. Thus we can state the above existence and uniqueness result in terms of λ as follows.

THEOREM 6.1. *Let the constitutive function of the material of the cylinder be given by* (6.1). *Then there exists a continuously differentiable function* $\hat{\lambda}_{\mathrm{BAR}}$: $(0, \infty) \to (0, 1]$ *such that* $\hat{\lambda}_{\mathrm{BAR}}(k_n h)$ *is the unique solution of* (4.55) *or equivalently* (6.6).

We now derive some properties of the function $\hat{\lambda}_{\mathrm{BAR}}$ in Theorem 6.1. We work with $\hat{e}$ and then use (6.2b), (6.5b) to get the corresponding results for $\hat{\lambda}_{\mathrm{BAR}}$. If we differentiate $f(\rho, \hat{e}(\rho)) = 0$ with respect to ρ, we get that

$$\frac{\partial f}{\partial \rho}\big(\rho, \hat{e}(\rho)\big) + \frac{\partial f}{\partial e}\big(\rho, \hat{e}(\rho)\big)\frac{\mathrm{d}\hat{e}}{\mathrm{d}\rho}(\rho) = 0. \tag{6.13}$$

But from (6.11b), (6.12) it follows that

$$\frac{\partial f}{\partial e}\big(\rho, \hat{e}(\rho)\big) < 0. \tag{6.14}$$

Hence it follows from (6.13) that

$$\operatorname{sign}\frac{\mathrm{d}\hat{e}}{\mathrm{d}\rho}(\rho) = \operatorname{sign}\frac{\partial f}{\partial \rho}\big(\rho, \hat{e}(\rho)\big). \tag{6.15}$$

From (6.8) we get that

$$\frac{\partial f}{\partial \rho}(\rho, e) = e\operatorname{sech}^2(\rho e) - \frac{q_1(e)}{q_2(e)}\operatorname{sech}^2(\rho), \tag{6.16}$$

where for simplicity we have set

$$q_1(e) = \big[(m+2)e^2 + m\big]^2, \qquad q_2(e) = 4(m+1)e\big[(m+2)e^2 - 1\big]. \tag{6.17}$$

LEMMA 6.2. *Let* $q(e) = eq_2(e) - q_1(e)$. *Then* $q(e) > 0$ *for all* $e > 1$.

Proof. By direct computation one easily gets that

$$\begin{aligned}
&q(1) = 0, \qquad q'(1) = 8m^2 + 16m + 8 > 0,\\
&q''(1) = 32m^2 + 80m + 40 > 0,\\
&q'''(e) = (72m^2 + 192m + 96)e > 0, \quad \text{for all } e > 1,
\end{aligned}$$

from which the result follows. □

PROPOSITION 6.3. *Let* $\hat{e}$: $(0, \infty) \to (1, \infty)$ *be the* C^1 *solution of the equation* $f(\rho, \hat{e}(\rho)) = 0$. *Then* $\hat{e}$ *have the following properties:*

$$\liminf_{\rho\to\infty} \hat{e}(\rho) > 1, \qquad \limsup_{\rho\to\infty} \hat{e}(\rho) < \infty, \tag{6.18}$$

$$\lim_{\rho\to 0} \hat{e}(\rho) = \infty, \qquad \liminf_{\rho\to 0} \rho\hat{e}(\rho) > 0, \qquad \limsup_{\rho\to 0} \rho\hat{e}(\rho) < \infty. \tag{6.19}$$

Proof. We argue by contradiction. To get (6.18a) suppose that there exists a sequence $\rho_j \to \infty$ such that $e_j \equiv \hat{e}(\rho_j) \to 1$. Using Taylor's Theorem we can expand $\tanh(\rho e)$ as

$$\tanh(\rho e) = \tanh(\rho) + \rho\, \mathrm{sech}^2(\rho e^*)(e-1)$$

where e^* is between e and 1. Using this in the expression for $f(\rho, e)$ and dividing by $e-1$ we get that

$$0 = \frac{f(\rho_j, e_j)}{e_j - 1} = \frac{(1 - q_1(e_j)/q_2(e_j))}{e_j - 1}\tanh(\rho_j) + \rho_j \mathrm{sech}^2(\rho_j e_j^*). \tag{6.20}$$

Since $e_j \to 1$ we have that $q_1(e_j)/q_2(e_j) \to 1$. Thus using L'Hopital's rule we get that

$$\begin{aligned}\lim_{j\to\infty} \frac{(1 - q_1(e_j)/q_2(e_j))}{e_j - 1} &= \lim_{j\to\infty} \frac{-p_1(e_j)}{4(m+1)e_j^2((m+2)e_j^2 - 1)^2} \\ &= \frac{-p_1(1)}{4(m+1)^3} = 1.\end{aligned}$$

Since $e_j^* \to 1$, the limit of the right-hand side of (6.20) is one, which gives a contradiction.

For (6.18b) suppose that that there exists a sequence $\rho_j \to \infty$ such that $e_j \to \infty$. Then we have that

$$0 = \frac{f(\rho_j, e_j)}{e_j} = \frac{\tanh(\rho_j e_j)}{e_j} - \frac{q_1(e_j)}{e_j q_2(e_j)}\tanh(\rho_j) \to \frac{-(m+2)}{4(m+1)},$$

which again leads to a contradiction.

To get (6.19a), let $\rho_j \to 0$ and with e_j bounded. Without loss of generality, we may assume that $e_j \to e_0 \geqslant 1$. If $e_0 = 1$, then dividing (6.20) by ρ_j and taking the limit as $j \to \infty$, we get that the right-hand side of that expression converges to

$$\frac{-p_1(1)}{(m+1)^3} + 1 = 2,$$

which yields a contradiction. If on the other hand $e_0 > 1$, then

$$\begin{aligned}0 = \frac{f(\rho_j, e_j)}{\tanh(\rho_j)} &= \frac{\tanh(\rho_j e_j)}{\tanh(\rho_j)} - \frac{q_1(e_j)}{q_2(e_j)} \\ &\to e_0 - \frac{q_1(e_0)}{q_2(e_0)} > 0,\end{aligned}$$

where the last inequality follows from Lemma 6.2, thus leading to another contradiction.

To argue (6.19b), let $\rho_j \to 0$ and $\rho_j e_j \to 0$. Then

$$0 = \frac{f(\rho_j, e_j)}{\rho_j e_j} = \frac{\tanh(\rho_j e_j)}{\rho_j e_j} - \left[\frac{q_1(e_j)}{e_j q_2(e_j)}\right]\frac{\tanh(\rho_j)}{\rho_j}. \tag{6.21}$$

Since $e_j \to \infty$ by (6.19a), we get that

$$\frac{q_1(e_j)}{e_j q_2(e_j)} \to \frac{m+2}{4(m+1)} < 1.$$

Thus letting $j \to \infty$ in (6.21) we get that the right-hand side converges to

$$1 - \frac{m+2}{4(m+1)} \neq 0,$$

which gives a contradiction. A similar argument using (6.21) but now with $\rho_j e_j \to \infty$, yields (6.19c). □

LEMMA 6.4. *Let*

$$\gamma(x) = \begin{cases} 1, & x = 0, \\ \dfrac{\tanh(x)}{x}, & x > 0. \end{cases}$$

Then $\gamma(x) > \operatorname{sech}^2(x)$ for all $x > 0$ and γ is one to one with a continuous inverse.

Proof. That $\gamma(x) > \operatorname{sech}^2(x)$ for $x > 0$ is equivalent to $\tanh(x) > x\operatorname{sech}^2(x)$. Letting $h(x) = \tanh(x) - x \operatorname{sech}^2(x)$, we get that $h(0) = 0$ and that

$$h'(x) = 2x \operatorname{sech}^2 \tanh(x) > 0,$$

for $x > 0$. Thus $h(x) > 0$ for $x > 0$ and the inequality follows. The statement about γ being 1–1 follows from the fact that $\gamma(x) > \operatorname{sech}^2(x)$ implies that $\gamma'(x) < 0$. The continuity of the inverse follows from the inverse function theorem. □

LEMMA 6.5. *The function $\hat{e}$: $(0, \infty) \to [1, \infty)$ satisfies that*

$$\lim_{\rho\to 0} \rho\hat{e}(\rho) = \mu, \qquad \lim_{\rho\to\infty} \frac{q_1(\hat{e}(\rho))}{q_2(\hat{e}(\rho))} = 1, \tag{6.22}$$

where μ is the unique positive solution of

$$\frac{\tanh(\mu)}{\mu} = \frac{m+2}{4(m+1)}. \tag{6.23}$$

Proof. Using Taylor's Theorem we can write that

$$\tanh(x) = x + \mathrm{o}(x^2). \tag{6.24}$$

It follows from (6.19a) and (6.17) that

$$\lim_{\rho\to 0} \frac{q_1(\hat{e}(\rho))}{\hat{e}(\rho) q_2(\hat{e}(\rho))} = \frac{m+2}{4(m+1)}. \tag{6.25}$$

Thus from $f(\rho, \hat{e}(\rho)) = 0$ and (6.8) we get that

$$\gamma\big(\rho\hat{e}(\rho)\big) = \frac{\tanh(\rho\hat{e}(\rho))}{\rho\hat{e}(\rho)} = \frac{q_1(\hat{e}(\rho))}{\hat{e}(\rho)q_2(\hat{e}(\rho))}\left[1 + \frac{\mathrm{o}(\rho^2)}{\rho}\right].$$

The result (6.22a) now follows from (6.24), (6.25) and the fact that γ^{-1} is continuous.

To get (6.22b) we note that $f(\rho, \hat{e}(\rho)) = 0$ is equivalent to

$$\frac{q_1(\hat{e}(\rho))}{q_2(\hat{e}(\rho))} = \frac{\tanh(\rho\hat{e}(\rho))}{\tanh(\rho)}.$$

But from (6.18a) it follows that $\rho\hat{e}(\rho) \to \infty$ as $\rho \to \infty$. Thus the result now follows from the above equation and the fact that $\tanh(x) \to 1$ as $x \to \infty$. □

PROPOSITION 6.6. *The function* $\hat{e}$: $(0, \infty) \to [1, \infty)$ *is strictly decreasing.*

Proof. We show first that $\hat{e}$ is decreasing for ρ small enough. According to (6.15) it is enough to show that

$$\frac{\partial f}{\partial \rho}\big(\rho, \hat{e}(\rho)\big) < 0, \tag{6.26}$$

for ρ sufficiently small. Combining (6.16), (6.22a), and (6.23) we get that

$$\begin{aligned}\frac{1}{\hat{e}(\rho)}\frac{\partial f}{\partial \rho}\big(\rho, \hat{e}(\rho)\big) &= \operatorname{sech}^2(\rho\hat{e}(\rho)) - \frac{q_1(\hat{e}(\rho))}{\hat{e}(\rho)q_2(\hat{e}(\rho))}\operatorname{sech}^2(\rho)\\ &\to \operatorname{sech}^2(\mu) - \frac{m+2}{4(m+1)}\\ &= \operatorname{sech}^2(\mu) - \frac{\tanh(\mu)}{\mu} < 0, \quad \text{as } \rho \to 0,\end{aligned}$$

where the last inequality follows from Lemma 6.4. Thus (6.26) holds for ρ small.

If $\hat{e}$ becomes increasing, there most be a $\bar{\rho} > 0$ such that $\hat{e}'(\bar{\rho}) = 0$ and $\hat{e}''(\bar{\rho}) > 0$, i.e., there must be a local minimum. But if $\hat{e}'(\bar{\rho}) = 0$, then

$$\frac{\partial f}{\partial \rho}\big(\bar{\rho}, \hat{e}(\bar{\rho})\big) = 0, \qquad \hat{e}''(\bar{\rho}) = -\frac{(\partial^2 f/\partial\rho^2)(\bar{\rho}, \hat{e}(\bar{\rho}))}{(\partial f/\partial e)(\bar{\rho}, \hat{e}(\bar{\rho}))}.$$

Using the first of these two equations and the definition of f we can easily get that

$$\frac{\partial^2 f}{\partial \rho^2}\big(\bar{\rho}, \hat{e}(\bar{\rho})\big) = 2\frac{q_1(\hat{e}(\bar{\rho}))}{q_2(\hat{e}(\bar{\rho}))}\operatorname{sech}^2(\bar{\rho})\big[\tanh(\bar{\rho}) - \hat{e}(\bar{\rho})\tanh(\bar{\rho}\hat{e}(\bar{\rho})\big] < 0,$$

which combined with (6.14) yields that $\hat{e}''(\bar{\rho}) < 0$. Thus $\hat{e}$ can not have a local minima and hence must be decreasing for all ρ. □

We study now the solutions of (4.56). For the material (6.1), this equation is equivalent to:

$$g(\rho, e) \equiv \coth(\rho e) - \coth(\rho)\frac{q_1(e)}{q_2(e)} = 0, \tag{6.27}$$

or also equivalently:

$$\hat{g}(\rho, e) \equiv \tanh(\rho e) - \tanh(\rho)\frac{q_2(e)}{q_1(e)} = 0, \tag{6.28}$$

where q_1, q_2 are given by (6.17). By direct computation we get that:

$$\hat{g}(\rho, 1) = 0, \qquad \lim_{e\to\infty} \hat{g}(\rho, e) = 1, \tag{6.29}$$

$$\frac{\partial \hat{g}}{\partial e}(\rho, 1) = \rho \mathrm{sech}^2(\rho) - \tanh(\rho) < 0 \quad \text{(cf. Lemma 6.4)}, \tag{6.30}$$

from which it follows that $\hat{g}(\rho, e) = 0$ has a solution for each $\rho > 0$. Let $\tilde{e}$: $(0, \infty) \to (1, \infty)$ be the smallest such solution. Again, an argument based on the Implicit Function Theorem shows that $\tilde{e}$ is a continuously differentiable function of ρ. Because of the 1–1 correspondence between e and η, and η and λ (see (6.2), (6.5b)), we get the following:

THEOREM 6.7. *Let the constitutive function of the material of the cylinder be given by* (6.1). *Then there exists a continuously differentiable function* $\hat{\lambda}_{\mathrm{BUC}}$: $(0, \infty) \to (0, 1]$ *such that* $\hat{\lambda}_{\mathrm{BUC}}(k_n h)$ *is the smallest solution of* (4.56).

The function $\tilde{e}$: $(0, \infty) \to (1, \infty)$ has the following properties:

PROPOSITION 6.8. *Let* $\tilde{e}$: $(0, \infty) \to (1, \infty)$ *be the* C^1 *smallest solution of either* (6.27) *or* (6.28). *Then* $\tilde{e}$ *has the following properties:*

$$\liminf_{\rho\to\infty} \tilde{e}(\rho) > 1, \qquad \limsup_{\rho\to\infty} \tilde{e}(\rho) < \infty, \tag{6.31}$$

$$\lim_{\rho\to 0} \tilde{e}(\rho) = 1, \qquad \lim_{\rho\to\infty} \frac{q_1(\tilde{e}(\rho))}{q_2(\tilde{e}(\rho))} = 1. \tag{6.32}$$

Proof. To get (6.31a), let $\rho_j \to \infty$ with $e_j = \tilde{e}(\rho_j) \to 1$. Using the Taylor expansion

$$\coth(\rho e) = \coth(\rho) - \rho\, \mathrm{csch}^2(\rho e^*)(e - 1),$$

we get that $g(\rho_j, e_j) = 0$ is equivalent to:

$$0 = \frac{g(\rho_j, e_j)}{e_j - 1} = \frac{(1 - q_1(e_j)/q_2(e_j))}{e_j - 1}\coth(\rho_j) - \rho_j\, \mathrm{csch}^2(\rho_j e_j^*).$$

As in the proof of Proposition 6.3, using L'Hopital's rule, we get that the right-hand side of this expression converges to one as $j \to \infty$, thus leading to a contradiction. The proof of (6.31b) is again similar to (6.18) but using (6.27).

To get (6.32a) first note that since $\tilde{e}(\rho) > 1$ for all $\rho > 0$, we have that $\liminf_{\rho\to 0} \tilde{e}(\rho) \geqslant 1$. Now let $a = \limsup_{\rho\to 0} \tilde{e}(\rho)$. Note that $a < \infty$ for if $\rho_j \to 0$ with $e_j = \tilde{e}(\rho_j) \to \infty$, then we have the following possibilities:

(i) The sequence $(\rho_j e_j)$ is unbounded. Without lost of generality, we may assume $\rho_j e_j \to \infty$. Then

$$0 = \hat{g}(\rho_j, e_j) = \tanh(\rho_j e_j) - \tanh(\rho_j)\frac{q_2(e_j)}{q_1(e_j)} \to 1 - (0)(0) = 1,$$

which yields a contradiction.

(ii) The sequence $(\rho_j e_j)$ is bounded. Without lost of generality, we may assume $\rho_j e_j \to \alpha \geqslant 0$.

(a) If $\alpha > 0$, then $0 = \hat{g}(\rho_j, e_j) \to \tanh(\alpha) - (0)(0) \neq 0$, which is a contradiction.

(b) If $\alpha = 0$, then

$$\begin{aligned} 0 &= \frac{\hat{g}(\rho_j, e_j)}{\rho_j e_j} \\ &= \frac{\tanh(\rho_j e_j)}{\rho_j e_j} - \frac{\tanh(\rho_j)}{\rho_j}\frac{q_2(e_j)}{e_j q_1(e_j)} \to 1 - 1(0) = 1, \end{aligned}$$

leading again to a contradiction.

Hence $a < \infty$ and clearly $a \geqslant 1$. If $a > 1$, then with $\rho_j \to 0$ and $e_j = \tilde{e}(\rho_j) \to a$, we get that

$$0 = \frac{\hat{g}(\rho_j, e_j)}{\rho_j} = \frac{\tanh(\rho_j e_j)}{\rho_j} - \frac{\tanh(\rho_j)}{\rho_j}\frac{q_2(e_j)}{q_1(e_j)} \to a - \frac{q_2(a)}{q_1(a)}.$$

But by direct computation one can see that $aq_1(a) - q_2(a) > 0$ for $a > 1$, thus leading to another contradiction. Thus we must have $a = 1$, i.e.,

$$\liminf_{\rho\to 0} \tilde{e}(\rho) = \limsup_{\rho\to 0} \tilde{e}(\rho) = 1,$$

from which (6.32a) follows. The proof of (6.32b) is as that of (6.22). □

PROPOSITION 6.9. *The function* $\tilde{e}$: $(0, \infty) \to [1, \infty)$ *is strictly increasing.*

Proof. From $\hat{g}(\rho, \tilde{e}(\rho)) = 0$, it follows that

$$\frac{q_2(\tilde{e}(\rho))}{q_1(\tilde{e}(\rho))} = \frac{\tanh(\rho\tilde{e}(\rho))}{\tanh(\rho)}.$$

Also

$$\frac{\partial\hat{g}}{\partial\rho}(\rho, e) = e\,\mathrm{sech}^2(\rho e) - \mathrm{sech}^2(\rho)\frac{q_2(e)}{q_1(e)}.$$

Combining the above two equations we get that

$$\rho \frac{\partial \hat{g}}{\partial \rho}\big(\rho, \tilde{e}(\rho)\big) = \tanh\big(\rho\tilde{e}(\rho)\big)\big[\chi(\rho\tilde{e}(\rho)) - \chi(\rho)\big], \tag{6.33}$$

where

$$\chi(\rho) = \frac{\rho\,\mathrm{sech}^2(\rho)}{\tanh(\rho)}.$$

By direct computation one gets that χ is strictly decreasing. Since $\rho\tilde{e}(\rho) > \rho$, it follows from (6.33) that

$$\frac{\partial \hat{g}}{\partial \rho}\big(\rho, \tilde{e}(\rho)\big) < 0.$$

Also from (6.29), (6.30), we get that

$$\frac{\partial \hat{g}}{\partial e}\big(\rho, \tilde{e}(\rho)\big) > 0.$$

Upon differentiating $\hat{g}(\rho, \tilde{e}(\rho)) = 0$ with respect to ρ we get that

$$\frac{\mathrm{d}\tilde{e}}{\mathrm{d}\rho}(\rho) = -\frac{(\partial\hat{g}/\partial\rho)(\rho, \tilde{e}(\rho))}{(\partial\hat{g}/\partial e)(\rho, \tilde{e}(\rho))},$$

which combined with the two inequalities above, yields that $\tilde{e}'(\rho) > 0$, i.e., $\tilde{e}$ is strictly increasing. □

We now have the main result of this section:

THEOREM 6.10. *Let the constitutive function of the material of the cylinder be given by* (6.1). *Then the roots of* (4.55) *are given by an increasing sequence* $\{\hat{\lambda}_n\}$ *and those of* (4.56) *by a decreasing sequence* $\{\tilde{\lambda}_n\}$ *where* $\hat{\lambda}_n = \hat{\lambda}_{\mathrm{BAR}}(k_n h)$ *and* $\tilde{\lambda}_n = \hat{\lambda}_{\mathrm{BUC}}(k_n h)$. *Moreover*

$$\hat{\lambda}_{\mathrm{BAR}}(k_n h) \neq \hat{\lambda}_{\mathrm{BAR}}(k_m h), \qquad \hat{\lambda}_{\mathrm{BUC}}(k_n h) \neq \hat{\lambda}_{\mathrm{BUC}}(k_m h),$$

for $n \neq m$. *Both sequences* $(\hat{\lambda}_n)$, $(\tilde{\lambda}_n)$ *converge to* λ_∞ *which is a solution** *of* (5.7), *i.e., a value of* λ *at which the complementing condition for the boundary value problem* (4.27)–(4.33) *fails. In particular*

* For this material, an elementary computation using Descartes rule of signs shows that (5.7) has a unique solution.

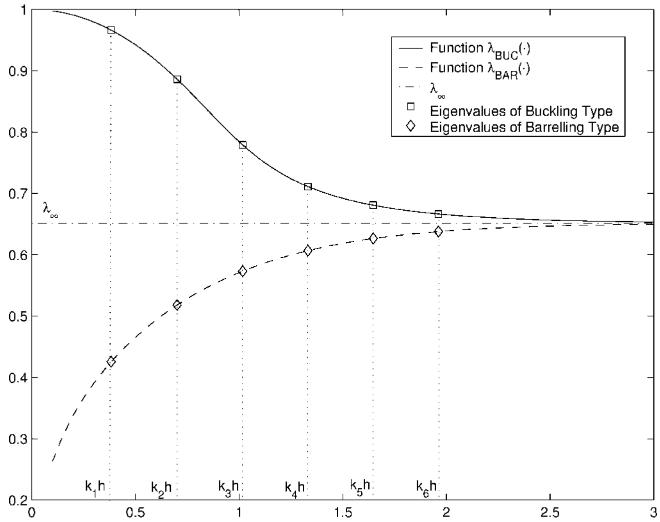

Figure 1. Graph illustrating the results of Theorem (6.10.) for the case $m = 13.3$ and cylinder half-height of $h = 0.1$.

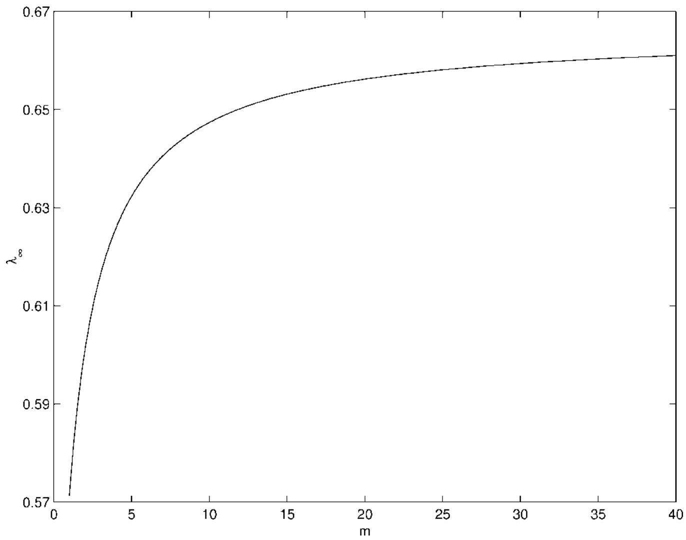

Figure 2. Graph of λ_∞ as given by (6.34) as a function of m.

$$\lambda_\infty^{2(3m+2/m+2)} = \frac{m+1}{(m+2)e_\infty^2 - 1}, \tag{6.34}$$

and e_∞ is a root of the equation $q_1(e) = q_2(e)$.

Proof. We only present the proof of the statements for (4.55) the others are similar. If we combine Propositions 6.3 and 6.6 we get that

$$\lim_{\rho\to\infty} \hat{e}(\rho) = e_\infty, \tag{6.35}$$

exists. From (6.2), (6.5b), and Theorem 6.1 we get that

$$\hat{\lambda}_{\mathrm{BAR}}(\rho) = \left[\frac{m+1}{(m+2)\hat{e}(\rho)^2 - 1} \right]^{(m+2)/(2(3m+2))}. \tag{6.36}$$

It follows now from Proposition 6.6 that $\hat{\lambda}_{\mathrm{BAR}}\colon (0, \infty) \to (0, 1]$ is strictly increasing. Thus, $\{\hat{\lambda}_n\}$ forms an strictly increasing sequence and combining (6.35) and (6.36) we get that $\hat{\lambda}_n \to \lambda_\infty$ where λ_∞ is given by (6.34). The statement that $\hat{\lambda}_{\mathrm{BAR}}(k_n h) \neq \hat{\lambda}_{\mathrm{BAR}}(k_m h)$ for $n \neq m$ follows from the monotonicity of $\hat{\lambda}_{\mathrm{BAR}}$. Now from (6.22b) and (6.35) we get that e_∞ must be a solution of $q_1(e) = q_2(e)$. But using the expressions (6.3)–(6.5) one can easily get that (5.7) reduces to $q_1(e) = q_2(e)$ and thus that λ_∞ represents a value at which the complementing condition for the boundary value problem (4.27)–(4.33) fails. □

The numerical results in [17] were for the case $m = 13.3$. In this case, λ_∞, which is the value at which the complementing condition fails for the linear problem in the above theorem, is approximately 0.6525. We show in Figure 1 this value as well as the corresponding eigenvalues of buckling and barrelling type for this material. Furthermore, in Figure 2 we show how λ_∞ varies for different values of m between 1 and 40.

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