A note on the Radon–Riesz property

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Abstract

The "Radon–Riez" property gives sufficient conditions for extracting strongly convergent subsequences out of weakly convergent ones in Banach spaces. The key assumption is that the norms of the elements of the weakly convergent sequence, converge to the norm of the weak limit. This result is well known for Hilbert spaces of which L_2 is a special case, but it is also true for uniformly convex Banach spaces of which L^p with $1 < p < \infty$ are special cases. In this expository paper I will present a proof of this more general result.

1 Introduction and problem formulation

In this note we consider the problem of under which conditions, a *weakly convergent* sequence, actually converges *strongly.* The *Radon–Riesz property* states that if in addition, the norms of the weakly convergent sequence converge, then the sequence converges strongly. This result is well known for *Hilbert spaces*, like L_2, but it is also true for more general *Banach spaces* like L_p where $1 < p < \infty$. The motivation for this note comes from [3] for a problem nonlinear elasticity, where the Radon–Riesz property is used to prove the convergence of certain approximating sequences in the context of computing singular minimizers of a stored energy functional.

2 Basic notions and definitions

2.1 Banach spaces and functionals

Let X be a normed linear space with norm denoted by $\|\cdot\|$. We say that X is a *Banach space* if every Cauchy sequence in X, with respect to the norm $\|\cdot\|$, converges to an element of X. In this case we say that is X *complete* in its norm.

A function $F : X \to \mathbb{R}$ on a Banach space X is called a (real) *bounded linear functional* on X if

i) $F(\alpha u + \beta v) = \alpha F(u) + \beta F(v)$ for all $u, v \in X$ and $\alpha, \beta \in \mathbb{R}$, and

ii) $\|F\| \equiv \sup_{u \neq 0} \dfrac{|F(u)|}{\|u\|} < \infty$.

The set of all bounded linear functionals on X is called the *dual* of X and is denoted by X^*. It follows from this definition that X^* is a Banach space.

A sequence of functionals (F_n) in X^* *converge weakly* to $F \in X^*$, written $F_n \rightharpoonup F$, if $F_n(u) \to F(u)$ for any $u \in X$.

2.2 Inner product spaces

A linear *real inner product space* X is a linear space with a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ such that for all $u, v, w \in X$ and $\alpha \in \mathbb{R}$,

i) $\langle u, v \rangle = \langle v, u \rangle$,

ii) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$,

iii) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$,

iv) $\langle u, u \rangle > 0$ for all $u \neq 0$.

An inner product space becomes a normed space with the norm $\|u\| = \langle u, u \rangle^{1/2}$. This is called the norm *induced* by the inner product. If the inner product space is *complete* in its induced norm, it is called a *Hilbert space*.

In an inner product space we always have the *parallelogram law*:

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

2.3 The L_p spaces

For $U \subset \mathbb{R}^n$ measurable and bounded, we define for $1 \leq p < \infty$

$$L_p(U) = \left\{ f : U \to \mathbb{R}, \text{ such that } \int_U |f(x)|^p \, \mathrm{d}x < \infty \right\}.$$

For $f \in L_p(U)$ we let

$$\|f\|_p = \left[\int_U |f(x)|^p \, \mathrm{d}x \right]^{\frac{1}{p}}.$$

$L_\infty(U)$ consists of functions that are bounded almost everywhere with the norm given by the *essential supremum*.

Facts about L_p spaces:

i) L_p is a Banach space for $1 \leq p \leq \infty$.

ii) L_2 is a Hilbert space with $\langle f, g \rangle = \int_U f(x)g(x)\,\mathrm{d}x$, for all $f, g \in L_2$.

iii) $(L_p)^*$ can be identified with L_q where $\frac{1}{p} + \frac{1}{q} = 1$ for $1 \leq p < \infty$. Moreover, for every $F \in (L_p)^*$ there exists $g \in L_q$ such that $\|F\| = \|g\|_q$ and

$$F(f) = \int_U f(x)g(x)\,\mathrm{d}x, \quad \forall f \in L_p.$$

A sequence (f_n) converges weakly to f in L_p, written $f_n \rightharpoonup f$, if

$$\lim_{n\to\infty} \int_U f_n(x)g(x)\,\mathrm{d}x = \int_U f(x)g(x)\,\mathrm{d}x, \quad \forall g \in L^q.$$

We can now state and prove the *Radon–Riesz property* for L_2:

Lemma 2.1. *Let $f_n \rightharpoonup f$ in L_2 and let $\|f_n\|_2 \to \|f\|_2$. Then $f_n \to f$ in L_2, i.e., $\|f_n - f\|_2 \to 0$.*

Proof: Using the inner product of L_2, the assumed weak convergence and that of the norms, this follows from:

$$\begin{aligned} \|f_n - f\|_2^2 &= \langle f_n - f, f_n - f \rangle, \\ &= \|f_n\|_2^2 - 2\langle f_n, f \rangle + \|f\|_2^2, \\ &\to \|f\|_2^2 - 2\langle f, f \rangle + \|f\|_2^2 = 0. \end{aligned}$$

□

3 The Radon–Riesz property for Banach spaces

We begin this section with the definition of a *uniformly convex* normed space. The definition basically says that in such space, if the length of the line segment joining any two points in the unit sphere is bounded away from zero, then the midpoint of that segment is bounded away from the boundary of the unit sphere.

Definition 3.1. A normed space X is called *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta > 0$ (that depends on ε) such that for all $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| > \varepsilon$, we have that

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

We state without proof the following result about the uniform convexity of the L_p spaces.

Theorem 3.2 ([1])**.** *L_p with $1 < p < \infty$ is a uniformly convex Banach space.*

We shall also need the following weak lower semi–continuity property of the norm in X^*.

Proposition 3.3. *Let X be a Banach space and assume that $F_n \rightharpoonup F$ in X^*. Then*

$$\|F\| \leq \liminf_n \|F_n\| .$$

Proof: For any $x \in X$ we have from $F_n \rightharpoonup F$ in X^* that $F_n(x) \to F(x)$. Also

$$|F_n(x)| \leq \|F_n\| \, \|x\| .$$

Letting $n \to \infty$ we get that

$$|F(x)| \leq \liminf_n \|F_n\| \, \|x\| .$$

Since $x \in X$ is arbitrary, the result follows. □

We now have the main result of this note.

Theorem 3.4 ([2], [4])**.** *Let X be a uniformly convex Banach space and assume that $F_n \rightharpoonup F$ in X^* and that $\limsup_n \|F_n\| \leq \|F\|$. Then $F_n \to F$ strongly in X^*.*

Proof: Without loss of generality we may assume that $F \neq 0$. By the previous proposition and the given hypothesis we have that $\lim_n \|F_n\| = \|F\|$. Define

$$y_n = \frac{F_n}{\max\{\|F_n\|, \|F\|\}}, \quad y = \frac{F}{\|F\|}.$$

It follows that $y_n \rightharpoonup y$ in X^*. We now show that $y_n \to y$ strongly in X^* from which $F_n \to F$ strongly in X^* follows. Now $\frac{1}{2}(y_n + y) \rightharpoonup y$ and so, by the previous proposition:

$$1 = \|y\| \leq \liminf \left\| \frac{1}{2}(y_n + y) \right\|. \tag{3.1}$$

Suppose, to argue by contradiction, that for some $\varepsilon_0 > 0$, passing to a subsequence if necessary,

$$\|y_n - y\| \geq \varepsilon_0, \quad \forall n.$$

By the uniform convexity, we would have that for some $\delta_0 > 0$,

$$\left\| \frac{1}{2}(y_n + y) \right\| \leq 1 - \delta_0 \quad \forall n.$$

But this would be a contradiction to (3.1), which completes the proof. □

Using the previous theorem and that the spaces L_p are *reflexive* for $1 < p < \infty$, we get the following:

Corollary 3.5. *Let (f_n) be a sequence in L_p, $1 < p < \infty$, such that $f_n \rightharpoonup f$ in L_p and with $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$. Then $f_n \to f$ in L_p.*

Proof: In the previous theorem, take

$$F_n(g) = \int_U f_n(x)g(x)\,\mathrm{d}x, \quad F(g) = \int_U f(x)g(x)\,\mathrm{d}x, \quad g \in L_p^*,$$

and $X = L_p^*$. It follows that $F_n \rightharpoonup F$ in X and that $\|F_n\| = \|f_n\|_p \to \|f\|_p = \|F\|$. By Theorem 3.4 we get that $\|F_n - F\| \to 0$, and the result follows now upon the observation that $\|F_n - F\| = \|f_n - f\|_p$. □

Remark 3.6. The Radon–Riesz property does not hold in L_1. The following example is from [4, Page 169]. Let

$$f_n(x) = 1 + \sin(nx), \quad f(x) = 1.$$

Then by the Riemann–Lebesgue Lemma, $f_n \rightharpoonup f$ in $L_1(-\pi, \pi)$. Moreover, as $f_n \geq 0$, we have

$$\|f_n\|_1 = \int_{-\pi}^{\pi} (1 + \sin(nx))\,\mathrm{d}x = 2\pi = \|f\|_1 .$$

However, (f_n) does not converge strongly to f in $L_1(-\pi, \pi)$ as $(\sin(nx))$ does not converge strongly in $L_1(-\pi, \pi)$.

References

[1] Clarkson, J.A., *Uniformly Convex Spaces*, Transactions of the American Mathematical Society Vol. 40, No. 3, pp. 396-414, 1936.

[2] Riesz, F. and Nagy, B., Functional Analysis, Dover Publishing, Mineola, NY, 1990. (Republication of 1955 edition)

[3] Negrón-Marrero, P. V. and Sivaloganathan, J., *On the convergence of a regularization scheme for approximating cavitation solutions with prescribed cavity volume*, SIAM J. Appl. Math., 80(1), 119-141, January 2020.

[4] Royden, H.L. and Fitzpatrick, P.M., Real Analysis, Fourth Ed., Pearson Education, Inc., 2010.