Nonplanar reflector arrays for capturing sun light

Pablo V. Negrón–Marrero* Jomarie Jiménez–González†
Department of Mathematics
University of Puerto Rico
Humacao, PR 00791-4300

Abstract

An array of mirrors that directs or reflects sunlight onto a collector tower is called a *heliostat*. Heliostats have successfully been employed in several countries (Spain and US) as a means of generating clean renewable energy. The usual arrays of heliostat mirrors have them set in a kind of semi-circular planar arrangement, each mirror facing the collecting tower. Recently it has been found that if the mirrors are arranged in a pattern reminiscent to the way leaves arrange in many plants (and related to the golden ratio), then the efficiency of the heliostat is greatly increased. In this project we study whether such arrangements or others increase efficiency as well, but in non planar arrangements of mirrors. After finding an expression for the total collected light by the tower in terms of the geometrical parameters of the mirror array (location of mirrors centroids and angles of inclination and rotation), we use a nonlinear optimization computer package to determine the values of these parameters that maximize the total collected light. We perform simulations for sunlight azimuthal and elevation data of regions in California (US), Spain and Dubai.

1 Introduction

Finding alternative energy sources has been a problem in this and last century. Modern technologies like the wind turbine and solar cell have help us but we still need to find other alternatives. In this paper we explore an alternative energy source based on an array of mirrors called heliostats. This array of mirrors directs or reflects sunlight into a collector tower. The energy collected can be used to power turbines that can generate electricity. This technology has been used by countries like Spain[1] and the United States.

*pablo.negron1@upr.edu
†jomarie.jimenez@upr.edu
[1]http://www.abengoasolar.com

Usually the mirrors are arrange in a semi-circular fashion around the tower on a flat terrain, with each mirror "facing" the tower. Recently it has been shown [6] that if the mirrors are arranged in a pattern, similar to the one leaves arrange in some plants [2], then the efficiency of the heliostat improves substantially. In this paper we study whether such arrangements or others increase efficiency as well, but in non planar arrangements of mirrors like those setup on the ladder of a hill.

In Section 2 we derive the basic equations for the reflected sun light for a single mirror. These expressions depend on the geometrical parameters of the mirror (angles of rotation and inclination, length and width, location in terms of its centroid), the height and dimensions of the tower and its collector, as well as the angles characterizing the incident ray. These equations are used in Section 3 to get an expression (cf. (12)) for the total collected light by the tower, now for a mirror array assuming no interference between mirrors. The mirrors in this model can be at different heights, either above ground level or placed on a hill. The problem now is to determine the angles of rotation and inclination, as well as positions of the centroids, that maximize the total collection by the tower from the mirror array.

2 Basic equations for a single mirror

Let P_0 be a plane containing the point (x_0, y_0, z_0). Assume that P_0 makes angle α_1 with the xy plane where $0 \leq \alpha_1 \leq \frac{\pi}{2}$, and that the line of intersection between P_0 and the xy plane makes an angle α_2 with the x axis (in the positive direction) where $0 \leq \alpha_2 \leq \pi$. It follows that this plane has unit normal given by:

$$\mathbf{N} = [\sin\alpha_1 \, \sin\alpha_2, -\sin\alpha_1 \, \cos\alpha_2, \cos\alpha_1] \,. \tag{1}$$

An equation for P_0 is thus given by

$$\mathbf{N} \cdot (x - x_0, y - y_0, z - z_0) = 0. \tag{2}$$

An incident sunlight ray with azimuthal and zenith angles θ and ψ respectively, can be described by

$$\mathbf{v}_i = [\sin\psi \, \sin\theta, \sin\psi \cos\theta, \cos\psi] \,, \tag{3}$$

where $\theta \in [0, 2\pi]$ and $\psi \in [0, \pi]$. The angle θ is measured clockwise from the positive y axis (the *north* direction). An easy calculation now shows that the reflected ray, using Snell's law, is given by

$$\mathbf{v}_r = 2(\mathbf{v}_i \cdot \mathbf{N})\mathbf{N} - \mathbf{v}_i. \tag{4}$$

We let $\theta_r \in [-\pi, \pi]$ and $\psi_r \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ be the azimuthal and elevation angles respectively for $\mathbf{v}_r$. Note that these angles are functions of α_1, α_2, θ, and ψ, independent of the point of reflection over P_0.

A mirror M with centroid (x_0, y_0, z_0) and angles α_1, α_2, is now given by

$$M = \{(x, y, z) \, : \, z = z_0 + \tan(\alpha_1)[-\sin(\alpha_2)(x - x_0) + \cos(\alpha_2)(y - y_0)]\} \,, \tag{5}$$

and (x, y) are restricted to

$$\left[\begin{array}{c} x \\ y \end{array} \right] = R_{\alpha_2} \left[\begin{array}{c} u \\ v \end{array} \right] + \left[\begin{array}{c} x_0 \\ y_0 \end{array} \right], \; u \in \left[-\frac{m_w}{2}, \frac{m_w}{2} \right], \; v \in \left[-\frac{m_l}{2} \cos(\alpha_1), \frac{m_l}{2} \cos(\alpha_1) \right]. \quad (6)$$

Here m_w, m_l are the width and length respectively of the mirror, and R_{α_2} is the rotation matrix

$$R_{\alpha_2} = \left[\begin{array}{cc} \cos(\alpha_2) & -\sin(\alpha_2) \\ \sin(\alpha_2) & \cos(\alpha_2) \end{array} \right].$$

Note that M is a function of α_1, α_2, and $\mathbf{c} = (x_0, y_0, z_0)$. Thus we occasionally write $M(\alpha_1, \alpha_2, \mathbf{c})$ to emphasize such dependence.

We assume the collecting tower is cylindrical with centre axis at the origin and diameter $2a_T$. The tower is of height H with the collector on top of height R. The reflected ray $\mathbf{v}_r$ does not necessarily hit the tower collector. There are two conditions for this ray to hit the reflector: one concerning the azimuthal angle of the reflecting ray, and the other on the elevation angle characterizing the height at which this ray potentially hits the tower. Let (x, y, z) be a point on M and in reference to Figure 1, which shows the projection onto the xy plane of this point and the tower, we define the following angles:

$$\gamma(x, y) = \sin^{-1} \left(\frac{a_T}{\sqrt{x^2 + y^2}} \right), \quad \delta(x, y) = \mathrm{Tan}^{-1}(-x, -y). \tag{7}$$

Here $\omega = \mathrm{Tan}^{-1}(a, b)$ (the four quadrant inverse tangent function) is the angle that the ray from the origin and containing the point (a, b) makes with the positive x axis, where $\omega \in (-\pi, \pi)$. The condition on the azimuthal angle of the reflecting ray now takes the form:

$$\delta(x, y) - \gamma(x, y) \leq \theta_r \leq \delta(x, y) + \gamma(x, y). \tag{8}$$

If θ_r satisfies this condition, then the line in the xy plane with this direction and containing the point (x, y), will intersect the circle of the tower in at most two points. If we write $\mathbf{v}_r = (v_{r,1}, v_{r,2}, v_{r,3})$, then these intersections are given by $(x + t\, v_{r,1}, y + t\, v_{r,2})$, where t is a solution of the quadratic

$$(x + t\, v_{r,1})^2 + (y + t\, v_{r,2})^2 = a_T^2. \tag{9}$$

We now have:

Lemma 2.1. *The quadratic equation in t given by* (9) *has real solutions if and only if* (8) *holds.*

Proof: Let ω be the angle between the vectors (x, y) and $(v_{r,1}, v_{r,2})$. The discriminant in (9) is non-negative if and only if $\cos^2 \omega \geq \cos^2 \gamma(x, y)$. In reference to Figure 1, we have that $\omega = \pi + \delta(x, y) - \theta_r$. Thus the discriminant is non-negative if and only if

$$\cos^2(\delta(x, y) - \theta_r) \geq \cos^2 \gamma(x, y).$$

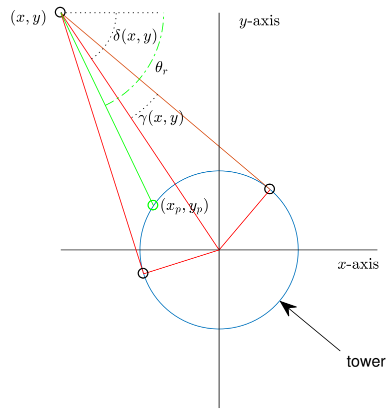

Figure 1: Angles $\gamma(x, y)$ and $\delta(x, y)$ for a given point (x, y, z) on the mirror. The green line segment represents the projection onto the xy plane of a reflected ray.

As $\delta(x,y)-\theta_r \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is necessary for the reflecting ray to hit the tower, and $\gamma(x,y) \in \left[0, \frac{\pi}{2}\right]$, the above inequality implies that $\delta(x,y)-\theta_r \leq \gamma(x,y)$ or $\delta(x,y)-\theta_r \geq -\gamma(x,y)$, i.e., that (8) holds. □

If (x_p, y_p) is the intersection point closest to (x, y), then in reference to Figure 2, we have that the height condition for the reflecting ray to hit the collector, is given by

$$H \leq z + \tan(\psi_r)\sqrt{(x-x_p)^2 + (y-y_p)^2} \leq H + R. \tag{10}$$

If $\alpha_1 < \frac{\pi}{2}$, we can solve for z in equation (2) to get that the inequality above is in terms of (x, y) only.

Let D be the set of all (x, y) satisfying (6) and the inequalities (8) and (10). If A represents the area of the region in M with $(x, y) \in D$, then the energy collected by the tower's collector from the mirror at (x_0, y_0, z_0), is proportional to A. In the following lemma we establish an expression for this area.

Lemma 2.2. *Let A be the area of the region in M with $(x, y) \in D$. If we write $\mathbf{N} = (N_1, N_2, N_3)$ in (1), then provided $N_3 \neq 0$, we have that*

$$A = \frac{1}{2N_3}\int_{\partial D}\left([(N_3 - N_1N_2)x - N_2^2 y]\,\mathrm{d}y - [(N_3 + N_1N_2)y + N_1^2 x]\,\mathrm{d}x\right). \tag{11}$$

Proof: We can parametrize M with $(x, y) \in D$ by

$$\mathbf{\Phi}(x,y) = \left(x, y, -\frac{1}{N_3}(N_1x + N_2y + d)\right), \quad (x,y) \in D,$$

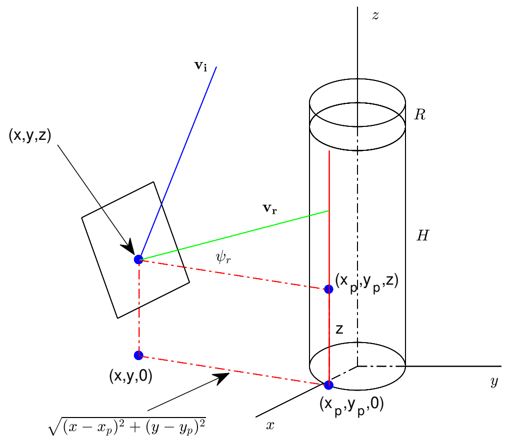

Figure 2: Diagram used for the height condition for a reflecting ray $\mathbf{v}_r$ hitting the tower.

for some constant d. Let $\mathbf{f} : D \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{f}(x, y, z) = (-y, x, N_1x + N_2y), \quad (x, y) \in D.$$

It follows now that $\nabla \times \mathbf{f} = (N_2, -N_1, 2)$ and the normal to M is given by $\hat{\mathbf{N}} = (N_1/N_3, N_2/N_3, 1)$. Hence $(\nabla \times \mathbf{f}) \cdot \hat{\mathbf{N}} = 2$, and

$$\iint_M (\nabla \times \mathbf{f}) \cdot \hat{\mathbf{N}} \, \mathrm{d}A = 2 \iint_M \mathrm{d}A = 2A.$$

Thus, by Stokes Theorem, we get

$$\begin{aligned} 2A &= \int_{\partial M} \mathbf{f} \cdot \mathrm{d}\mathbf{r} = \int_{\partial M} (-y \, \mathrm{d}x + x \, \mathrm{d}y + (N_1x + N_2y) \, dz), \\ &= \int_{\partial D} (-y \, \mathrm{d}x + x \, \mathrm{d}y + (N_1x + N_2y)(-(N_1/N_3) \, \mathrm{d}x - (N_2/N_3) \, \mathrm{d}y), \\ &= \frac{1}{N_3} \int_{\partial D} \left([(N_3 - N_1N_2)x - N_2^2 y] \, \mathrm{d}y - [(N_3 + N_1N_2)y + N_1^2 x] \, \mathrm{d}x \right). \end{aligned}$$

□

Note that both A and D are functions of α_1, α_2, θ, ψ, and $\mathbf{c} = (x_0, y_0, z_0)$. Thus occasionally we shall write $A(\alpha_1, \alpha_2, \mathbf{c}, \theta, \psi)$, the same with D, to emphasize such dependence. It is interesting to note that in the case $N_1 = N_2 = 0$ and $N_3 = 1$, equation (11) reduces to the standard one for the area of a plane region, obtained from Green's Theorem.

3 Equations for the mirror array

We now use the equations from the previous section to define a total collected energy function for a mirror array. We will assume that the centroids are specified, and the mirror angles vary from mirror to mirror and during the day. Thus, with indexes i for the mirror number and k for the hour of the day, we consider a finite collection of mirrors[2] $\{M^{(i,k)}\}$ characterized by the angles and centroids:

$$\alpha_1^{(i,k)}, \quad \alpha_2^{(i,k)}, \quad \mathbf{c}^{(i)} = (x_0^{(i)}, y_0^{(i)}, z_0^{(i)}), \quad 1 \le i \le N, \quad 1 \le k \le M.$$

In reference to (5) and (6), we have that $M^{(i,k)} = M(\alpha_1^{(i,k)}, \alpha_2^{(i,k)}, \mathbf{c}^{(i)})$. We use the notation

$$\boldsymbol{\alpha}_1 = (\alpha_1^{(i,k)}), \quad \boldsymbol{\alpha}_2 = (\alpha_2^{(i,k)}), \quad \mathbf{c} = (\mathbf{c}^{(1)}, \ldots, \mathbf{c}^{(N)}).$$

Let $(\theta^{(k)}, \psi^{(k)})$, $k = 1, \ldots, M$, represent pairs of azimuthal and zenith angles for incident sunlight rays during a particular day. We define

$$D^{(i,k)} = D(\alpha_1^{(i,k)}, \alpha_2^{(i,k)}, \mathbf{c}^{(i)}, \theta^{(k)}, \psi^{(k)}).$$

This corresponds to the set of all (x, y) such that $(x, y, z) \in M^{(i,k)}$ for some z, and such that the pair $(\theta_r^{(i,k)}, \psi_r^{(i,k)})$ satisfies the inequalities (8) and (10). Here $(\theta_r^{(i,k)}, \psi_r^{(i,k)})$ are the azimuthal and elevation angles of the reflecting ray from mirror $M^{(i,k)}$ and incident sunlight ray with angles $(\theta^{(k)}, \psi^{(k)})$.

Let $A^{(i,k)} = A(\alpha_1^{(i,k)}, \alpha_2^{(i,k)}, \mathbf{c}^{(i)}, \theta^{(k)}, \psi^{(k)})$ be the area of the region in $M^{(i)}$ with $(x, y) \in D^{(i,k)}$. The energy collected by the tower's collector from the mirror array[3] is given by

$$E(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = \sum_{i=1}^{N} \sum_{k=1}^{M} I_s(\theta^{(k)}, \psi^{(k)}, \alpha_1^{(i,k)}, \alpha_2^{(i,k)}) A^{(i,k)} \Delta t_k \equiv \sum_{i=1}^{N} \sum_{k=1}^{M} E^{(i,k)} \Delta t_k, \tag{12}$$

where I_s accounts in part for the *insolation*, in units of kilowatts per square meter per day, and Δt_k is a time increment (in fraction of a day).

The insolation I_s includes the effects of the atmosphere air mass and the so called cosine effects. The air mass, for a given zenith angle ψ, is defined by

$$AM = \frac{1}{\cos(\psi)}.$$

The insolation due to the air mass effect is modeled by (cf. [1], [3]):

$$I_{AM} = 1.353 \left[0.7^{AM^{0.678}} \right].$$

The cosine effect for incident sunlight ray $\mathbf{v}_i$ and mirror with normal direction $\mathbf{N}$, is proportional to $\mathbf{v}_i \cdot \mathbf{N}$. Thus for the net insolation I_s we take:

$$I_s(\theta, \psi, \alpha_1, \alpha_2) = (\mathbf{v}_i \cdot \mathbf{N}) I_{AM} = 1.353 (\mathbf{v}_i \cdot \mathbf{N})\, 0.7^{AM^{0.678}}.$$

[2] We assume all the mirrors have the same dimensions m_w, m_l.

[3] At this point we are neglecting possible interference between mirrors.

4 Numerical simulations

For the simulations, since we are not considering mirror interference, we can maximize each of the functions $E^{(i,k)}$ in (12) individually. Each of this optimizations is performed using the function `fmincon` of MATLAB$^{\mathrm{TM}}$. The selection of the initial point for the minimization process is rather important as these functions are very "flat", that is, they are zero in a large part of its domain, with the nonzero part of the graph in a rather small region. Thus for the initial angles of a given mirror, we work with the inequalities (8) and (10), with (x, y, z) set to the centroid of the mirror, and solving for all angles satisfying these inequalities. We then pick any point in this region.

The specific values for the different parameters in the modeled were chosen as follows (all lengths in meters), for the mirrors:

$$m_w = 10, \quad m_l = 10,$$

with the centroids in a rectangular array with three rows of 17 mirrors, each row higher than the one in front. Specifically:

$$\mathbf{c}^{(j+17(s-1))} = (-50 + r_j, -125 + \rho(s-1), z_j), \quad j = 1, 2, 3, \quad s = 1, 2, \ldots, 17,$$

where $\rho = 250/16$, $(r_1, r_2, r_3) = (0, -40, -80)$, and $(z_1, z_2, z_3) = (5, 15, 25)$. The tower is taken in the form of a right circular cylinder with radius $a_T = 7$, solar collector of height $R = 14$, and tower height $H = 100$. The inequalities (8) and (10) were solved using meshes for the local mirror coordinates uv in (6) with 40×40 points. Once the mesh points (x, y) satisfying these inequalities are determined, we use the function `boundary` of MATLAB$^{\mathrm{TM}}$, to extract the boundary points of this discrete region. After constructing splines for the boundary points, we use these together with the midpoint rule to approximate the areas given by (11). In the simulations we used data of azimuthal and zenith angles from Dubai, March 31st, during the hours of 7:00AM to 6:00PM, in increments of one hour. The results are summarized in Figures 3–5. The dots in these figures correspond each to a mirror on a given row, with color indicating the magnitude associated with the dot on the corresponding scale. Figures 3–4 show the optimal mirror angles, while Figure 5 show the associated mirror energies (in kilowatts), each per row and during the day. The optimal total array energy (cf. (12)) computed was approximately 964 kw. Finally in Figure 6 we show a diagram of the mirror array at noon with mirrors in their optimal positions. Black vectors correspond to the plane normals and the blue ones represent incoming sunlight rays.

References

[1] Kasten, F. and Young, A. T., *Revised optical air mass tables and approximation formula*, Applied Optics, vol. 28, pp. 47354738, 1989.

[2] King, S., Beck, F., and Lüttge, U., *On the mystery of the golden angle in phyllotaxis*, Plant, Cell and Environment, **27**: 685-695, 2004.

[3] Laue, E. G., *The measurement of solar spectral irradiance at different terrestrial elevations*, Solar Energy, vol. 13, pp. 43-50, IN1-IN4, 51-57, 1970.

[4] Mancini, T. R., *Catalog of Solar Heliostats*, SolarPACES, Technical Report No. III-1/00, June 2000.

[5] Negrón-Marrero, P. V., Montes-Pizarro, E., and Velázquez, J., *Arreglos de espejos no planares para la recolección de luz solar*, Informe Técnico, Departamento de Matemáticas, UPR Humacao, Febrero 2013.

[6] Noone, C.J., Torrilhon, M., and Mitsos, A., *Heliostat field optimization: A new computationally efficient model and biomimetic layout*, Solar Energy, **86**: 792-803, 2012.

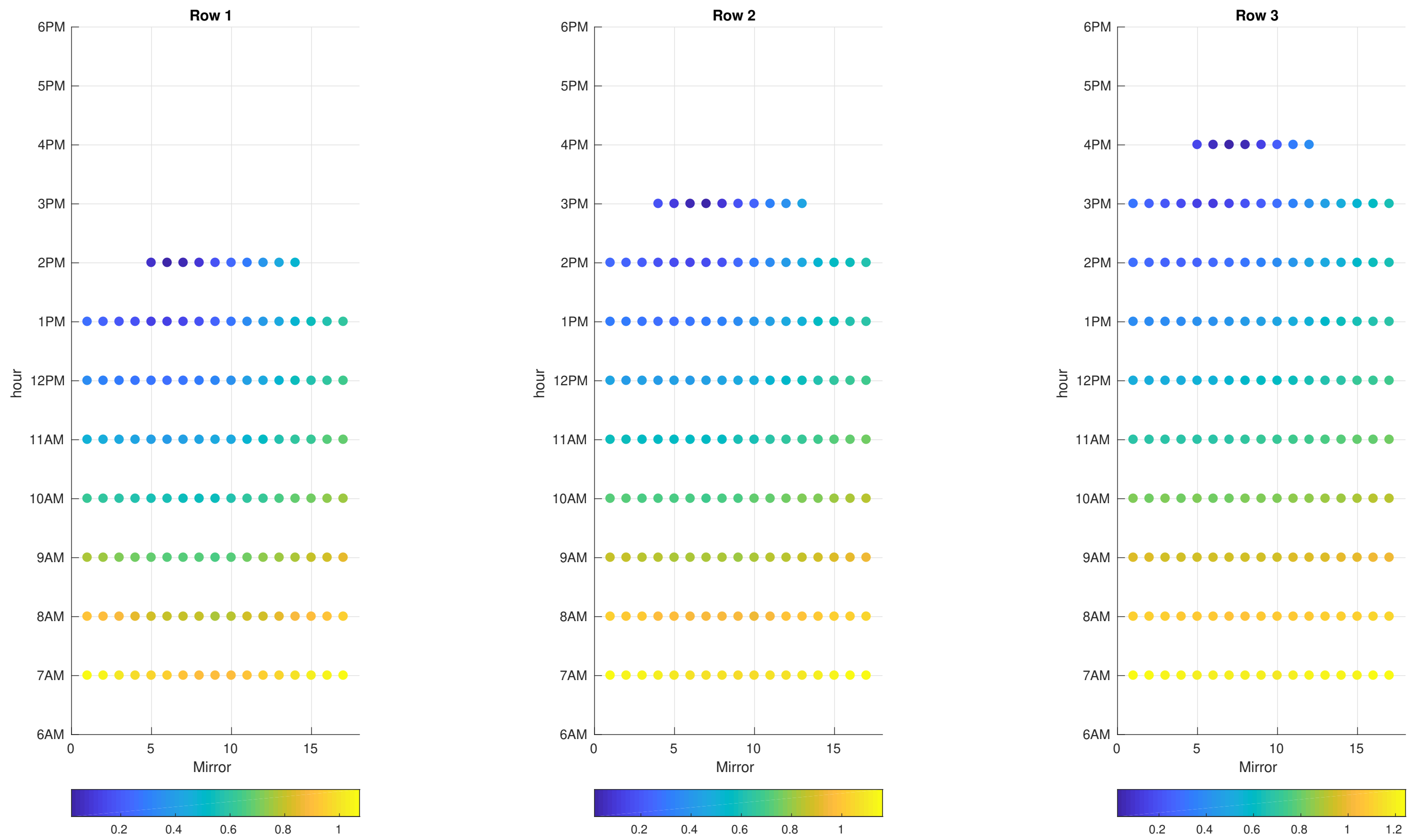

Figure 3: Diagram of the computed optimal α_1 angles for the array as functions of the hour of the day.

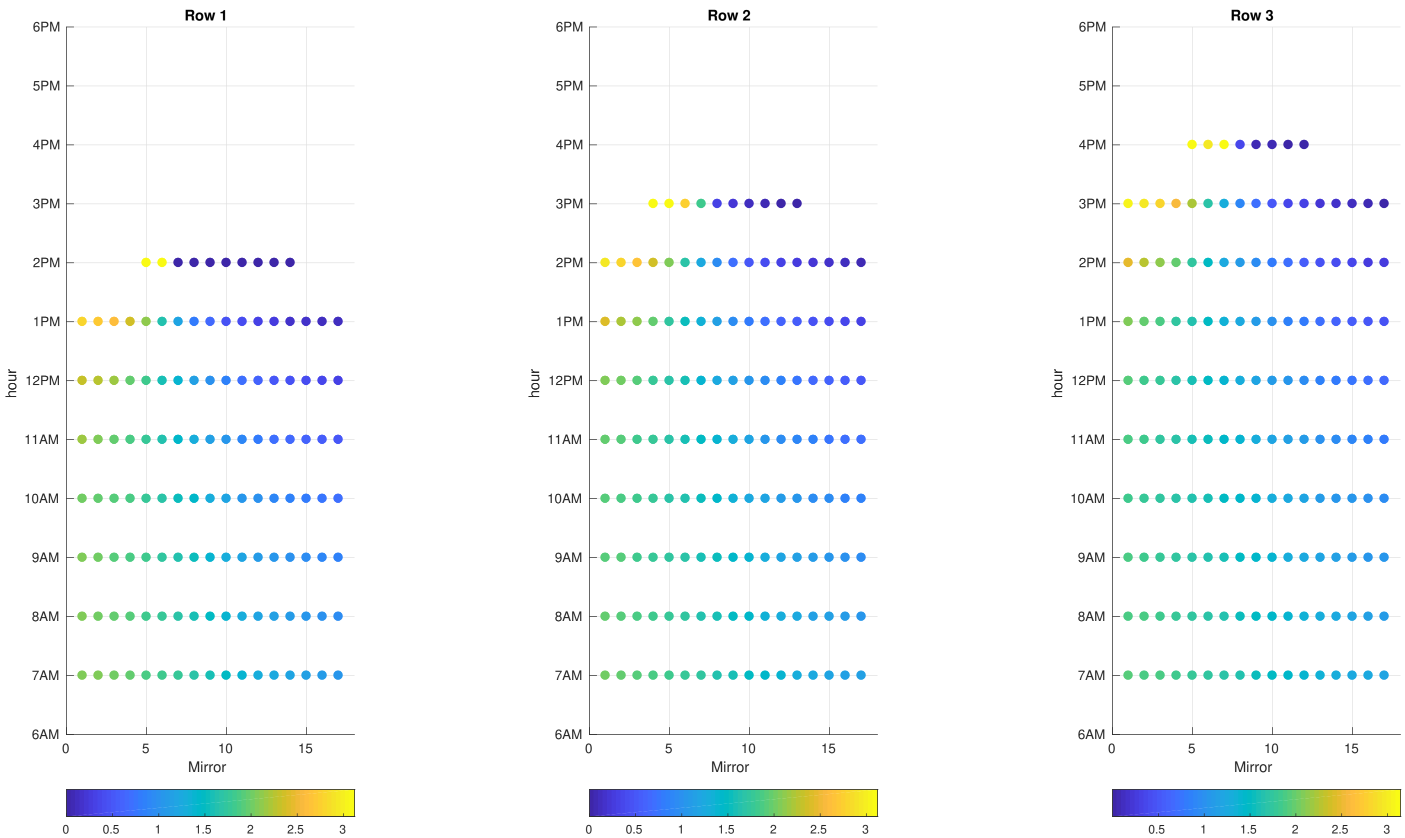

Figure 4: Diagram of the computed optimal α_2 angles for the array as functions of the hour of the day.

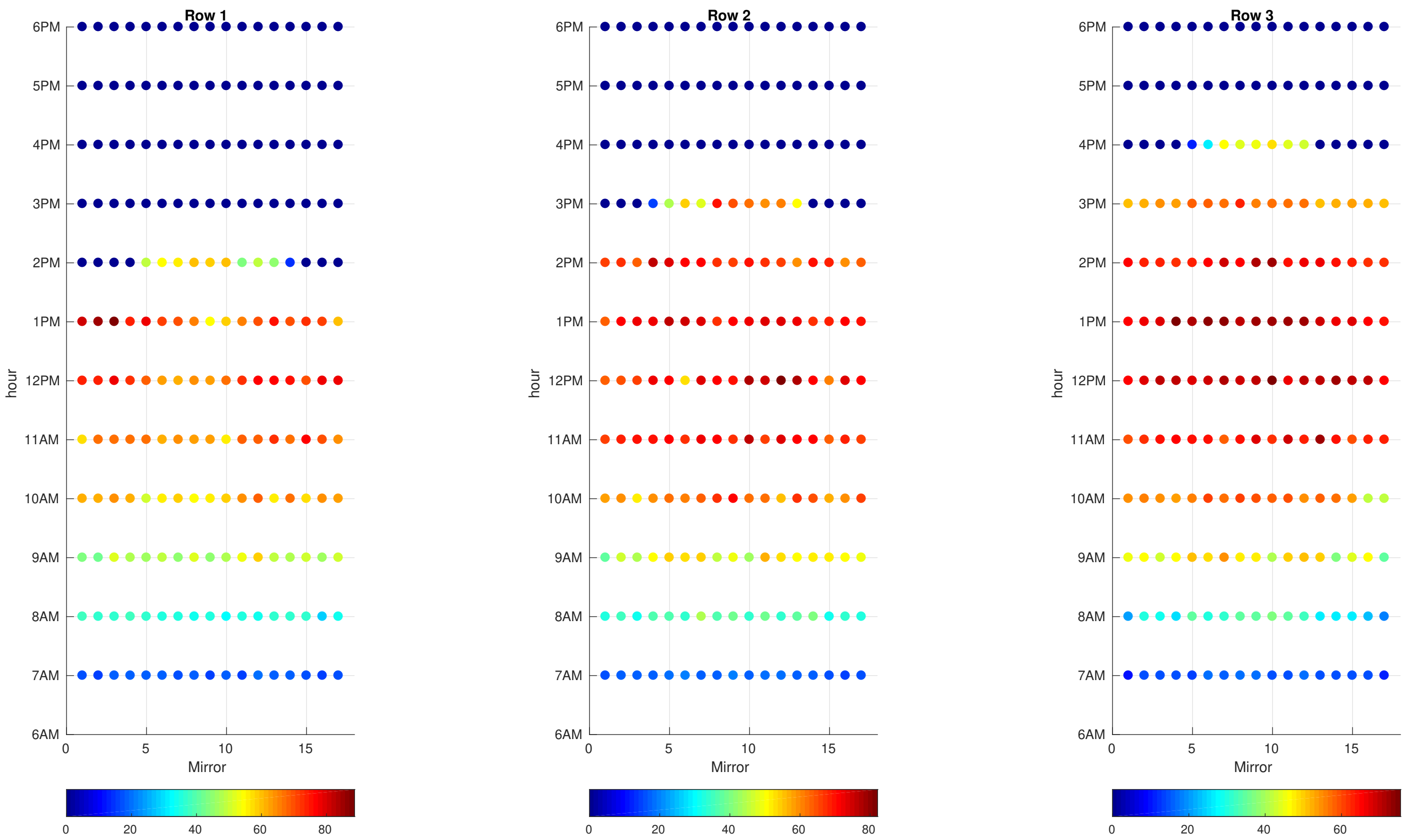

Figure 5: Diagram of the computed optimal energies (kw) for the array as functions of the hour of the day.

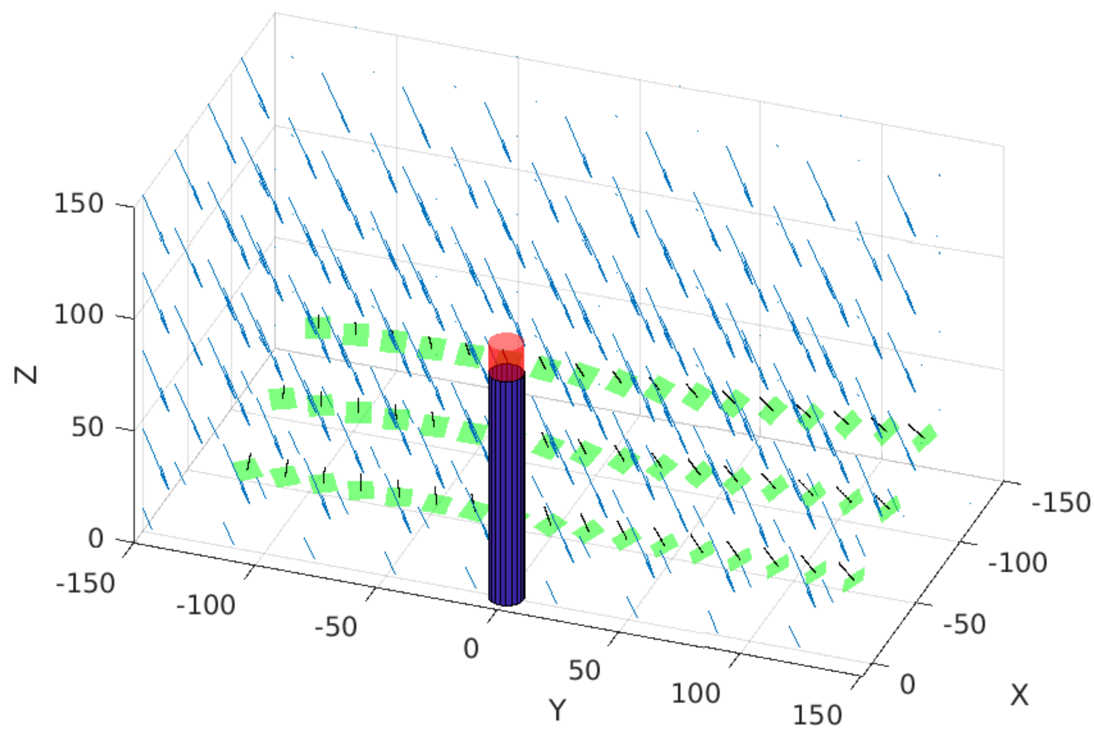

Figure 6: Diagram of the mirror array with mirrors in their optimal positions at noon. Black vectors correspond to the plane normals and the blue ones represent incoming sunlight rays.