What is the value of $\lim_{x \to 0^\circ} \dfrac{\sin x}{x}$?

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March, 2013

Abstract

The basic fact that the derivative of the sine function (argument radians) is the cosine, depends in a very fundamental way on the value of the limit $\lim_{x\to 0} \frac{\sin x}{x}$ where the argument x is in radians. Failure to account for this might lead to erroneous calculations in the solution of certain problems involving trigonometric functions of angles. In this short note we review this basic result and give two examples

1 Introduction

If you answer quickly to this question, you might say the value of this limit is one, which is incorrect! However, if you pay attention to details, you might have noticed the sign for degrees in the limit, and probably have correctly answered $\frac{\pi}{180}$ for the value of the limit. The result that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, which requires that x be measured in radians, is essential to get that the derivative of $\sin x$ is $\cos x$ (again x in radians). If one does not pay attention to this fact, it is very easy to make mistakes in derivations that could lead to erroneous answers or conclusions. In this note I will describe two such instances. In the first case, in an article published in the journal *Mathematics Teacher* of December 2005 and having to do with an application of the Euler–Lagrange multiplier rule. The other

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example of computing or applying this limit incorrectly, is on a solution to a problem about approximations with differentials, generated by the test generator on the Pearson website for Vanberg et al. calculus book.

2 A first impression

In the article [5] the author considers the problem of finding the polygon of maximal area that can be inscribed in a circle of radius R. (See Figure 1.) In his presentation, the author poses the problem as follows: find nonnegative angles $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$A(\alpha_1, \ldots, \alpha_n) = \frac{1}{2}\sum_{k=1}^{n} R^2 \sin\alpha_k,$$

is maximal subject to the constraint that $\sum_{k=1}^{n} \alpha_i = 360°$. Note that clearly it is assumed that angles are measured in degrees!

Let

$$g(\alpha_1, \ldots, \alpha_n) = \sum_{k=1}^{n} \alpha_i - 360°,$$

and define

$$F(\alpha_1, \ldots, \alpha_n) = A(\alpha_1, \ldots, \alpha_n) - \lambda g(\alpha_1, \ldots, \alpha_n),$$

where $\lambda \in \mathbb{R}$ is called the *Lagrange multiplier*. The *Lagrange–multiplier rule* (see e.g. [2]) states that at such a maximum, it is necessary that $\nabla F = \mathbf{0}$. (Here the gradient is taken with respect to the variables $(\alpha_1, \ldots, \alpha_n)$.) Assuming that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha_k}(\sin\alpha_k) = \cos\alpha_k, \tag{1}$$

the author in [5] then claims that the Lagrange–multiplier rule in this case reduces to:

$$\frac{1}{2}R^2\cos\alpha_k = \lambda, \quad k = 1, \ldots, n. \tag{2}$$

However this equation is wrong by a factor of $\frac{\pi}{180}$ because the derivative in (1) is also missing this factor.

The correct application of the Lagrange–multiplier rules leads to:

$$\frac{\pi}{360}R^2\cos\alpha_k = \lambda, \quad k = 1, \ldots, n.$$

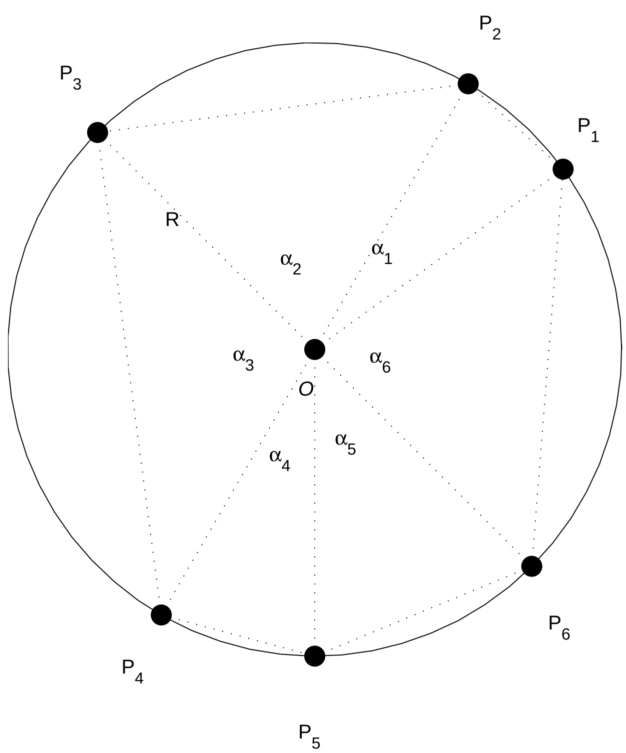

Figure 1: Polygon of six vertex inscribed in a circle of radius R.

If we define

$$\beta = \cos^{-1}\left[\frac{360\lambda}{\pi R^2}\right],$$

then

$$\alpha_k = \beta \ \text{ or } \ 360^\circ - \beta, \quad k = 1, \ldots, n.$$

Since $0^\circ \leq \beta \leq 180^\circ$, we have that $360^\circ - \beta \in [180^\circ, 360^\circ]$. Thus we can not have two or more of the α_k's equal to $360^\circ - \beta$ because this would violate the constraint that $\sum_{k=1}^{n} \alpha_i = 360^\circ$. If one α_k is $360^\circ - \beta$ and the other angles are equal to β, then using that $n \geq 3$ we get that

$$\sum_{k=1}^{n} \alpha_i \geq (360^\circ - \beta) + \beta + \beta = 360^\circ + \beta,$$

which violates the constraint[1] since $\beta > 0$. Hence all angles must be equal and using the constraint we get that

$$\alpha_k = \frac{360^\circ}{n}, \quad k = 1, \ldots, n,$$

In addition one can check that this solution satisfies the sufficient condition for a maximum of the constrained problem.

The factor $\frac{\pi}{180}$ in this context is actually a conversion factor! My first experience missing this conversion factor in a calculation was as a TA for a numerical analysis course. The result that the maximal area inscribed polygon is obtained when all the angles are equal to $\frac{360^\circ}{n}$, can be obtained even with the incorrect formula (2). However, the computed Lagrange multiplier would still be incorrect. Lagrange multipliers have physical meaning in many applications of the multiplier rule. I wrote a note for the column in the *Mathematics Teacher* about this issue [1], but never got a reply! Conversion factors in calculations are important! See for example the article [6] in which a misused conversion factor lead to the crashing of a Mars orbiter.

3 Back to basics

We now show that for α in degrees:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}(\sin\alpha) = \frac{\pi}{180}\cos\alpha\,.$$

[1]Note that if $\beta = 0$ in this case, then strictly speaking, we would have a one point polygon which can not happen as $n \geq 3$.

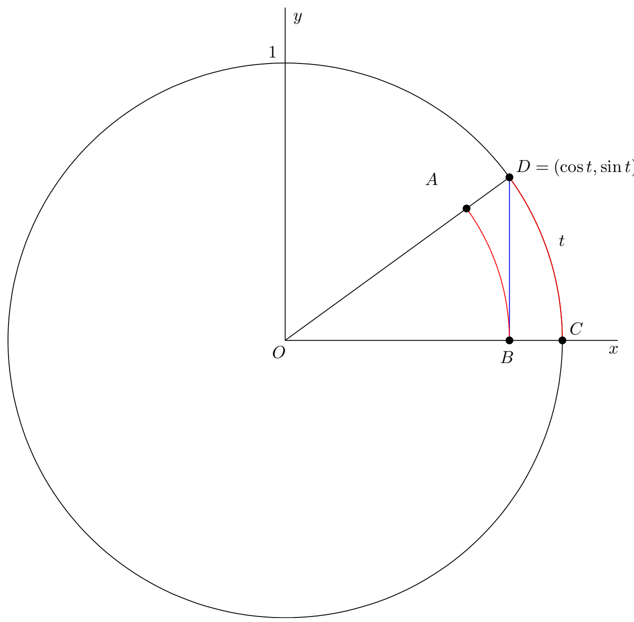

Figure 2: Three sector inside the unit circle with increasing areas.

It all hinges on the basic limit:

$$\lim_{t\to 0} \frac{\sin t}{t} = 1,$$

where t now is in radians. From Figure 2 we have that for $t > 0$ and sufficiently small:

$$\text{area}(AOB) \leq \text{area}(DOB) \leq \text{area}(DOC).$$

Computing the areas and simplifying, we get that

$$\cos t \leq \frac{\sin t}{t} \leq \frac{1}{\cos t},$$

which holds as well for $t < 0$ and small. Using now that $\lim_{t\to 0} \cos t = 1$, we get that

$$\lim_{t\to 0} \frac{\sin t}{t} = 1.$$

Using this result one can show that $\lim_{t\to 0}\frac{1-\cos t}{t}=0$:

$$\lim_{t\to 0}\frac{1-\cos t}{t}=\lim_{t\to 0}\frac{\sin^2 t}{t(1+\cos t)}=\lim_{t\to 0}\left[\frac{\sin t}{t}\right]\left[\frac{\sin t}{1+\cos t}\right]=1\cdot 0=0.$$

Now for x in radians:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sin x &= \lim_{t\to 0}\frac{\sin(x+t)-\sin x}{t}=\lim_{t\to 0}\frac{\sin x\cos t+\sin t\cos x-\sin x}{t}\\ &= \lim_{t\to 0}\left[\sin x\,\frac{\cos t-1}{t}+\cos x\,\frac{\sin t}{t}\right]\\ &= (\sin x)(0)+(\cos x)(1)=\cos x.\end{aligned}$$

What happens now when we try to compute $\frac{\mathrm{d}}{\mathrm{d}\alpha}\sin\alpha$ with α in degrees? First, we point out that there is a notational problem regarding this question, namely

$$\begin{aligned}f(\alpha) &= \sin\alpha, \quad \alpha \text{ in degrees},\\ g(t) &= \sin t, \quad t \text{ in radians},\end{aligned}$$

are different functions! Let's fix this temporally by defining the *sine function of degrees* $\sin_d$ as[2]:

$$\sin_d(\alpha)=\sin\left(\frac{\pi}{180}\,\alpha\right), \quad \alpha \text{ in degrees},$$

where the argument of the "sin" function on the right is in radians. Now computing $\frac{\mathrm{d}}{\mathrm{d}\alpha}\sin_d(\alpha)$ reduces to an application of the chain rule together with the result for the derivative of $\sin t$ with t in radians:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\sin_d(\alpha)=\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\sin\left(\frac{\pi}{180}\,\alpha\right)\right]=\frac{\pi}{180}\cos\left(\frac{\pi}{180}\,\alpha\right),$$

which could be written as

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\sin_d(\alpha)=\frac{\pi}{180}\cos_d(\alpha).$$

We can actually check this result with a hand calculator. Recall that for α in degrees:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\sin_d(\alpha)=\lim_{\Delta\alpha\to 0^\circ}\frac{\sin_d(\alpha+\Delta\alpha)-\sin_d(\alpha)}{\Delta\alpha}.$$

[2]The function $\sin_d$ can be computed with a hand calculator after setting the calculator to *degree mode*.

Example 3.1. For $\alpha = 45°$ we can approximate the derivative by difference quotients. In the table below we show the computed difference quotients which clearly converge to $\frac{\pi}{180}\cos_d(45°)$ as stated above.

$\Delta\alpha$	$\frac{\sin_d(45°+\Delta\alpha)-\sin_d(45°)}{\Delta\alpha}$	$\cos_d(45°)$	$\frac{\pi}{180}\cos_d(45°)$
0.1	0.01233056538	0.7071067812	0.01234134149
0.01	0.01234026445	0.7071067812	0.01234134149
0.001	0.01234123380	0.7071067812	0.01234134149
0.0001	0.01234133073	0.7071067812	0.01234134149
0.00001	0.01234134043	0.7071067812	0.01234134149
0.000001	0.01234134128	0.7071067812	0.01234134149

We can now compute the limit in the title of this note:

$$\lim_{\alpha\to 0°}\frac{\sin_d(\alpha)}{\alpha} = \lim_{\alpha\to 0°}\frac{\sin\left(\frac{\pi}{180}\alpha\right)}{\alpha} = \lim_{\alpha\to 0°}\left[\frac{\pi}{180}\right]\left[\frac{\sin\left(\frac{\pi}{180}\alpha\right)}{\frac{\pi}{180}\alpha}\right] = \left(\frac{\pi}{180}\right)(1) = \frac{\pi}{180}.$$

Again, we can check this result with a hand calculator:

α	$\frac{\sin_d(\alpha)}{\alpha}$	$\frac{\pi}{180}$
1	0.01745240644	0.01745329252
0.1	0.01745328366	0.01745329252
0.01	0.01745329243	0.01745329252
0.001	0.01745329252	0.01745329252

4 An approximation problem

The following problem and solution was generated using the test generator on the Pearson website for the book [4]:

> A surveyor is standing 35 ft from the base of a building. She measures the angle of elevation to the top of the building to be 65°. How accurately must the angle be measured for the percentage error in estimating the height of the building to be less than 5%?
>
> A) To within -0.005° B) To within -0.02%
> C) To within -0.02° D) To within -0.47°

The solution set states the the answer to this problem is alternative C). This is incorrect and as we will see, it is once again a case of the factor $\frac{\pi}{180}$ missing in the calculation.

We first review some definitions about measures of errors. Let x be some numerical (exact) quantity and let x_A denote an approximation of x. We recall the following definitions (see [3]):

$$\mathrm{Abs}(x_A) = x - x_A, \text{ (absolute error in } x_A),$$

$$\mathrm{Rel}(x_A) = \frac{\mathrm{Abs}(x_A)}{x}, \quad x \neq 0, \text{ (relative error in } x_A).$$

In general, if $\mathrm{Rel}(x_A) \approx 10^{-t}$ implies that x_A has roughly t *correct digits* with respect to x. This rule can be made more precise (see [3]). Relatives errors are usually expressed as a percentage.

Example 4.1. Take $x = \frac{\pi}{180} = 0.01745329252\ldots$ and $x_A = 0.017$. Then

$$\begin{aligned} \mathrm{Abs}(x_A) &= \frac{\pi}{180} - 0.017 = 0.00045329252\ldots \approx 4.53 \times 10^{-4}, \\ \mathrm{Rel}(x_A) &= \frac{0.00045329252\ldots}{0.01745329252\ldots} \approx 2.6 \times 10^{-2} = 2.6\%. \end{aligned}$$

The original problem can now be rephrased as:

> A surveyor is standing 35 ft from the base of a building. She measures the angle of elevation to the top of the building to be 65°. How accurately (*absolute error*) must the angle be measured for the percentage error (*relative error*) in estimating the height of the building to be less than 5%?
>
> A) To within -0.005° B) To within -0.02%
> C) To within -0.02° D) To within -0.47°

The correct solution: The height of the building for an angle of elevation α in degrees is given by $h(\alpha) = 35 \tan \alpha$. (See Figure 3.) We now have that[3]

$$\Delta h \approx h'(\alpha)\Delta\alpha = \frac{35\pi}{180}(\sec^2 \alpha)\Delta\alpha. \tag{3}$$

Thus

$$\mathrm{Rel}(h_A) = \frac{\Delta h}{h} \approx \frac{35\pi}{180h}(\sec^2 \alpha)\Delta\alpha.$$

[3]Note the factor $\frac{\pi}{180}$ after differentiating the tangent function of angles in degrees.

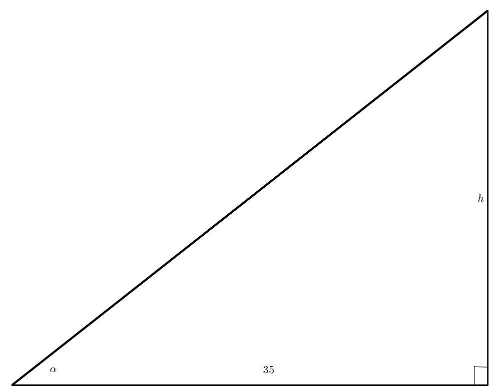

Figure 3: Triangle for the solution of the surveyor problem.

For $\mathrm{Rel}(h_A) \approx 5\%$ when $\alpha = 65°$, setting $h = 35\tan 65°$, we must have that

$$\Delta\alpha \approx \frac{0.05(35\tan 65°)}{\frac{35\pi}{180}\sec^2 65°} \approx 1.1°, \quad \frac{\Delta\alpha}{\alpha} \approx 1.7\%.$$

Neither 1.1 or 1.7% appear in the alternatives for the problem. However, if one takes the factor $\frac{\pi}{180}$ from equation (3), one gets the answer $\Delta\alpha \approx 0.02°$, the proposed answer in the website, which is incorrect!

5 Closing remarks

There are arithmetic errors and conceptual errors. Both are bad! The missing conversion factor in both cases discussed is one of the latter. It is important to emphasize the basic assumptions and applicability of many of the formulas that we teach in our courses. (This is not the same as been more rigorous!) I think emphasizing more applications can help in this respect as the student will be lead more frequently into checking the meaning of their calculations in the context of the application. We insist again that conversion factors are important and bad use of them can lead to disastrous errors, like that for the Mars orbiter [6]. What consequence do you think the incorrect answer of $\Delta\alpha \approx 0.02°$ would have for the surveyor in the problem? Well, measuring angle to within $1.1°$ requires a less sophisticated equipment than measuring an angle to within $0.02°$. Thus is has some consequences in terms of the equipment required to do the measuring.

References

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