An Algebraic Criterion for Cavitation

Pablo V. Negrón-Marrero*
Department of Mathematics
University of Puerto Rico
Humacao, PR 00791-4300

Abstract

We present the equations of isotropic elasticity in two (for a circular domain) and three dimensions (for a sphere) for the special case of radial solutions. This is equivalent to a nonlinear (quasilinear) boundary value problem depending parametrically on the boundary displacement. By using a formal linearization procedure about the trivial (affine) solution, we show that the critical load for cavitation (the opening of a hole at the center) can be characterized by an algebraic equation involving the boundary displacement and the constitutive functions. We give examples for specific materials and compare our formal results with some previous numerical experiments.

1 Introduction

The phenomena of void formation on bodies in tension have been observed among others by [7] in laboratory experiments. (See also [6] for a review on cavitation in rubber.) Ball [2] showed in the context of nonlinear elasticity, that void formation or "cavitation" can decreased the (potential) energy of a body in tension when the tension is sufficiently large. In fact for a spherical body composed of isotropic material, when the tension is sufficiently large, the purely radial deformation that opens a hole at the center of the ball, is a global minimizer among such deformations. We refer to [9] for a nice account of cavitation in nonlinear elasticity.

*pnm@cuhwww.upr.clu.edu

A very important problem here is that of characterizing or computing the critical tension at which cavitation occurs. As cavittation can point the initiation of fracture or rupture on a body, the computation of such critical tension is a very important one from the structural design point of view. This problem has been studied extensively in the past but we mention here the works of [8], [5], [15], and [13]. However most of the formulas for the critical tension involve complicated expresions like, improper integrals, which upon solution require further approximations. In this paper we develop a criterion for the critical tension in terms of a purely algebraic equation that involves the boundary displacement and the parameters of the corresponding constitutive equations. This is done for the displacement boundary value problem in which the outer radius of the ball is specified, and for a general class of nonlinear constitutive equations (cf. (5), (6), (7)). The free boundary condition on the inner surface when a cavity forms, specifies that the normal component of the Piola- Kirchkoff stress tensor is zero on the cavity surface.

In Section 2 of the paper we describe the model problem from the point of view of the Calculus of Variations and derive the corresponding Euler-Lagrange equations. In Sections 3 and 4 we do a formal linearization of the Euler Lagrange equations and get necessary conditions for the existence of eigenvalues which represent possible bifurcation points for the fully nonlinear problem. This bifurcation points represent the critical displacements at which cavitation occurs. For a particular example we compare the critical displacement predicted by the algebraic criterion with that of previous numerical experiments.

2 A Model for Nonlinear Elasticity

We consider a body which in its reference configuration occupies the region

$$\Omega = \{x \in \Re^n \ : \ ||x|| < 1\} \tag{1}$$

where $n = 2, 3$ and $||\cdot||$ denotes the Euclidean norm. Let $p : \Omega \to \Re^n$ denote a deformation of the body and let its *deformation gradient* be

$$F(x) = \frac{dp}{dx}(x) \tag{2}$$

The requirement that $p(x)$ *preserves orientation* takes the form

$$\det F(x) > 0 \ , \quad x \in \Omega \tag{3}$$

Let $W : M_+^{n\times n} \to \Re$ be the *stored energy function* of the material of the body where $M_+^{n\times n} = \{F \in \Re^{n\times n} \ : \ \det F > 0\}$. Note that physically reasonable W's must satisfy that $W \to \infty$ as either $\det F \to 0^+$ or $||F|| \to \infty$. The total stored energy on the body due to the deformation p is given by

$$I(p) = \int_\Omega W(F(x))dx \tag{4}$$

The equilibrium configuration of the body satisfies (3) and minimizes (4) among all functions belonging to an appropriate Sobolev space and satisfying appropriate boundary conditions. (See Ball (1977a,b)).

A physically reasonable model for W for an *isotropic and homogeneouos material* is as follows. Let $v_1, \ldots, v_n$ be the eigenvalues of $(F^tF)^{1/2}$ which are called the *principal stretches.* We take

$$W(F) = \Phi(v_1, \ldots, v_n) \quad , \quad F \in M_+^{n\times n} \tag{5}$$

where for $n = 2$

$$\begin{aligned} \Phi(v_1, v_2) \quad &= \quad A(v_1^\alpha + v_2^\alpha) + B(v_1^{-\beta} + v_2^{-\beta}) \\ &+ \quad C(v_1v_2)^\gamma + D(v_1v_2)^{-\delta} \end{aligned} \tag{6}$$

and for $n = 3$

$$\begin{aligned} \Phi(v_1, v_2, v_3) \quad &= \quad A(v_1^\alpha + v_2^\alpha + v_3^\alpha) + B(v_1^{-\beta} + v_2^{-\beta} + v_3^{-\beta}) \\ &+ \quad C((v_1v_2)^\gamma + (v_1v_3)^\gamma + (v_2v_3)^\gamma) \\ &+ \quad D((v_1v_2)^{-\delta} + (v_1v_3)^{-\delta} + (v_2v_3)^{-\delta}) \\ &+ \quad E(v_1v_2v_3)^\epsilon + G(v_1v_2v_3)^{-\phi} \end{aligned} \tag{7}$$

and all the coefficients and variables in the exponents are nonnegative except for α which is usually taken greater or equal to one. The different terms in (7) (similar comments apply to (6)) satisfy the requirement that infinite expansions or compressions of fibers, surface or volume elements within the material, must be accompanied by an infinite energy. Note that (6)–(7) satisfies

$$\Phi(v_1, v_2) \quad = \quad \Phi(v_2, v_1) \quad , \quad \text{for } n = 2 \tag{8}$$

$$\Phi(v_1, v_2, v_3) \quad = \quad \Phi(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}) \quad , \quad \text{for } n = 3 \tag{9}$$

where σ is any permutation of $\{1, 2, 3\}$. This condition is equivalent to the material of the body been isotropic.

We study the special case of (4) in which the deformation $p(\cdot)$ is *radially symmetric*, i.e.,

$$p(x) = \rho(||x||)\frac{x}{||x||} \quad , \quad x \in \Omega \tag{10}$$

for some scalar function ρ. In this case one can easily check that

$$v_1 = \rho'(s) \quad , \quad v_2, \ldots, v_n = \frac{\rho(s)}{s} \quad , \quad s = ||x|| \tag{11}$$

Thus (4) reduces (up to a constant multiple) to

$$I(p) = \int_0^1 s^{n-1}\Phi\left(\rho'(s), \frac{\rho(s)}{s}, \ldots, \frac{\rho(s)}{s}\right) ds \tag{12}$$

From (3) we get the inequalities

$$\rho'(s) \ , \ \frac{\rho(s)}{s} > 0 \quad , \quad 0 < s < 1 \tag{13}$$

We assume that the boundary is uniformly displaced which for (10) takes the form

$$\rho(1) = \lambda \quad , \quad \lambda > 0 \tag{14}$$

The Euler-Lagrange equation for (12) is

$$\left(s^{n-1}\Phi_{,1}\left(\rho'(s), \frac{\rho(s)}{s} \ldots, \frac{\rho(s)}{s}\right)\right)' = (n-1)s^{n-2}\Phi_{,2}\left(\rho'(s), \frac{\rho(s)}{s}, \ldots, \frac{\rho(s)}{s}\right) \tag{15}$$

where $0 < s < 1$, subject to (14) and

$$\rho(0) \geq 0 \quad , \quad \lim_{s\to 0^+} s^{n-1}\Phi_{,1}\left(\rho'(s), \frac{\rho(s)}{s} \ldots, \frac{\rho(s)}{s}\right) = 0 \tag{16}$$

(See Ball (1982) for conditions under which the minimizer of (12), (13), and (14) satisfies (15), (16)). The second condition in (16) states that if a hole opens at the center ($\rho(0) > 0$), then the component of the stress normal to the surface of the hole, is zero. For $\alpha < n$, Ball (1982) proved that for λ sufficiently large, the minimizer of (11) satisfies (16) with $\rho(0) > 0$, i.e., a hole opens up at the center of the ball. This phenomena of void formation is called *cavitation.*

3 The Linearized Euler-Lagrange Equations

Note that from (8)–(9) it follows that

$$\rho(s) = \lambda s \tag{17}$$

is a solution of (14)–(16) for all values of λ. We call (17) the *trivial* or *affine solution*. We now study the linearization of (14)–(16) about the trivial solution (17). We shall use the notation $\Phi_{,1}(\lambda)$ for the derivative of Φ with respect to the first argument evaluated at $(\lambda, \ldots, \lambda)$, etc. By using (8)–(9), one can easily check now that the linearized equation is given by

$$(s^{n-1}v')' + (n-1)\left[(n-2)\frac{\Phi_{,12}(\lambda)}{\Phi_{,11}(\lambda)} - (n-1)\right] s^{n-2}\frac{v}{s} = 0 \tag{18}$$

subject to

$$\lim_{s\to 0^+} s^{n-1}\left[\Phi_{,11}(\lambda)v' + (n-1)\Phi_{,12}(\lambda)\frac{v}{s}\right] = 0 \quad , \quad v(1) = 0 \tag{19}$$

The solutions of (18) are of the form s^r where r is a root of the quadratic equation

$$r^2 + (n-2)r + (n-1)\left[(n-2)\frac{\Phi_{,12}(\lambda)}{\Phi_{,11}(\lambda)} - (n-1)\right] = 0 \tag{20}$$

We now study conditions under which (18), (19) have nontrivial solutions. The values of λ for which this happens, called *eigenvalues*, represent possible bifurcation points for nontrivial branches of solutions (not equal to (17)) for (14)–(16). Those eigenvalues for which $v(0) > 0$ represent possible bifurcation points for branches of solutions with $\rho(0) > 0$, i.e., cavitating solutions. We consider the cases $n = 2$ and $n = 3$ separate.

4 Two Dimensional Cavitation

We study now the case where $n = 2$ in (18)–(20). It follows that $r = \pm 1$ are the roots of (20). Hence

$$v(s) = c_1 s + \frac{c_2}{s} \tag{21}$$

is the general solution of (18). Now (19a) reduces to

$$\lim_{s\to 0^+}\left[(\Phi_{,11}(\lambda)+\Phi_{,12}(\lambda))c_1 s+(\Phi_{,12}(\lambda)-\Phi_{,11}(\lambda)\frac{c_2}{s}\right]=0 \tag{22}$$

Note that if $c_2 = 0$, then (22) is satisfied. However since (19b) implies that $c_1 = -c_2$, we would have $c_1 = 0$ also and (21) would be trivial. Thus for nontrivial solutions of the form (21), we must have $c_2 \neq 0$. Thus any eigenvalue when $n = 2$ represent a possible bifurcation point for a branch of cavitating solutions. In this case (22) reduces to $\Phi_{,11}(\lambda) = \Phi_{,12}(\lambda)$, which because of our notation, is equivalent to

$$\Phi_{,11}(\lambda,\lambda)=\Phi_{,12}(\lambda,\lambda) \tag{23}$$

Upon recalling (6) and after some simplifications, it follows that (23) is equivalent to the following algebraic equation

$$\alpha(\alpha-1)A\lambda^{\alpha-2}+\beta(\beta+1)B\lambda^{-\beta-2}=\gamma C\lambda^{2(\gamma-1)}-\delta D\lambda^{-2(\delta+1)} \tag{24}$$

We study the solution set of (24) graphically by considering the intersection of the graphs of the functions on the left and right sides of this equation. The multiplicity of solutions of this equation depends essentially on the values of the exponents α and γ.

Case 1: ($\alpha < 2$ and $\gamma > 1$) In this case there is always a unique solution of (24). We show in Figure (4.1) a typical graph for this situation. In this and similar figures, the dotted curve represents the left side of (24) and the solid curve the right side.

Case 2: ($\alpha < 2$ and $\gamma < 1$) In this case there can be none or several solutions depending on the relative sizes of the coefficients and exponents in (24). We show in Figure (4.2) a graph in which we get two intersections.

Case 3: ($\alpha > 2$) In this case also there can be none or several solutions depending on the relative sizes of the coefficients and exponents in (24). We show in Figure (4.3) a typical graph in which $\gamma > 1$ and there is one intersection. In Figure (4.4) we show a case in which $\gamma < 1$ and there are two intersections.

Note that when $\alpha > 2$, an examination of (6) and (11) shows that any solution of (13)–(15) with $\rho(0) > 0$ has infinite energy. Thus the only physically reasonable bifurcating cavitating branches would be in Cases 1 and 2 above.

The particular example, $A = C = D = 1.0$, $B = 0$, $\alpha = \gamma = \delta = 1.5$ was studied in detail by Negrón-Marrero and Betancourt (1993) using an accelerated steepest descent method to minimize (11) directly. We show in Figure (4.5) the results they obtained for the cavity size of the minimizer of (11) as a function of $\lambda - 1$. Thus in this figure there is a bifurcation point for λ in [1.08,1.09] approximately. This particular example corresponds to Case 1 above. If we solve equation (24) numerically, we get a root at $\lambda = 1.0994$ approximately which agrees very well with the numerical results.

5 Three Dimensional Cavitation

We study now the case where $n = 3$ in (18)–(20). One can easily check in this case that the singular point $s = 0$ for (18) is in the limit point case (see Stakgold (1979)). Thus we expect a mixed spectrum having possibly continuous and discrete parts. The roots of (20) are now given by

$$r_\pm = -\frac{1}{2} \pm \frac{1}{2} g(\lambda)^{1/2} \quad , \quad g(\lambda) = 17 - 8\frac{\Phi_{,12}(\lambda)}{\Phi_{,11}(\lambda)} \tag{25}$$

Consider first the case in which $g(\lambda) < 0$. In this case the general solution of (18) is given by

$$v(s) = s^{-1/2}\left[c_1 \cos\left(\sqrt{-g(\lambda)}\ln\sqrt{s}\right) + c_2 \sin\left(\sqrt{-g(\lambda)}\ln\sqrt{s}\right)\right] \tag{26}$$

Condition (19b) implies that $c_1 = 0$ in (26). An easy computation shows now that (19a) is satisfied for any value of c_2 and λ such that $g(\lambda) < 0$. Thus $\{\lambda : g(\lambda) < 0\}$ belongs to the continuous spectrum.

When $g(\lambda) > 0$, the general solution of (18) is given by

$$v(s) = c_1 s^{r_+} + c_2 s^{r_-} \tag{27}$$

Condition (19b) implies that $c_1 = -c_2$ and since $r_+ + 1 > 0$, (19a) reduces to

$$\lim_{s\to 0^+} c_2 \left(\Phi_{,11}(\lambda) r_- + 2\Phi_{,12}(\lambda)\right) s^{r_- + 1} = 0 \tag{28}$$

Note that if $r_- + 1 > 0$, equation (28) is satisfied with no restrictions on c_2. Thus we are in the continuous spectrum again. On the other hand, if $r_- + 1 < 0$, then to get solutions without $c_2 = 0$, we need that

$$\Phi_{,11}(\lambda) r_- + 2\Phi_{,12}(\lambda) = 0 \tag{29}$$

which upon simplification reduces to the algebraic equation

$$\Phi_{,11}(\lambda,\lambda,\lambda) = \Phi_{,12}(\lambda,\lambda,\lambda) \tag{30}$$

We summarize our results so far in the following theorem.

Theorem 5.1 *When $n = 3$ the continuous spectrum of (18), (19) consists of*

$$\{\lambda : g(\lambda) < 1\} = \{\lambda : 2\Phi_{,11}(\lambda) < \Phi_{,12}(\lambda)\} \tag{31}$$

and the discrete part is given by the solutions of (30).

Note that from (7) it follows that (30) with the plus sign is equivalent to

$$\begin{aligned}\alpha(\alpha-1)A\lambda^{\alpha-2} + \beta(\beta+1)B\lambda^{-(\beta+2)} + \gamma(\gamma-2)C\lambda^{2(\gamma-1)} \\ = \epsilon E\lambda^{3\epsilon-2} - \phi G\lambda^{-(3\phi+2)} - \delta(\delta+2)D\lambda^{-(2\delta+2)}\end{aligned} \tag{32}$$

Again we study the solution set of (32) graphically by considering the intersections of the graphs of the functions on the left and right sides of this equation. Motivated by the growth hypotheses in Ball (1982), we assume that

$$\gamma = \frac{\alpha}{\alpha-1} \tag{33}$$

It then follows that

$$2(\gamma-1) < \alpha - 2 \text{ iff } \alpha > 3 \tag{34}$$

We then have the following cases:

Case 1: ($1 < \alpha < 2$) It follows now that $\gamma > 2$ and $2(\gamma-1) > 2$. Thus the third term to the left of (32) and the first to the right are the dominant ones. We get at least one intersection if $3\epsilon - 2 > 2(\gamma-1)$, i.e., $3\epsilon > 2\gamma$. A typical graph is like the situation depicted in Figure (4.3).

Case 2: ($2 < \alpha < 3$) We now have that $\gamma < 2$ and $2(\gamma-1) < 2$. The third term to left of (32) is still the dominant one but now with a negative coefficient. Thus we have an intersection for any $\epsilon > 1$. A typical graph is shown in Figure (5.1).

Case 3: ($\alpha > 3$) It follows from (34) that the first term to the left of (32) and the first to the right are the dominant terms. Thus we have an intersection if $3\epsilon - 2 > \alpha - 2$, i.e., $3\epsilon > \alpha$. A typical graph is like the situation depicted in Figure (4.3).

Note that an examination of (7) and (11) shows that any bifurcating cavitating branch in Case 3 has infinite energy.

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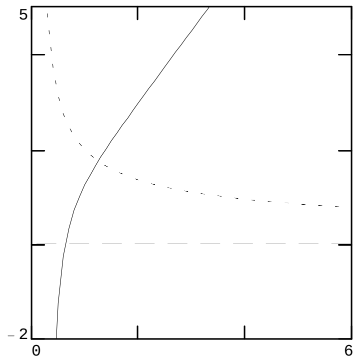


Fig. (4.1): A typical graph for Case 1 in which $\alpha < 2$ and $\gamma > 1$.

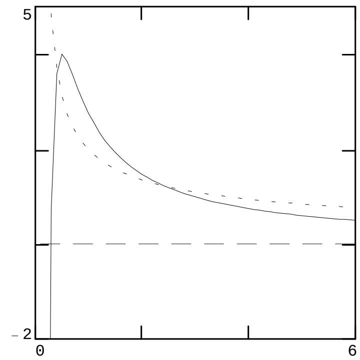


Fig. (4.2): A typical graph for Case in which $\alpha < 2$ and $\gamma < 1$.

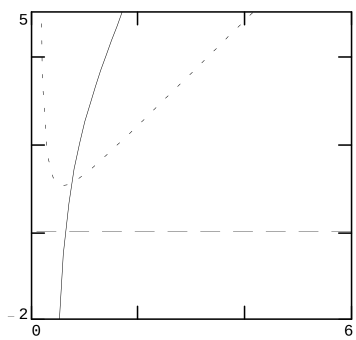


Fig. (4.3): A typical graph for Case 3 in which $\alpha > 2$ and $\gamma > 1$.

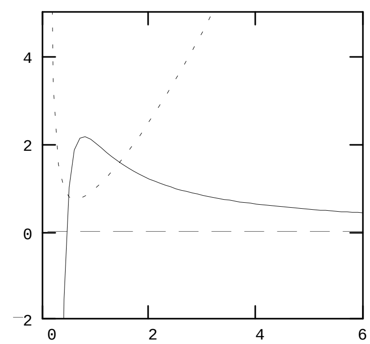
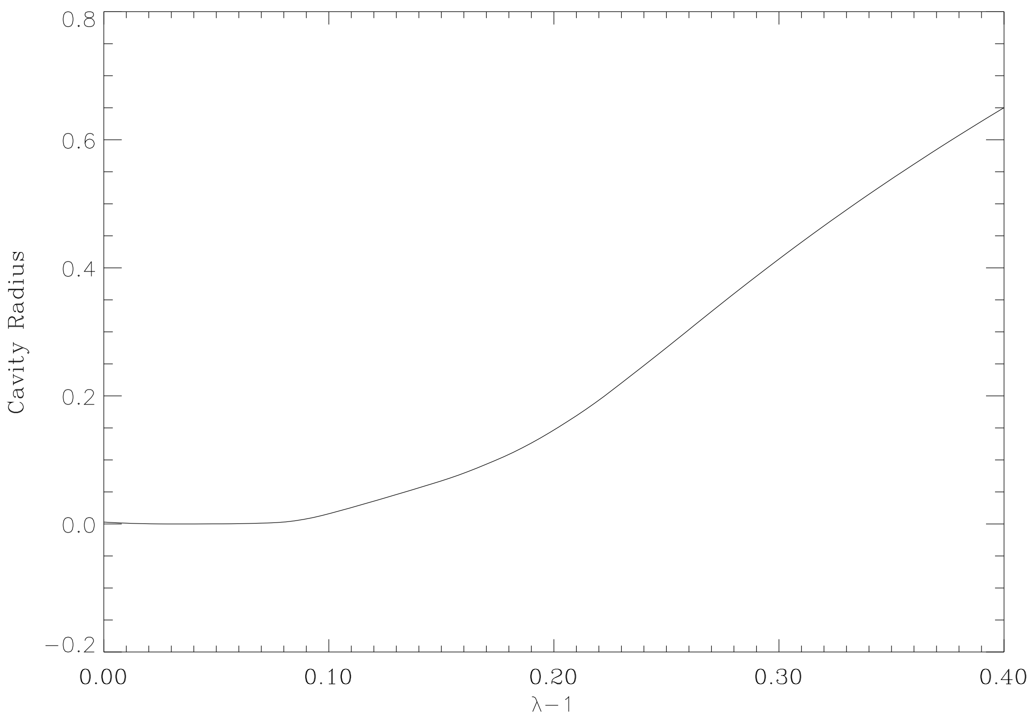


Fig. (4.4): A typical graph for Case 3 in which $\alpha > 2$ and $\gamma < 1$.


0.8
0.6
0.4
0.2
0.0
−0.2
Cavity Radius
0.00
0.10
0.20
0.30
0.40
λ−1