Violation of the Complementing Condition and Local Bifurcation in Nonlinear Elasticity

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Abstract. The complementing condition (CC) is an algebraic compatibility requirement between the principal part of a linear elliptic differential operator and the principal part of the corresponding boundary operators. We study the implications of failure of the CC in the context of nonlinear elasticity. In particular we show that for axisymmetric deformations of cylinders and for any isotropic material, failure of the CC is equivalent to the existence of sequences of possible bifurcation points accumulating at the point where the CC fails. For non axisymmetric deformations and for Hadamard–Green type materials, we show for axial compressions of the cylinder that the CC fails on a full interval of values of the loading parameter, and for the lateral compression problem it fails at least once.

Keywords: nonlinear elasticity, complementing condition, global bifurcation, wrinkling

1. Introduction

Many problems in elasticity can be conveniently written abstractly as:

$$G(\lambda, \mathbf{u}) = \mathbf{0}, \quad \lambda \in (0, \infty),$$

where $\mathbf{u}$ denotes the displacement from a corresponding trivial solution, λ is some physical parameter (typically with values in a closed interval in the real line), and G is a differentiable nonlinear operator between appropriate Banach spaces with $G(\lambda, \mathbf{0}) = \mathbf{0}$. A necessary condition for local bifurcation at $(\lambda_*, \mathbf{0})$ is that the linearized problem

$$G_{\mathbf{u}}(\lambda_*, \mathbf{0}) \cdot \mathbf{v} = \mathbf{0},$$

has nontrivial solutions $\mathbf{v}$. In addition Fredholm properties of the linearized operator $G_{\mathbf{u}}(\lambda_*, \mathbf{0})$ play an important role for obtaining sufficient conditions for local bifurcation. For problems over a bounded set Ω, if we assume enough regularity on $\partial\Omega$ and the coefficients of the differential and boundary operators, the Fredholm properties for the linearized operator and some apriori estimates on the solutions of the associated boundary value problem can be obtained from strong ellipticity and the complementing condition. The complementing condition (CC) was first introduced by Agmon, Douglis, and Nirenberg in [1], [2]. It is an algebraic compatibility requirement between the principal part of a linear elliptic differential operator and the principal part of the corresponding boundary operators. (See also [10], [20], [21], [22] and the references therein.) We should mention that as far as we know, the first who used the CC and considered some of its consequences in elasticity was Thompson [21] who studied the linearized traction boundary value problem of the equations of three dimensional elasticity. Thompson also showed that violations of the CC are equivalent to the existence of Rayleigh waves of certain type providing a physical interpretation for violation of the CC.

More recently, Healey and Simpson in [8] constructed a generalized degree that has all the properties of the Leray–Schauder degree and from which global continuation results were obtained for a general class of problems in nonlinear elasticity in the presence of traction boundary conditions. The global continuation branch satisfies Rabinowitz' type alternatives in which the CC figures prominently. In particular they showed that a global solution branch can be characterized, in addition to the two Rabinowitz alternatives, cf. [15], by the possibility that it "terminates" due to loss of local injectivity; and/or ellipticity; and/or the failure of the complementing condition. Often, failures of local injectivity and ellipticity can be ruled out by imposing physically reasonable constitutive assumptions, e.g. [7] and [6]. However, the complementing condition can not be enforced as a constitutive assumption on the stored energy function in the context of (first gradient) elasticity because that would rule out many interesting materials. In addition, the CC is a property of the governing equations linearized at some solution of the corresponding nonlinear problem, and since in general we do not have an explicit linearization at a solution belonging to a global solution branch, we cannot check the CC directly along those branches.

When the CC fails the global continuation method developed by Healey and Simpson cannot be applied, hence the global branch that is being continued cannot been shown to exist using their methods, pass points where the CC fails. However that by itself does not necessarily implies that the global branch stops. In fact the results in [6], [17], [12], and [14] show several examples of problems that admit a trivial solution branch along which the corresponding linearized BVP fails to satisfy the CC, nevertheless using the explicit analytical expression for the trivial branch, it can easily be shown that it can be continued beyond the point where the CC fails.

The results in [6], [17], [12], and [14] also suggest that in the context of elasticity there is a recurrent relation between violation of the CC and the existence of sequences of possible bifurcation points accumulating at a point where the CC fails. This observation motivates the following two conjectures which are the main focus of this paper:

i) If λ_c is an accumulation point of the set:

$$\{\lambda : G_{\mathbf{u}}(\lambda, \mathbf{0}) \cdot \mathbf{v} = \mathbf{0} \text{ has nontrivial solutions}\}, \tag{1}$$

then the linear operator $G_{\mathbf{u}}(\lambda_c, \mathbf{0})$ fails to satisfy the complementing condition.

ii) If λ_c is a boundary point and a member of the set

$$\{\lambda : G_{\mathbf{u}}(\lambda, \mathbf{0}) \text{ does not satisfy the CC}\},$$

then there exists a sequence (λ_n) in the set (1) such that $\lim_n \lambda_n = \lambda_c$.

If true, (i) would imply that bifurcating branches of nontrivial solutions would locally accumulate at points along the trivial solution branch where the CC fails. On the other hand, if statement (ii) is true, one would have a very powerful bifurcation theorem as failure of the CC would imply the existence of a sequence of possible bifurcation points accumulating at the point where the CC fails, and in general checking whether the CC fails or not could be a much simpler problem than characterizing the values of λ for which the linearized problem $G_{\mathbf{u}}(\lambda, \mathbf{0}) \cdot \mathbf{v} = \mathbf{0}$ has nontrivial solutions $\mathbf{v}$. These questions are actually consistent with previous physical interpretations of the complementing condition as associated with oscillatory instabilities at the boundary, and may also suggest a limitation in the theory of elasticity based on first order gradients to model such phenomena.

In general, if the CC always holds, then one expects no accumulation of possible bifurcation points. In fact in [9] we studied the problem of axial compression of a rectangular slab with stored energy function:

$$\hat{W}(\nabla \mathbf{f}, \nabla^2 \mathbf{f}) = \frac{\varepsilon}{2} \nabla^2 \mathbf{f} \vdots \nabla^2 \mathbf{f} + W(\nabla \mathbf{f}),$$

for which the CC is always satisfied. Moreover for a W (i.e., $\varepsilon = 0$) corresponding to a Blatz–Ko material,

$$W(\mathbf{F}) = \frac{1}{2} \|\mathbf{F}\|^2 + \frac{1}{m} (\det \mathbf{F})^{-m}, \quad m > 0,$$

we have that for $\varepsilon > 0$ there is only a finite number of possible bifurcation points; and as $\varepsilon \to 0^+$, the number of possible bifurcation points monotonically increases and tend to accumulate at the value of the compression ratio for which the CC fails for the corresponding problem with $\varepsilon = 0$.

In this paper we study the CC, in particular failure of it, for the problems of axial and lateral compressions of a cylinder. In Section (2) we give the basic definitions leading to that of the CC together with its algebraic characterization in terms on the vanishing of a certain determinant, and in Section (3) we derive the equations of elasticity and discuss some implications of the CC in this context. In Section (4) we specialize the equations of elasticity to those for the deformations of cylinders composed of a Hadamard–Green type material, and derive the corresponding linearized equations about certain types of diagonal deformations. In preparation for the analysis of the CC, we collect in this section as well some results that apply to both boundary value problems, regarding the auxiliary problem (8) or equivalently (25) in the context of elasticity.

We call the function $\mathbf{v}$ in (1) a *variation* and focus attention on the types of variations. In particular when the variations in (1) are completely arbitrary in the sense that we make no explicit assumption on symmetry on them, we refer to this scenario as the *full problem*. In Section (5.1) we characterize the values of the compression ratio λ for which the CC fails for the axial problem for the cylinder, for variations completely arbitrary and for Hadamard–Green type materials. In particular for the full axial problem we show that given a crossing type condition (cf. (56)), the CC fails for a full interval of values of λ. This shows that if there are values of λ belonging to (1) and to this interval where the CC fails, then the global continuation method of [8] can not be applied at least within the context of the full problem. On the case of a Blatz–Ko type material, this crossing condition is always satisfied and the interval where the CC fails can be explicitly obtained. For the full lateral compression problem and for the Hadamard–Green type material, we show in Section (6.1) that the CC fails for at least one value of λ. For a Blatz–Ko type material this value of λ at which the CC fails is unique.

We call the *reduced problem* the one in which the variations $\mathbf{v}$ in (1) are axisymmetric. Both boundary value problems in this case have been extensively studied. (See for example [16] for both problems; [18], [19], [6] for the axial problem; and [4], [11], [12] for the lateral problem.) In Theorem (5.2) we show that the two previously proposed conjectures are true for the reduced axial problem. Although the analysis there is for Hadamard–Green type materials, the result actually holds for any homogeneous and isotropic material. Moreover, generically for a Hadamard–Green type material, we show that for certain aspect ratios of the cylinder, there exists a finite number of compression ratios with nontrivial solution branches bifurcating from them, with the CC for the reduced problem holding at each of them, despite the fact that the CC fails for the full problem at each of these values of λ. This shows that even after failure of the CC for the full problem, there may exist branches of nontrivial solutions in a reduced space (in this case: the space of axisymmetric deformations), bifurcating from the trivial branch.

For the reduced lateral compression problem (see [12]), there are two characteristic equations for the values of λ in (1): one of barreling type solutions and the other of buckling type. In Theorem (6.2) we show that both conjectures (i)–(ii) are true in this context as well. As for the axial problem this result is derived for Hadamard–Green type materials, but it actually holds for any homogeneous and isotropic material. For a Blatz–Ko type material it follows from the results in [12] that the buckling type compression ratios are always greater than the barreling type. In Section (6.2) we give a numerical example of a Hadamard–Green material for which this holds for the corresponding reduced lateral problem, suggesting that at least for Hadamard–Green materials with stored energy functions close to that of a Blatz–Ko material, the structure of the possible bifurcation points corresponding to the Hadamard–Green material should resemble that of the Blatz–Ko material.

2. The Complementing Condition

Let $\Omega \subset \mathbb{R}^r$ be an open and bounded set with a smooth boundary $\partial\Omega$. Let A be an elliptic (scalar) linear partial differential operator given by:

$$A[u](\mathbf{x}) \equiv \sum_{|\mathbf{s}|\leq 2m} a_{\mathbf{s}}(\mathbf{x}) D^{\mathbf{s}} u(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{2}$$

where $u : \Omega \to \mathbb{R}$ and $m \geq 1$. Here $\mathbf{s} = (s_1, \ldots, s_r)$ is a multi–index of length r so that $D^{\mathbf{s}} u$ represents a partial derivative of order $|\mathbf{s}| = s_1 + \cdots + s_r$. The principal part of A is given by:

$$A^H[u](\mathbf{x}) \equiv \sum_{|\mathbf{s}|=2m} a_{\mathbf{s}}(\mathbf{x}) D^{\mathbf{s}} u(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{3}$$

The boundary operators are given by:

$$B_i[u](\mathbf{x}) \equiv \sum_{|\mathbf{s}|\leq m_i} b_{i\,\mathbf{s}}(\mathbf{x}) D^{\mathbf{s}} u(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{4}$$

where $m_i \leq 2m-1$, for $i = 1, \ldots, m$, with corresponding principal part:

$$B_i^H[u](\mathbf{x}) \equiv \sum_{|\mathbf{s}|=m_i} b_{i\,\mathbf{s}}(\mathbf{x}) D^{\mathbf{s}} u(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{5}$$

for $i = 1, \ldots, m$. For smooth function f, g, we consider the boundary value problem:

$$A[u] \;=\; f \quad \text{in } \Omega, \tag{6a}$$

$$B[u] \;=\; g, \quad \text{over } \partial\Omega, \tag{6b}$$

where $B = (B_1, \ldots, B_m)$.

Let $\mathbf{x}_0$ be any boundary point and $\mathbf{n}(\mathbf{x}_0)$ the corresponding unit normal at $\mathbf{x}_0$. We consider the half–space $\mathcal{H}$ defined by:

$$\mathcal{H} = \left\{\mathbf{x} \,:\, (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}_0) < 0\right\}, \tag{7}$$

and the following auxiliary problem over $\mathcal{H}$:

$$A_0^H[v] \;=\; 0, \quad \text{in } \mathcal{H}, \tag{8a}$$

$$B_0^H[v] \;=\; 0, \quad \text{over } \partial\mathcal{H}, \tag{8b}$$

where

$$A_0^H[v](\mathbf{x}) \;\equiv\; \sum_{|\mathbf{s}|=2m} a_{\mathbf{s}}(\mathbf{x}_0) D^{\mathbf{s}} v(\mathbf{x}), \quad \mathbf{x} \in \mathcal{H}, \tag{9a}$$

$$B_{0,i}^H[v](\mathbf{x}) \;\equiv\; \sum_{|\mathbf{s}|=m_i} b_{i\,\mathbf{s}}(\mathbf{x}_0) D^{\mathbf{s}} v(\mathbf{x}), \quad \mathbf{x} \in \partial\mathcal{H}, \quad i = 1, \ldots, m. \tag{9b}$$

If $\boldsymbol{\xi}$ represents any nonzero vector in $\mathbb{R}^r$ with $\boldsymbol{\xi} \cdot \mathbf{n}(\mathbf{x}_0) = 0$, then we look for solutions of (8) of the form

$$v(\mathbf{x}) = w((\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}_0)) \mathrm{e}^{i\boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}_0)}, \quad \mathbf{x} \in \mathcal{H}, \tag{10}$$

where $w(s) \to 0$ as $s \to -\infty$. One says that for the boundary value problem (6), the *complementing condition* (CC) holds for the pair $(\mathbf{x}_0, \mathbf{n}(\mathbf{x}_0))$, if the only solution of (8) of the form (10) is the one with $w \equiv 0$.

What are the consequences of the complementing condition? Let

$$L[u] \equiv (A[u], B[u]),$$

and define the spaces:

$$X = C^{2m,\beta}(\overline{\Omega}; \mathbb{R}), \quad Z = C^{0,\beta}(\overline{\Omega}; \mathbb{R}), \tag{11a}$$

$$Y = Z \times \Pi_{i=1}^{m} C^{2m-m_i,\beta}(\partial\Omega; \mathbb{R}). \tag{11b}$$

If the coefficients in (2) and (4) are sufficiently smooth, then $L : X \to Y$ is continuous. We have now (see e.g. [1], [13], [20], [22]):

Theorem 2.1. *Let the operator $L : X \to Y$ be elliptic and assume that the CC holds. Then L is a Fredholm operator of index zero, and there exists a constant $c > 0$ such that for all $u \in X$,*

$$\|u\|_X \leq c\,[\|L[u]\|_Y + \|u\|_Z]\,.$$

For ease of exposition we have stated the CC for the scalar operators (2) and (4). This definition, as well as Theorem 2.1, can be extended to systems of partial differential equations (see [2]), where w in (10) is now replaced by a vector valued function $\mathbf{w}$. In Section (3) we discuss the details of these notions for systems in the particular case of the linearized system of elasticity, as well as some other consequences of the CC in this context (c.f. Theorem (3.1)). Also in the context of elasticity, the operator $L[\cdot]$ will depend on a parameter λ, like the operator $G_{\mathbf{u}}(\lambda, \mathbf{0})$ mentioned in the introduction, and we study some of the consequences of failure of the CC as the parameter λ changes.

3. The BVP's of Nonlinear Elasticity

We consider a body which in its reference configuration occupies the region $\Omega \subset \mathbb{R}^3$. Let $\mathbf{f} : \Omega \to \mathbb{R}^3$ denote a deformation of the body and let its *deformation gradient* be

$$\nabla\mathbf{f}(\mathbf{x}) = \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\mathbf{x}}(\mathbf{x}). \tag{12}$$

For smooth deformations, the requirement that $\mathbf{f}(\mathbf{x})$ is locally *invertible and preserves orientation* takes the form

$$\det \nabla\mathbf{f}(\mathbf{x}) > 0, \quad \mathbf{x} \in \Omega. \tag{13}$$

Let $W : \mathrm{Lin}^+ \to \mathbb{R}$ be the *stored energy function* of the material of the body where $\mathrm{Lin}^+ = \{\mathbf{F} \in M^{3\times3} \;:\; \det\mathbf{F} > 0\}$ and $M^{3\times3}$ denotes the space of real 3×3 matrices. The total energy stored in the body due to the deformation $\mathbf{f}$ is given by

$$I(\mathbf{f}) = \int_\Omega W(\nabla\mathbf{f}(\mathbf{x}))\,\mathrm{d}\mathbf{x}. \tag{14}$$

We assume (see e.g. [3], [5]) that the stored energy function W is frame–indifferent and satisfies that $W \to \infty$ as either $\det\mathbf{F} \to 0^+$ or $\|\mathbf{F}\| \to \infty$. The derivatives

$$\mathbf{S}(\mathbf{F}) = \frac{\mathrm{d}}{\mathrm{d}\mathbf{F}}W(\mathbf{F}), \quad \mathbf{C}(\mathbf{F}) = \frac{\mathrm{d}^2}{\mathrm{d}\mathbf{F}^2}W(\mathbf{F}), \tag{15}$$

are the usual (Piola–Kirchhoff) *stress* and *elasticity* tensors, respectively. We assume that $\mathbf{C}(\mathbf{F})$ is *strongly elliptic*, i.e. that

$$\mathbf{ab} : \mathbf{C}(\mathbf{F})[\mathbf{ab}] > 0, \tag{16}$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$ and all $\mathbf{F} \in \mathrm{Lin}^+$.

The material of the body is *homogenous* and *isotropic* if there exists a function $\sigma : (0, \infty)^3 \to \mathbb{R}$ such that:

$$W(\mathbf{F}) = \sigma\left(\frac{1}{2}\,\mathbf{F}\cdot\mathbf{F}, \frac{1}{4}\,\mathbf{F}\mathbf{F}^t \cdot \mathbf{F}\mathbf{F}^t, \det\mathbf{F}\right). \tag{17}$$

We now have that ([18]):

$$\mathbf{S}(\mathbf{F}) = \sigma_{,1}\mathbf{F} + \sigma_{,2}\mathbf{F}\mathbf{F}^t\mathbf{F} + (\det\mathbf{F})\sigma_{,3}\mathbf{F}^{-t}, \tag{18}$$

and the elasticity tensor is given by

$$\begin{aligned} \mathbf{C}(\mathbf{F})[\mathbf{H}] \;=\; & \sigma_{,1}\,\mathbf{H} + \sigma_{,2}\,(\mathbf{H}\mathbf{F}^t\mathbf{F} + \mathbf{F}\mathbf{H}^t\mathbf{F} + \mathbf{F}\mathbf{F}^t\mathbf{H}) \\ & + (\det\mathbf{F})\,\sigma_{,3}\,((\mathbf{F}^{-t}\cdot\mathbf{H})\mathbf{I} - \mathbf{F}^{-t}\mathbf{H}^t)\mathbf{F}^{-t} \\ & + \sum_{i\,,j=1}^{3} (\mathbf{G}^i\cdot\mathbf{H})\,\sigma_{,ij}\,\mathbf{G}^j, \end{aligned} \tag{19}$$

where

$$\mathbf{G}^1 = \mathbf{F} \quad , \quad \mathbf{G}^2 = \mathbf{F}\mathbf{F}^t\mathbf{F} \quad , \quad \mathbf{G}^3 = (\det\mathbf{F})\mathbf{F}^{-t}. \tag{20}$$

Let $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$, and for any given $\mathbf{f}_0$ consider the problem of minimizing (14) over the admissible set

$$\mathrm{Def} = \left\{\mathbf{f} \in C^2(\overline{\Omega}; \mathbb{R}^3) \,:\, \det\nabla\mathbf{f} > 0, \quad \mathbf{f} = \mathbf{f}_0 \text{ on } \Gamma_1\right\}. \tag{21}$$

The Euler–Lagrange equations for this problem are given by:

$$\operatorname{div}\mathbf{S}(\nabla\mathbf{f}) = \mathbf{0}, \quad \text{in } \Omega, \tag{22a}$$

$$\mathbf{f} = \mathbf{f}_0 \quad \text{on } \Gamma_1, \quad \mathbf{S}(\nabla\mathbf{f})\cdot\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_2, \tag{22b}$$

where $\mathbf{n}$ is a unit normal vector to $\partial\Omega$. The linearization of this problem about any solution $\mathbf{f}$ is given by:

$$\operatorname{div}\mathbf{C}(\nabla\mathbf{f})[\nabla\mathbf{u}] = \mathbf{0} \quad \text{in } \Omega, \tag{23a}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \quad \mathbf{C}(\nabla\mathbf{f})[\nabla\mathbf{u}]\cdot\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_2. \tag{23b}$$

This linearized problem will play the role of the linear problem (6) in the analysis that follows.

Remark: We could have other types of boundary conditions in (22). For example, if instead of $\mathbf{f} = \mathbf{f}_0$ in (21), we specify $\mathbf{f} \cdot \mathbf{n}$ on Γ_1, then we would get as a natural boundary condition that $(\mathbf{S}(\nabla \mathbf{f}) \cdot \mathbf{n}) \cdot \mathbf{t} = 0$ for any tangential direction $\mathbf{t}$ to Γ_1.

Let $\mathbf{x}_0 \in \partial\Omega$. If $\mathbf{x}_0 \in \Gamma_1$, then it is easy to show that the problem (23) satisfies the CC for the pair $(\mathbf{x}_0, \mathbf{n}(\mathbf{x}_0))$. For the case $\mathbf{x}_0 \in \Gamma_2$, the auxiliary problem (8) is given by:

$$\operatorname{div} \mathbf{C}(\nabla \mathbf{f}(\mathbf{x}_0))[\nabla \mathbf{v}] = \mathbf{0} \quad \text{in } \mathcal{H}, \tag{24a}$$

$$\mathbf{C}(\nabla \mathbf{f}(\mathbf{x}_0))[\nabla \mathbf{v}] \cdot \mathbf{n}(\mathbf{x}_0) = \mathbf{0} \quad \text{on } \partial\mathcal{H}. \tag{24b}$$

For the exponential type function (10), upon setting $\mathbf{C}_0 = \mathbf{C}(\nabla \mathbf{f}(\mathbf{x}_0))$, $\mathbf{n}_0 = \mathbf{n}(\mathbf{x}_0)$, and $s = (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}_0)$, we have that the boundary value problem (24) reduces to the initial value problem:

$$\begin{aligned} \mathbf{C}_0 \left[\mathbf{w}''(s) \otimes \mathbf{n}_0\right] \cdot \mathbf{n}_0 \;&+\; i\mathbf{C}_0 \left[\mathbf{w}'(s) \otimes \mathbf{n}_0\right] \cdot \boldsymbol{\xi} + i\mathbf{C}_0 \left[\mathbf{w}'(s) \otimes \boldsymbol{\xi}\right] \cdot \mathbf{n}_0 \\ &-\; \mathbf{C}_0 \left[\mathbf{w}(s) \otimes \boldsymbol{\xi}\right] \cdot \boldsymbol{\xi} = \mathbf{0}, \quad s < 0, \end{aligned} \tag{25a}$$

$$\mathbf{C}_0 \left[\mathbf{w}'(0) \otimes \mathbf{n}_0\right] \cdot \mathbf{n}_0 \;+\; i\mathbf{C}_0 \left[\mathbf{w}(0) \otimes \boldsymbol{\xi}\right] \cdot \mathbf{n}_0 = \mathbf{0}. \tag{25b}$$

If we look for solutions of (25) of the form $\mathbf{w}(s) = \mathrm{e}^{rs}\mathbf{p}$, where $\mathbf{p} \in \mathbb{R}^3$, then (25a) reduces to the matrix equation:

$$\left(r^2 M + ir(N_{\boldsymbol{\xi}} + N_{\boldsymbol{\xi}}^t) - Q_{\boldsymbol{\xi}}\right) \mathbf{p} = \mathbf{0}, \tag{26}$$

where

$$\begin{aligned} M\mathbf{p} \;&=\; \mathbf{C}_0 \left[\mathbf{p} \otimes \mathbf{n}_0\right] \cdot \mathbf{n}_0, \\ N_{\boldsymbol{\xi}}\mathbf{p} \;&=\; \mathbf{C}_0 \left[\mathbf{p} \otimes \mathbf{n}_0\right] \cdot \boldsymbol{\xi}, \\ Q_{\boldsymbol{\xi}}\mathbf{p} \;&=\; \mathbf{C}_0 \left[\mathbf{p} \otimes \boldsymbol{\xi}\right] \cdot \boldsymbol{\xi}. \end{aligned} \tag{27}$$

Note that M and $N_{\boldsymbol{\xi}}$ depend on $\mathbf{n}_0$ as well. The ellipticity condition (16) implies that equation (26) has exactly three roots r_1, r_2, r_3 with positive real part each, which for simplicity we assume to be different, and with corresponding nonzero vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$. The general solution of (25a) satisfying that $\mathbf{w}(s) \to \mathbf{0}$ as $s \to -\infty$, is given by $\mathbf{w}(s) = \sum_k \alpha_k \mathrm{e}^{r_k s}\mathbf{p}_k$. The substitution of this expression into (25b), leads to the matrix equation:

$$\sum_k \alpha_k \left(r_k M + iN_{\boldsymbol{\xi}}^t\right) \mathbf{p}_k = \mathbf{0}. \tag{28}$$

Thus the complementing condition fails for the problem (23) at $\mathbf{x}_0 \in \Gamma_2$ with normal $\mathbf{n}_0$ if:

$$\det\left[MPD + iN_{\boldsymbol{\xi}}^t P\right] = 0, \tag{29}$$

for some $\boldsymbol{\xi}$, where

$$P = [\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3], \quad D = \mathrm{diag}(r_1, r_2, r_3).$$

We already mentioned in Section (2) some important consequences of the complementing condition. We discuss now another result in the context of elasticity. For any solution $\mathbf{f}$ of (22), a sufficient condition for $\mathbf{f}$ to be a weak local minimizer of (14) is that the *second variation*:

$$\delta^2 I(\mathbf{f})[\mathbf{u}] = \int_\Omega \nabla\mathbf{u}(\mathbf{x}) \cdot \mathbf{C}(\nabla\mathbf{f}(\mathbf{x}))[\nabla\mathbf{u}(\mathbf{x})]\,\mathrm{d}\mathbf{x},$$

be *uniformly positive*, i.e., that there exists a constant $c > 0$ such that:

$$\delta^2 I(\mathbf{f})[\mathbf{u}] \geq c\left[\int_\Omega |\mathbf{u}|^2\,\mathrm{d}\mathbf{x} + \int_\Omega |\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\right],$$

for all $\mathbf{u} \in C^1(\Omega)$, $\mathbf{u} = \mathbf{0}$ on Γ_1. We have now:

Theorem 3.1 (Simpson and Spector [20])**.** *Let* $\delta^2 I(\mathbf{f})[\mathbf{u}] > 0$ *for all nonzero* $\mathbf{u} \in C^1(\Omega)$ *vanishing over* Γ_1*. Then* $\delta^2 I(\mathbf{f})$ *is uniformly positive if and only if* $\mathbf{C}$ *is strongly elliptic and the corresponding linearization of* (22) *about* $\mathbf{f}$ *satisfies the complementing condition.*

4. Deformations of cylinders

We consider a body which in its reference configuration occupies the region $\bar{\Omega}$, where

$$\Omega = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_1^2 + x_2^2 < 1, \quad 0 < x_3 < L\right\}. \tag{30}$$

We write $\partial\Omega = \partial\Omega_B \cup \partial\Omega_S \cup \partial\Omega_T$, where

$$\begin{aligned}
\partial\Omega_B &= \left\{(x_1, x_2, x_3) \,:\, x_1^2 + x_2^2 \leq 1, \quad x_3 = 0\right\},\\
\partial\Omega_T &= \left\{(x_1, x_2, x_3) \,:\, x_1^2 + x_2^2 \leq 1, \quad x_3 = L\right\},\\
\partial\Omega_S &= \left\{(x_1, x_2, x_3) \,:\, x_1^2 + x_2^2 = 1, \quad 0 \leq x_3 \leq L\right\}.
\end{aligned}$$

Consider a (homogeneous) deformation of the cylinder of the form

$$\mathbf{f}_h = (\mu^{1/2} x_1, \mu^{1/2} x_2, \omega x_3),$$

for some $\mu, \omega > 0$. It follows now that

$$\nabla\mathbf{f}_h = \begin{pmatrix} \mu^{1/2} & 0 & 0 \\ 0 & \mu^{1/2} & 0 \\ 0 & 0 & \omega \end{pmatrix}. \tag{31}$$

For a *Hadamard–Green type material* the function σ in (17) is given by:

$$\sigma(v_1, v_2, v_3) = av_1 + b(v_1^2 - v_2) + \Psi(v_3), \tag{32}$$

where $a > 0$, $b \geq 0$, and $\Psi : (0, \infty) \to \mathbb{R}_+$ satisfies that

$$\lim_{v \to 0+} \Psi(v) = \lim_{v \to \infty} \Psi(v) = \infty. \tag{33}$$

We also assume that $\Psi'(1) = -a - 2b$ which guarantees that the reference configuration is stress–free. For our numerical examples, we shall use as a model for Ψ the following function:

$$\Psi(v) = cv^{\gamma} + dv^{-\delta}, \tag{34}$$

where $c \geq 0$, $d > 0$, and $\gamma, \delta > 1$. Note that with $a = 1$, $b = 0$, $c = 0$, $d = 1/m$, and $\delta = m$, equations (32) and (34) reduce to the *Blatz–Ko material*:

$$\sigma(v_1, v_2, v_3) = v_1 + \frac{1}{m} v_3^{-m}. \tag{35}$$

Let

$$\begin{gathered} t = \mu\omega^{-2}, \quad q = \mu\omega^2\Psi''(\mu\omega), \quad \tau_1 = a + b(\mu + \omega^2) + q, \\ \tau_3 = a + 2b\mu + tq, \quad \beta_1 = a + b\mu, \quad \beta_3 = a + b\omega^2, \\ X = t^{1/2}\left[\omega\Psi'(\mu\omega) + 2b\omega^2 + q\right]. \end{gathered}$$

For the Hadamard–Green material (32), equations (18) and (19) reduce to

$$\begin{aligned} \mathbf{S}(\nabla\mathbf{f}_h) \;&=\; \operatorname{diag}\Big[\mu^{1/2}(a + b(\mu + \omega^2) + \omega\Psi'(\mu\omega)), \\ &\qquad \mu^{1/2}(a + b(\mu + \omega^2) + \omega\Psi'(\mu\omega)), \\ &\qquad \omega(a + 2b\mu) + \mu\Psi'(\mu\omega)\Big], \qquad \text{(36a)} \\ \mathbf{C}(\nabla\mathbf{f}_h)[\mathbf{H}] \;&=\; \mathbf{B}, \qquad \text{(36b)} \end{aligned}$$

where

$$\begin{aligned} B_{11} &= \tau_1 H_{11} + (\omega\Psi'(\mu\omega) + 2b\mu + q)H_{22} + XH_{33}, && \text{(37a)} \\ B_{22} &= \tau_1 H_{22} + (\omega\Psi'(\mu\omega) + 2b\mu + q)H_{11} + XH_{33}, && \text{(37b)} \\ B_{33} &= \tau_3 H_{33} + X(H_{11} + H_{22}), && \text{(37c)} \\ B_{12} &= \beta_3 H_{12} - (b\mu + \omega\Psi'(\mu\omega))H_{21}, && \text{(37d)} \\ B_{21} &= \beta_3 H_{21} - (b\mu + \omega\Psi'(\mu\omega))H_{12}, && \text{(37e)} \\ B_{i3} &= \beta_1 H_{i3} - \mu^{1/2}(b\omega + \Psi'(\mu\omega))H_{3i}, \quad i = 1, 2, && \text{(37f)} \\ B_{3i} &= \beta_1 H_{3i} - \mu^{1/2}(b\omega + \Psi'(\mu\omega))H_{i3}, \quad i = 1, 2. && \text{(37g)} \end{aligned}$$

Using (36b), (37) we now explicitly determine the equations for the initial value problem (25) for the cylindrical region (30) and $\mathbf{C}_0 = \mathbf{C}(\nabla \mathbf{f}_h)$. Since we will be discussing two boundary value problems for the cylinder (lateral compression or axial compression) at this point we compute only the general solution of (25a) and leave the application of the initial condition (25b) for the discussion of each individual problem.

Let $\mathbf{x}_0 \in \partial\Omega_B \cup \partial\Omega_T$. In this case we have that $\mathbf{n}_0 = \pm\mathbf{e}_3$. Since $\boldsymbol{\xi} \cdot \mathbf{n}_0 = 0$ in (10), we get that $\boldsymbol{\xi} = (\xi_1, \xi_2, 0)$. Using (36b)-(37) we get upon setting $\mathbf{w}(s) = (w_1(s), w_2(s), w_3(s))$ that equation (25a) reduces to the system:

$$\begin{aligned} \beta_1 w_1'' + it^{1/2}\xi_1(b\omega^2 + q)w_3' &- (\tau_1\xi_1^2 + \beta_3\xi_2^2)w_1 \\ &- \xi_1\xi_2(b\mu + q)w_2 = 0, \end{aligned} \tag{38a}$$

$$\begin{aligned} \beta_1 w_2'' + it^{1/2}\xi_2(b\omega^2 + q)w_3' &- (\beta_3\xi_1^2 + \tau_1\xi_2^2)w_2 \\ &- \xi_1\xi_2(b\mu + q)w_1 = 0, \end{aligned} \tag{38b}$$

$$\tau_3 w_3'' + it^{1/2}(b\omega^2 + q)(\xi_1 w_1' + \xi_2 w_2') - \beta_1 |\boldsymbol{\xi}|^2 w_3 = 0, \tag{38c}$$

where $|\boldsymbol{\xi}| = \sqrt{\xi_1^2 + \xi_2^2}$. The initial condition (25b) is now given by:

$$\beta_1 w_1'(0) - i\mu^{1/2}(b\omega + \Psi'(\mu\omega))\xi_1 w_3(0) = 0, \tag{39a}$$

$$\beta_1 w_2'(0) - i\mu^{1/2}(b\omega + \Psi'(\mu\omega))\xi_2 w_3(0) = 0, \tag{39b}$$

$$\tau_3 w_3'(0) + iX(\xi_1 w_1(0) + \xi_2 w_2(0)) = 0. \tag{39c}$$

The system (26) is now given by $A\mathbf{p} = \mathbf{0}$ where:

$$A = \begin{bmatrix} \beta_1 r^2 - (\tau_1\xi_1^2 + \beta_3\xi_2^2) & -\xi_1\xi_2(b\mu + q) & i\xi_1 t^{1/2}(b\omega^2 + q)r \\ -\xi_1\xi_2(b\mu + q) & \beta_1 r^2 - (\beta_3\xi_1^2 + \tau_1\xi_2^2) & i\xi_2 t^{1/2}(b\omega^2 + q)r \\ i\xi_1 t^{1/2}(b\omega^2 + q)r & i\xi_2 t^{1/2}(b\omega^2 + q)r & \tau_3 r^2 - \beta_1 |\boldsymbol{\xi}|^2 \end{bmatrix}.$$

For nontrivial solutions of the system $A\mathbf{p} = \mathbf{0}$ we require that $\det A = 0$. Note that $A = [\beta_1 r^2 - \beta_3 |\boldsymbol{\xi}|^2]I + D$ where:

$$D = \begin{bmatrix} -(b\mu + q)\xi_1^2 & -\xi_1\xi_2(b\mu + q) & i\xi_1 t^{1/2}(b\omega^2 + q)r \\ -\xi_1\xi_2(b\mu + q) & -(b\mu + q)\xi_2^2 & i\xi_2 t^{1/2}(b\omega^2 + q)r \\ i\xi_1 t^{1/2}(b\omega^2 + q)r & i\xi_2 t^{1/2}(b\omega^2 + q)r & (b\mu + tq)r^2 + b(\omega^2 - \mu)|\boldsymbol{\xi}|^2 \end{bmatrix}.$$

Thus $\det A = 0$ if and only if $\beta_3 |\boldsymbol{\xi}|^2 - \beta_1 r^2$ is an eigenvalue of D. It follows after computing the eigenvalues of D, that

$$r = \pm |\boldsymbol{\xi}| \sqrt{\frac{\beta_3}{\beta_1}}, \quad \pm |\boldsymbol{\xi}|, \quad \pm |\boldsymbol{\xi}| \sqrt{\frac{\tau_1}{\tau_3}}. \tag{40}$$

Since $\mathbf{w}(s) \to \mathbf{0}$ as $s \to -\infty$, we consider only the positive roots and labeled them as

$$r_1 = \rho_1 |\boldsymbol{\xi}|, \quad \rho_1 = \sqrt{\frac{\beta_3}{\beta_1}}, \tag{41a}$$

$$r_2 = |\boldsymbol{\xi}|, \tag{41b}$$

$$r_3 = \rho_2 |\boldsymbol{\xi}|, \quad \rho_2 = \sqrt{\frac{\tau_1}{\tau_3}}. \tag{41c}$$

From the eigenvectors of D we further get that

$$\ker(A_{r=r_1}) = \operatorname{span}\{\mathbf{p}_1\}, \quad \mathbf{p}_1 = [-\xi_2, \xi_1, 0]^t, \tag{42a}$$

$$\ker(A_{r=r_2}) = \operatorname{span}\{\mathbf{p}_2\}, \quad \mathbf{p}_2 = \left[\xi_1, \xi_2, -i|\boldsymbol{\xi}|\, t^{-1/2}\right]^t, \tag{42b}$$

$$\ker(A_{r=r_3}) = \operatorname{span}\{\mathbf{p}_3\}, \quad \mathbf{p}_3 = \left[\xi_1, \xi_2, -it^{1/2} r_3\right]^t. \tag{42c}$$

Thus for the case $\mathbf{x}_0 \in \partial\Omega_B \cup \partial\Omega_T$, the general solution of (38) satisfying $\mathbf{w}(s) \to \mathbf{0}$ as $s \to -\infty$ is given by

$$\mathbf{w}(s) = \sum_{k=1}^{3} \alpha_k \mathrm{e}^{r_k s} \mathbf{p}_k. \tag{43}$$

We consider now the case $\mathbf{x}_0 \in \partial\Omega_S$. Now the normal vector is given by $\mathbf{n}_0 = (\cos\alpha, \sin\alpha, 0)$ where $\alpha \in [0, 2\pi]$. By the symmetry of our problem is enough to consider the case $\alpha = 0$. Thus we take $\mathbf{n}_0 = (1, 0, 0)$ and now $\boldsymbol{\xi}$ with $\boldsymbol{\xi} \cdot \mathbf{n}_0 = 0$ reduces to $\boldsymbol{\xi} = (0, \xi_2, \xi_3)$. Equations (25a) using (36b)-(37) reduce now to:

$$\begin{aligned}
\tau_1 w_1'' - (\beta_3\xi_2^2 + \beta_1\xi_3^2)w_1 &+ i\xi_2(b\mu + q)w_2' \\
&+ i\xi_3 t^{1/2}(b\omega^2 + q)w_3' = 0, \qquad (44a)\\
i\xi_2(b\mu + q)w_1' + \beta_3 w_2'' &- (\tau_1\xi_2^2 + \beta_1\xi_3^2)w_2 \\
&- \xi_2\xi_3 t^{1/2}(b\omega^2 + q)w_3 = 0, \qquad (44b)\\
i\xi_3 t^{1/2}(b\omega^2 + q)w_1' &- \xi_2\xi_3 t^{1/2}(b\omega^2 + q)w_2 + \beta_1 w_3'' \\
&- (\beta_1\xi_2^2 + \tau_3\xi_3^2)w_3 = 0, \qquad (44c)
\end{aligned}$$

and the initial condition (25b) to:

$$\tau_1 w_1'(0) + i\xi_2(\omega\Psi'(\mu\omega) + 2b\mu + q)w_2(0) + i\xi_3 X w_3(0) = 0, \tag{45a}$$

$$\beta_3 w_2'(0) - i\xi_2(b\mu + \omega\Psi'(\mu\omega))w_1(0) = 0, \tag{45b}$$

$$\beta_1 w_3'(0) - i\mu^{1/2}\xi_3(b\omega + \Psi'(\mu\omega))w_1(0) = 0. \tag{45c}$$

If we look for solutions of (44) of the form $\mathbf{w}(s) = \mathrm{e}^{rs}\mathbf{p}$, then $A\mathbf{p} = \mathbf{0}$ where:

$$A = \begin{bmatrix} \tau_1 r^2 - (\beta_3\xi_2^2 + \beta_1\xi_3^2) & i\xi_2(b\mu + q)r & i\xi_3 t^{1/2}(b\omega^2 + q)r \\ i\xi_2(b\mu + q)r & \beta_3 r^2 - (\tau_1\xi_2^2 + \beta_1\xi_3^2) & -\xi_2\xi_3 t^{1/2}(b\omega^2 + q) \\ i\xi_3 t^{1/2}(b\omega^2 + q)r & -\xi_2\xi_3 t^{1/2}(b\omega^2 + q) & \beta_1 r^2 - (\beta_1\xi_2^2 + \tau_3\xi_3^2) \end{bmatrix}.$$

For nontrivial solutions of the system $A\mathbf{p} = \mathbf{0}$ we require that $\det A = 0$. Note that $A = [(\tau_1 - q)r^2 - \beta_1 |\boldsymbol{\xi}|^2]I + D$ where now $|\boldsymbol{\xi}| = \sqrt{\xi_2^2 + \xi_3^2}$ and:

$$D = \begin{bmatrix} qr^2 - b(\omega^2 - \mu)\xi_2^2 & i\xi_2(b\mu + q)r & i\xi_3 t^{1/2}(b\omega^2 + q)r \\ i\xi_2(b\mu + q)r & -b\mu r^2 - (b\omega^2 + q)\xi_2^2 & -\xi_2\xi_3 t^{1/2}(b\omega^2 + q) \\ i\xi_3 t^{1/2}(b\omega^2 + q)r & -\xi_2\xi_3 t^{1/2}(b\omega^2 + q) & -b\omega^2 r^2 - (b\mu + tq)\xi_3^2 \end{bmatrix}.$$

Thus $\det A = 0$ if and only if $\beta_1 |\boldsymbol{\xi}|^2 - (\tau_1 - q)r^2$ is an eigenvalue of D. It follows after computing the eigenvalues of D, that

$$\begin{aligned} r_1^{\pm} &= \pm\sqrt{\frac{\tau_1\xi_2^2 + (\beta_1 + t(b\omega^2 + q))\xi_3^2}{\tau_1}}, \\ r_2^{\pm} &= \pm\sqrt{\frac{\beta_3\xi_2^2 + \beta_1\xi_3^2}{\beta_3}}, \quad r_3^{\pm} = \pm |\boldsymbol{\xi}| . \end{aligned} \tag{46}$$

Since $\mathbf{w}(s) \to \mathbf{0}$ as $s \to -\infty$, we consider only the positive roots and label then r_1, r_2, r_3 respectively. From the eigenvectors of D we get that:

$$\ker(A_{r=r_1}) = \operatorname{span}\{\mathbf{p}_1\}, \quad \mathbf{p}_1 = \left[r_1, i\xi_2, it^{1/2}\xi_3\right]^t, \tag{47a}$$

$$\ker(A_{r=r_2}) = \operatorname{span}\{\mathbf{p}_2\}, \quad \mathbf{p}_2 = [\xi_2, ir_2, 0]^t, \tag{47b}$$

$$\ker(A_{r=r_3}) = \operatorname{span}\{\mathbf{p}_3\}, \quad \mathbf{p}_3 = \left[|\boldsymbol{\xi}|, i\xi_2, i\xi_3 t^{-1/2}\right]^t. \tag{47c}$$

Thus for the case $\mathbf{x}_0 \in \partial\Omega_S$, the general solution of (44) satisfying $\mathbf{w}(s) \to \mathbf{0}$ as $s \to -\infty$ has the form (43) using the positive values in (46) for the r's and (47) for the $\mathbf{p}$'s.

5. The Axial Problem

We consider now the problem in which the cylinder is compressed axially by λ units. The Euler–Lagrange equations for (14) in this case are given by (22a) over the set (30) with boundary conditions::

$$f_3 = 0, \quad \partial\Omega_B, \quad f_3 = \lambda L \ \text{ on } \ \partial\Omega_T, \tag{48a}$$

$$(\mathbf{S}(\nabla\mathbf{f}) \cdot \mathbf{n}) \cdot \mathbf{t} = 0 \ \text{ on } \ \partial\Omega_B \cup \partial\Omega_T \quad \mathbf{t} \cdot \mathbf{n} = 0, \tag{48b}$$

$$\mathbf{S}(\nabla\mathbf{f}) \cdot \mathbf{n} = \mathbf{0} \ \text{ on } \ \Omega_S . \tag{48c}$$

The conditions (48a)–(48b) are the so called *sliding* boundary conditions. Since this is not strictly a Dirichlet–type boundary condition, we study in the subsection (5.1) whether or not the CC holds on $\partial\Omega_B \cup \partial\Omega_T$.

The homogenous deformation (31) with $\omega = \lambda$ is a solution of (22a) over (30) with boundary conditions (48) if and only if

$$a + b(\mu + \lambda^2) + \lambda\Psi'(\mu\lambda) = 0. \tag{49}$$

By specifying the growth conditions on the function Ψ for extreme values of its argument, one can show (see [18]) that this equation has a unique solution $\mu(\lambda)$ for each $\lambda \in (0, 1]$.

The linearization of (22a), (30), (48) about the trivial solution $\mathbf{f}_h$ is given by:

$$\operatorname{div} \mathbf{C}(\nabla \mathbf{f}_h)[\nabla \mathbf{u}] = \mathbf{0} \quad \text{in} \quad \Omega, \tag{50a}$$

$$u_3 = 0 \quad \text{on} \quad \partial\Omega_B \cup \partial\Omega_T, \tag{50b}$$

$$(\mathbf{C}(\nabla \mathbf{f}_h)[\nabla \mathbf{u}] \cdot \mathbf{n}) \cdot \mathbf{t} = 0 \quad \text{on} \quad \partial\Omega_B \cup \partial\Omega_T, \quad \mathbf{t} \cdot \mathbf{n} = 0, \tag{50c}$$

$$\mathbf{C}(\nabla \mathbf{f}_h)[\nabla \mathbf{u}] \cdot \mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\Omega_S\,, \tag{50d}$$

We now characterize the values of λ for which this linearized problem satisfies or violates the complementing condition. When the CC fails, we study some of the consequences of this failure.

5.1. The full axial problem

We first consider the case in which $\mathbf{x}_0 \in \partial\Omega_B \cup \partial\Omega_T$. In this case we need to consider the problem

$$\operatorname{div} \mathbf{C}(\nabla \mathbf{f}_\lambda)[\nabla \mathbf{v}] = \mathbf{0}, \quad \text{in } \mathcal{H}, \tag{51a}$$

$$v_3 = 0, \quad \text{on } \partial\mathcal{H}, \tag{51b}$$

$$(\mathbf{C}(\nabla \mathbf{f}_h)[\nabla \mathbf{v}] \cdot \mathbf{n}_0) \cdot \mathbf{t} = 0, \quad \text{on } \partial\mathcal{H}, \quad \mathbf{t} \cdot \mathbf{n}_0 = 0, \tag{51c}$$

where $\mathbf{n}_0 = \pm\mathbf{e}_3$. For the exponential type function (10), equation (51a) reduces to the system (38) whose solution is given by (41), (42), and (43). The initial condition (51c) is equivalent to (39a) and (39b). Thus (51b)–(51c) are now equivalent to $C\mathbf{q} = \mathbf{0}$ where $\mathbf{q} = (\alpha_1, \alpha_2, \alpha_3)^t$ and:

$$C = \begin{bmatrix} -\rho_1\xi_2 & \xi_1 & \rho_2\xi_1 \\ \rho_1\xi_1 & \xi_2 & \rho_2\xi_2 \\ 0 & -it^{-1/2} & -i\rho_2 t^{1/2} \end{bmatrix}.$$

For nontrivial solutions of $C\mathbf{q} = \mathbf{0}$, we must have that $\det C = 0$, i.e.,

$$\rho_1\rho_2\,|\boldsymbol{\xi}|^2\, t^{-1/2}(t-1) = 0.$$

Since $t > 1$ for $\lambda < 1$, it follows that this determinant is never zero for such λ's and thus that the only solution of the form (10) has $\mathbf{w}(s) = \mathbf{0}$ for all s. Thus the complementing condition holds for every point on the boundary $\partial\Omega_B \cup \partial\Omega_T$.

Now consider the case in which $\mathbf{x}_0 \in \partial\Omega_S$. Any $\mathbf{x}_0 \in \partial\Omega_S$ can be written as $\mathbf{x}_0 = (\cos\beta, \sin\beta, z)$ and the corresponding normal vector $\mathbf{n}_0 = (\cos\beta, \sin\beta, 0)$. Thus the auxiliary problem (8) is given now by:

$$\operatorname{div}\mathbf{C}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{v}] = \mathbf{0}, \quad \text{in } \mathcal{H}, \tag{52a}$$

$$\mathbf{C}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{v}]\cdot\mathbf{n}_0 = \mathbf{0}, \quad \text{on } \partial\mathcal{H}, \tag{52b}$$

where

$$\mathcal{H} = \{(x_1, x_2, x_3) \,:\, (\cos\beta)x_1 + (\sin\beta)x_2 < 1\}.$$

Is enough to consider the case $\beta = 0$. For the exponential type function (10), equation (52a) reduces now to the system (44) whose solution is given by (43), (46) (positive roots), and (47). With $\beta = 0$ the boundary condition (52b) is equivalent to (45). With $\mathbf{q} = (\alpha_1, \alpha_2, \alpha_3)$ we get that $C\mathbf{q} = \mathbf{0}$ where:

$$C = \begin{bmatrix} 2\beta_3 r_2^2 + (t-1)\beta_1\xi_3^2 & 2\beta_3 r_2\xi_2 & 2\beta_3 r_2^2 \\ 2\xi_2 r_1 & r_2^2 + \xi_2^2 & 2\xi_2\,|\boldsymbol{\xi}| \\ 2t^{1/2} r_1 & t^{1/2}\xi_2 & t^{-1/2}(1+t)\,|\boldsymbol{\xi}| \end{bmatrix}.$$

For nontrivial solutions $\mathbf{w}(s)$ we must have that $\det C = 0$, i.e.,

$$\frac{|\boldsymbol{\xi}|^5}{t^{1/2}\beta_3}\left[(\beta_3\hat{r}_2^2 + \beta_3\eta_2^2 + t\beta_1\eta_3^2)^2 - 4\hat{r}_1\hat{r}_2\beta_3^2((1-t)\eta_2^2 + t\hat{r}_2^3)\right] = 0. \tag{53}$$

where $\eta_k = \xi_k/\,|\boldsymbol{\xi}|$, $k = 2, 3$ and

$$\hat{r}_1 = \sqrt{\frac{\tau_1\eta_2^2 + (\beta_1 + t(b\omega^2 + q))\eta_3^2}{\tau_1}}, \quad \hat{r}_2 = \sqrt{\frac{\beta_3\eta_2^2 + \beta_1\eta_3^2}{\beta_3}}. \tag{54}$$

If we let $\eta_2 = \cos\alpha, \eta_3 = \sin\alpha$, then this equation has the form $g(\lambda, \alpha) = 0$ where

$$g(\lambda,\alpha) = (\beta_3\hat{r}_2^2 + \beta_3\eta_2^2 + t\beta_1\eta_3^2)^2 - 4\hat{r}_1\hat{r}_2\beta_3^2((1-t)\eta_2^2 + t\hat{r}_2^3). \tag{55}$$

From (54) we get that $g(\lambda, \pi/2+u) = g(\lambda, \pi/2-u)$. It follows now that the solution set of $g(\lambda, \alpha) = 0$ is symmetric about $\alpha = \pi/2$. Is easy to check that

$$g(\lambda, \pi/2) = \beta_1^2\left[(1+t)^2 - 4t\hat{r}_1\right].$$

The equation $g(\lambda, \pi/2) = 0$ was the one considered in [6] when studying violations of the complementing condition for the axial problem only for axisymmetric deformations. In [6] it is shown that this equation has at least one solution $\lambda_{crit} < 1$. If in addition we assume[1] that:

$$\frac{\partial g}{\partial \lambda}(\lambda_{crit}, \pi/2) \neq 0, \tag{56}$$

we have the following:

Theorem 5.1. *Assume that* $g(\lambda_{crit}, \pi/2) = 0$ *and that* (56) *holds. Then there exists a smooth curve* $\alpha \to \hat{\lambda}(\alpha)$ *such that for some* $\varepsilon > 0$,

$$g(\hat{\lambda}(\alpha), \alpha) = 0, \quad \alpha \in (\pi/2 - \varepsilon, \pi/2 + \varepsilon), \quad \hat{\lambda}(\pi/2) = \lambda_{crit}. \tag{57}$$

Moreover

$$\hat{\lambda}(\pi/2 + u) = \hat{\lambda}(\pi/2 - u), \quad |u| < \varepsilon, \tag{58}$$

and $\hat{\lambda}'(\pi/2) = 0$. *Thus the complementing condition for the problem* (50) *fails on* $\partial\Omega_S$ *for all values of* λ *in an interval of the form* $(\lambda_{low}, \lambda_{crit}]$ *for some* $\lambda_{low} > 0$ *or* $[\lambda_{crit}, \lambda_{up})$ *for some* $\lambda_{up} > \lambda_{crit}$.

Proof: The existence of the smooth function $\alpha \to \hat{\lambda}(\alpha)$ satisfying (57) follows from $g(\lambda_{crit}, \pi/2) = 0$, condition (56) and by an application of the Implicit Function Theorem. Since $g(\lambda, \pi/2+u) = g(\lambda, \pi/2-u)$, we get that

$$0 = g(\hat{\lambda}(\pi/2 + u), \pi/2 + u) = g(\hat{\lambda}(\pi/2 + u), \pi/2 - u), \quad u \in (-\varepsilon, \varepsilon).$$

But $\hat{\lambda}(\pi/2 - u)$ is the unique solution of $g(\lambda, \pi/2 - u) = 0$ on the curve $\hat{\lambda}(\cdot)$ when $|u| < \varepsilon$. It follows that $\hat{\lambda}(\pi/2 + u) = \hat{\lambda}(\pi/2 - u)$, and thus that (58) holds. From this we get at once that $\hat{\lambda}'(\pi/2) = 0$. Thus either λ_{crit} is a relative maximum or minimum of $\hat{\lambda}(\cdot)$ near $\pi/2$ and the interval in which the complementing condition fails must be of the form $(\lambda_{low}, \lambda_{crit}]$ or $[\lambda_{crit}, \lambda_{up})$. □

Remark: The interval in the theorem in which the complementing condition fails is not optimal in the sense that it might be contained on a larger interval of values of λ for which the CC fails, or it might be disjoint from another interval in which the CC fails.

For the Blatz–Ko material (35) we can actually determine the function $\hat{\lambda}(\cdot)$ explicitly. In this case $\beta_1 = \beta_3 = \hat{r}_2 = 1$, and is easy to see that equation (53) reduces to

$$(1+\phi)^2 - 4\phi\left[\frac{1+(m+1)\phi}{m+2}\right]^{1/2} = 0, \quad \phi = \eta_2^2 + t\eta_3^2. \tag{59}$$

[1] In Section (5.2) we check this condition numerically for a particular example of a Hadamard–Green type material.

This equation can be transformed into a polinomial equation in ϕ that has exactly one positive root $\phi(m) > 1$. Since in this case $t = \lambda^{-(3m+2)/(m+1)}$ (see [19]), it follows from the definition above of ϕ that:

$$\hat{\lambda}(\alpha) = \left[\frac{\sin^2\alpha}{\phi(m) - \cos^2\alpha}\right]^{\frac{m+1}{3m+2}}. \qquad (60)$$

Thus the complementing condition fails for $\lambda \in (0, \lambda_{crit}]$ where λ_{crit} is obtained from (60) when $\alpha = \pi/2$ and is the maximum value of $\hat{\lambda}(\cdot)$. The value λ_{crit} is the value at which the complementing condition fails when considering only axisymmetric deformations of the cylinder. (See [6].)

5.2. The reduced axial problem

We call the *reduced axial problem* to that of looking for axisymmetric solutions of (22a) over the set (30) with boundary conditions (48). This problem has been studied extensively, but we will rely mostly on the results in [6], [18], [19]. The linearization of the reduced problem is still given by (50), but with $\mathbf{u}$ now restricted to axisymmetric deformations.

We write equation (49) as:

$$F(\lambda, \mu) \equiv a + b(\mu + \lambda^2) + \lambda\Psi'(\mu\lambda) = 0. \qquad (61)$$

As we mentioned before, this equation has a unique solution $\mu(\lambda)$ for each $\lambda \in (0, 1]$. If we let

$$\rho_n = \frac{n\pi}{L}, \quad f^2 = 1 + (t-1)\frac{\beta_3}{\beta_1 + \beta_3}, \quad v(r) = r\frac{I_0(r)}{I_1(r)},$$

where I_0, I_1 are modified Bessel functions, then the linearized problem (50) has nontrivial axisymmetric solutions for those values of λ that satisfy[2]:

$$H(\lambda, \mu(\lambda), \rho_n) \equiv v(\rho_n f) - \frac{4t}{(1+t)^2} f^2 v(\rho_n) + 2\frac{(t-1)}{(1+t)^2}\frac{\beta_3}{\beta_1} f^2 = 0. \quad (62)$$

In [18] it is shown that this equation has at least one solution λ_n for each n. One can show now (see [6]) that the complementing condition for the reduced problem fails for those values of λ such that:

$$h(\lambda, \mu(\lambda)) \equiv 1 - \frac{4t}{(1+t)^2} f = 0, \qquad (63)$$

[2] This equation corrects a misprint in equation 4.5 of [6].

and that this equation has at least one solution $\lambda_{crit} \in (0,1)$. Note that

$$h(\lambda,\mu(\lambda)) = \beta_1^{-2}(1+t)^{-2} g(\lambda,\pi/2), \tag{64}$$

where the function g is given by (55). Thus violation of the complementing condition for the reduced problem corresponds to violation of the complementing condition of the full problem in the vertical direction.

For any fixed value of λ, since $v(r)/r \to 1$ as $r \to \infty$, is easy to see that:

$$\lim_{n\to\infty} \frac{1}{v(\rho_n f)} H(\lambda,\mu(\lambda),\rho_n) = h(\lambda,\mu(\lambda)),$$

where the convergence is uniform for λ on any compact interval not containing zero. (This follows from the continuity of $\lambda \to \mu(\lambda)$.) Thus we have the following result.

Theorem 5.2. *Let λ_n be a solution of* (62) *for each n. If λ^* is an accumulation point of (λ_n), then the complementing condition for the reduced problem fails for $\lambda = \lambda^*$, i.e., λ^* is a solution of* (63). *On the other hand, if λ^* is a simple solution of* (63), *then for n sufficiently large, the equation $H(\lambda,\mu(\lambda),\rho_n) = 0$ has at least one solution λ_n and the sequence (λ_n) can be chosen such as to converge to λ^*.*

This result is actually true for the general stored energy function (17). One would have to replace H and h with equations 3.20 and 3.32 respectively from [6]. However the existence of solutions of the corresponding equation (62) for the stored energy function (17) is not known.

For the following discussion we use the following values for the parameters in (32) and (34):

$$a = 1, \quad b = 0.1, \quad c = 0.1, \quad \gamma = 2, \quad \delta = 2, \tag{65}$$

with d chosen such that $\Psi'(1) = -a - 2b$. The numerical results for other parameter values are qualitatively similar. Note that this stored energy function represents a “small” perturbation from a Blatz–Ko type energy function. In Figure (1) we show a sketch of the function $\lambda \to h(\lambda,\mu(\lambda))$ in (63) for the values (65). This function has a simple root at approximately $\lambda_{crit} = 0.4653$ which represents the only value (apart from the trivial solution $\lambda = 1$) at which the complementing condition fails for the reduced problem. In particular, it follows from (64) that condition (56) is satisfied. Thus even though the CC for the reduced problem fails at just one value of $\lambda \in (0,1)$, it fails for a full interval interval of values of λ for the full problem. We show in Figure (2) a sketch of the function $\hat{\lambda}(\cdot)$ of Theorem (5.1) where it appears that

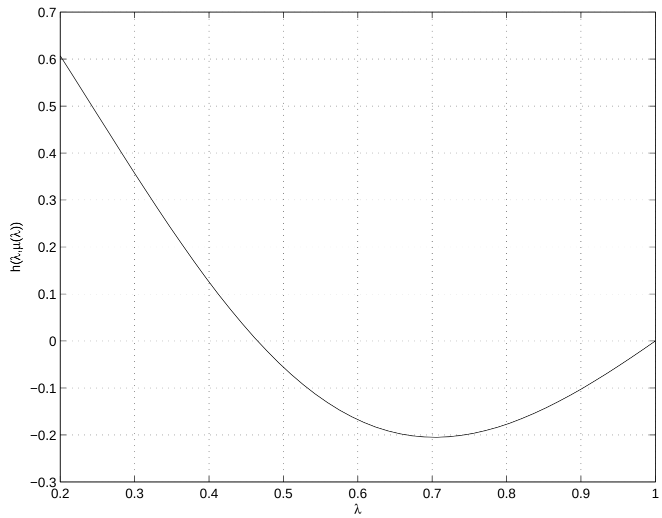

Figure 1. Sketch of the function $h(\lambda, \mu(\lambda))$ whose roots are the values of λ at which the complementing condition fails for the reduced axial problem.

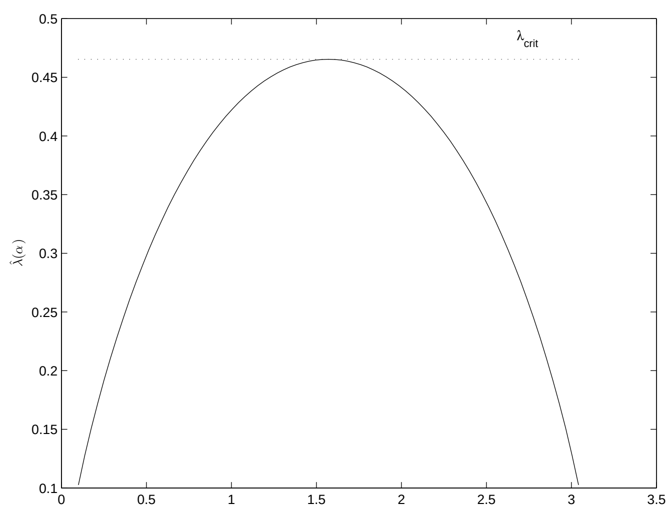

Figure 2. Sketch of the function $\hat{\lambda}(\cdot)$ of Theorem (5.1) for the Green–Hadamard material given by (65).

$\lambda_{low} = 0$ in this case, reminiscent to the Blatz-Ko case. Thus the CC fails for the full problem for all values of λ in $(0, \lambda_{crit}]$.

We now consider Equation (62) but replacing ρ_n with a real variable ρ. In this way we can consider arbitrary cylinder aspect ratios (radius to length) and modes n. We consider both (61) and (62) as a system:

$$\begin{cases} F(\lambda, \mu) &= 0, \\ H(\lambda, \mu, \rho) &= 0, \end{cases} \tag{66}$$

and solve it numerically via a continuation method using the variable ρ as the continuation parameter. We show in Figure (3) the computed solution curve $\rho \to \lambda(\rho)$ for the values (65). Note that the graph is qualitatively the same as for the Blatz-Ko material obtained in [19]. So in particular for this problem we have that there exists a sequence (λ_n) for which the linearized problem (50) has nontrivial axisymmetric solutions, and $\lambda_n \searrow \lambda_{crit}$. Since the $\lambda_n > \lambda_{crit}$, the complementing condition is satisfied for each λ_n for both the reduced and full problem. It follows then from the results in [6] that there exist global branches of nontrivial axisymmetric solutions bifurcating from the trivial branch at each λ_n. These branches "accumulate" at λ_{crit}, the only value of λ at which the complementing condition fails for the reduced, and the maximum value of λ at which it fails for the full problem. In addition, for certain aspect ratios, we have a finite number of eigenvalues of the reduced problem less than λ_{crit}. For these other eigenvalues the complementing condition is satisfied for the reduced problem. Once again the results in [6] apply and we get branches of nontrivial axisymmetric solutions bifurcating from the trivial branch at each of these eigenvalues, even though the complementing condition fails at these λ's for the full problem. (See Figure (4).) This results is significant because, as we remarked in the introduction, it means that although the continuation theorem of Healey and Simpson, as used in [6], does not apply when the complementing condition fails, that by itself does not necessarily imply that there are no branches of nontrivial solutions bifurcating from the trivial branch at values of the parameter where the CC fails.

6. The Lateral Compression Problem

We consider the problem of minimizing (14) subject to the condition of a radial displacement of λ units on $\partial\Omega_S$, $\lambda \in (0, 1]$. The Euler–Lagrange equations in this case are given by (22a) over the set (30)

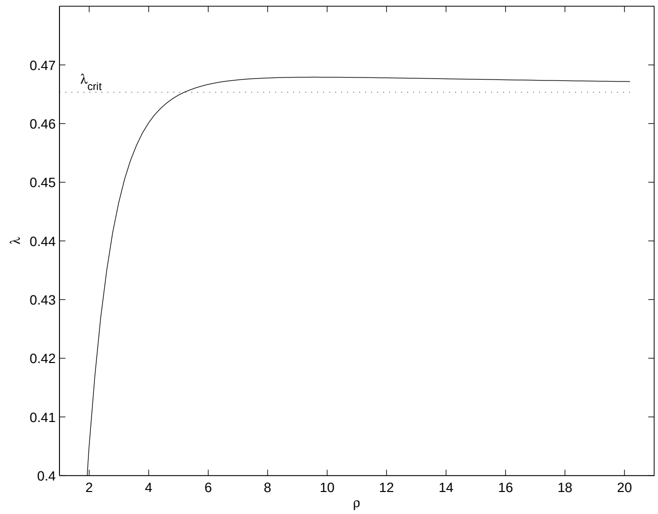

Figure 3. Sketch of the λ-component of the solution curve $\rho \rightarrow (\lambda(\rho), \mu(\rho))$ of the system (66) for the values (65).

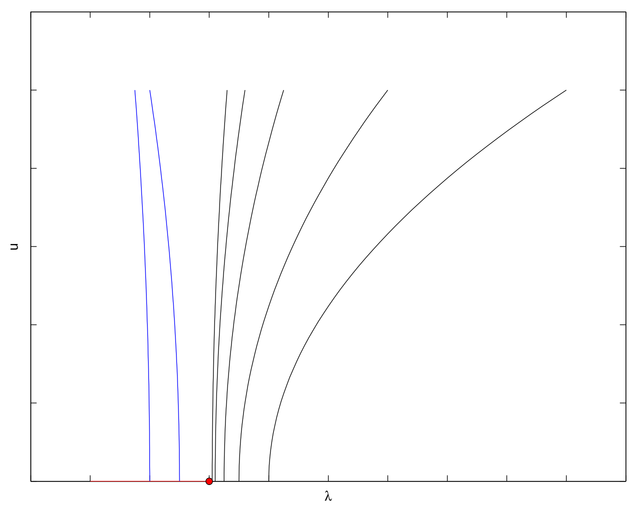

Figure 4. The CC fails for the full problem on an interval (red). In this interval there are possible bifurcation points for which the reduced problem satisfy the CC and from which branches of nontrivial axisymmetric solutions bifurcate (blue).

with boundary conditions:

$$f_1^2 + f_2^2 = \lambda^2 \ \text{ on } \ \partial\Omega_S, \tag{67a}$$

$$(\mathbf{S}(\nabla\mathbf{f})\cdot\mathbf{n})\cdot\mathbf{t} = 0 \ \text{ on } \ \partial\Omega_S, \quad \mathbf{t}\cdot\mathbf{n} = 0, \tag{67b}$$

$$\mathbf{S}(\nabla\mathbf{f})\cdot\mathbf{n} = \mathbf{0} \ \text{ on } \ \partial\Omega_B \cup \partial\Omega_T\,. \tag{67c}$$

Note that the conditions (67a)–(67b) are the so called *sliding* boundary conditions. As in the axial problem, since this is not strictly a Dirichlet–type boundary condition, we must check whether or not the CC holds on $\partial\Omega_S$.

It follows from (36a) that the homogenous deformation (31) with $\mu^{1/2} = \lambda$ is a solution of (22a) over (30) and boundary conditions (67) if and only if

$$(a + 2b\lambda^2)\omega + \lambda^2\Psi'(\lambda^2\omega) = 0. \tag{68}$$

By specifying the growth conditions on the function Ψ for extreme values of its argument, one can show (see [11]) that this equation has a unique solution $\omega = \omega(\lambda)$ for each $\lambda \in (0, 1]$. Although in general one can not determine the function $\omega(\lambda)$ explicitly, for the function (34) one can do so for certain combinations of the exponents γ and δ.

The linearization of (22a), (30), (67) about the trivial solution (31) is given by[3]:

$$\operatorname{div}\mathbf{C}(\nabla\mathbf{f}_h)[\nabla\mathbf{u}] = \mathbf{0} \ \text{ in } \ \Omega, \tag{69a}$$

$$x_1u_1 + x_2u_2 = 0 \ \text{ on } \ \partial\Omega_S, \tag{69b}$$

$$(\mathbf{C}(\nabla\mathbf{f}_h)[\nabla\mathbf{u}]\cdot\mathbf{n})\cdot\mathbf{t} = 0 \ \text{ on } \ \partial\Omega_S, \quad \mathbf{t}\cdot\mathbf{n} = 0, \tag{69c}$$

$$\mathbf{C}(\nabla\mathbf{f}_h)[\nabla\mathbf{u}]\cdot\mathbf{n} = \mathbf{0} \ \text{ on } \ \partial\Omega_B \cup \partial\Omega_T\,, \tag{69d}$$

We now characterize the values of λ for which this linearized problem satisfies or violates the complementing condition.

6.1. The full lateral problem

We begin with the case where $\mathbf{x}_0 \in \partial\Omega_B \cup \partial\Omega_T$. We look for solutions of the form (10) for the problem:

$$\operatorname{div}\mathbf{C}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{v}] = \mathbf{0}, \quad \text{in } \mathcal{H}, \tag{70a}$$

$$\mathbf{C}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{v}]\cdot\mathbf{n}_0 = \mathbf{0}, \quad \text{over } \partial\mathcal{H}, \tag{70b}$$

where $\mathbf{n}_0 = \pm\mathbf{e}_3$. Equation (70a) reduces to the system (38) whose solution is given by (41), (42), (43). The initial condition (39) is now

[3] Equation (69b) corrects equation (4.2) in [12].

equivalent to $C\mathbf{q} = \mathbf{0}$ where $\mathbf{q} = (\alpha_1, \alpha_2, \alpha_3)^t$ and:

$$C = \begin{bmatrix} -\rho_1\xi_2 & (1+t^{-1})\xi_1 & 2\rho_2\xi_1 \\ \rho_1\xi_1 & (1+t^{-1})\xi_2 & 2\rho_2\xi_2 \\ 0 & 2\beta_1 & (1+t)\beta_1 \end{bmatrix}.$$

For nontrivial solutions of $C\mathbf{q} = \mathbf{0}$, we must have that $\det C = 0$, i.e.,

$$\rho_1\beta_1 \, |\boldsymbol{\xi}|^2 \left[4\rho_2 - \frac{(1+t)^2}{t} \right] = 0.$$

Thus the complementing condition fails for those values of λ such that

$$4\rho_2 - \frac{(1+t)^2}{t} = 0. \tag{71}$$

Note that this result is independent of the tangent direction $\boldsymbol{\xi}$! We now show that this equation has at least one solution for the special case of the function (34). We need this function only to determine the asymptotic behavior of the function $\omega(\lambda)$ for $\lambda \to 0^+$.

Theorem 6.1. *Let the function* Ψ *in* (32) *be given by* (34)*. Then equation* (71) *has at least one solution* $\lambda_{crit} \in (0,1)$*. Thus the complementing condition fails for the problem* (69) *at* $\lambda = \lambda_{crit}$*.*

Proof: If we differentiate (68) with respect to λ we get that:

$$\omega'(\lambda) = -2\lambda \, \frac{b\omega(\lambda) + \Psi'(\lambda^2\omega(\lambda)) + \lambda^2\omega(\lambda)\Psi''(\lambda^2\omega(\lambda))}{a + 2b\lambda^2 + \lambda^4\Psi''(\lambda^2\omega(\lambda))} < 0,$$

where we used that the function (32) satisfies that $(s\Psi'(s))' \geq 0$. Using this we get as well that

$$t'(\lambda) = \frac{2\lambda\omega^2(\lambda) - \lambda^2\omega'(\lambda)}{\omega^4(\lambda)} > 0.$$

After some manipulations, we get that equation (71) is equivalent to:

$$g(\lambda) \equiv (t-1)\left(t^3 + \left[5 + \frac{16(b\omega^2 + q)}{\tau_3} \right] t^2 - 5t - 1 \right) = 0.$$

(Note that this equation is not a polynomial in t because both t and the coefficient of t^2 depend on λ.) Clearly $g(1) = 0$. Let $H(\lambda)$ be the factor of $g(\lambda)$ with the t^3. Then

$$g'(1) = t'(1)H(1) = t'(1) \left. \frac{16(b\omega^2 + q)}{\tau_3} \right|_{\lambda=1} > 0.$$

Also since $t(\lambda) \to 0$ as $\lambda \to 0^+$, we have that

$$\lim_{\lambda\to 0^+} g(\lambda) = (-1)\left(16 \lim_{\lambda\to 0^+} \frac{t^2 q}{\tau_3} - 1\right).$$

From equation (68) and the function (34) we get that for some constant $C > 0$,

$$\omega(\lambda) \sim C\lambda^{-\frac{2\delta}{\delta+2}}, \quad \lambda \to 0^+,$$

from which it follows that tq is bounded as $\lambda \to 0^+$. Thus

$$\lim_{\lambda\to 0^+} g(\lambda) = 1,$$

which together with $g(1) = 0$ and $g'(1) > 0$, imply that $g(\lambda) = 0$ has at least one solution in $(0, 1)$. $\square$

For the Blatz–Ko material (35), equation (71) reduces to the one obtained in [12] (considering only axisymmetric deformations) where it is shown that in this case it has exactly one solution λ_{crit}.

Now we look at the case where $\mathbf{x}_0 \in \partial\Omega_S$. Any $\mathbf{x}_0 \in \partial\Omega_S$ can be written as $\mathbf{x}_0 = (\cos\beta, \sin\beta, z)$ and the corresponding normal vector $\mathbf{n}_0 = (\cos\beta, \sin\beta, 0)$. Thus the auxiliary problem (8) is given now by:

$$\operatorname{div} \mathbf{C}(\nabla \mathbf{f}_\lambda)[\nabla \mathbf{v}] = \mathbf{0} \ \text{ in } \ \mathcal{H}, \tag{72a}$$

$$(\cos\beta)v_1 + (\sin\beta)v_2 = 0 \ \text{ on } \ \partial\mathcal{H}, \tag{72b}$$

$$(\mathbf{C}(\nabla \mathbf{f}_\lambda)[\nabla \mathbf{v}] \cdot \mathbf{n}_0) \cdot \mathbf{t} = 0 \ \text{ on } \ \partial\mathcal{H}, \quad \mathbf{t} \cdot \mathbf{n}_0 = 0, \tag{72c}$$

where

$$\mathcal{H} = \{(x_1, x_2, x_3) \,:\, (\cos\beta)x_1 + (\sin\beta)x_2 < 1\}.$$

Is enough to consider the case $\beta = 0$. Equation (72a) reduces now to the system (44) whose solution is given by (43), (46) (positive roots), and (47). With $\beta = 0$ the boundary condition (72b) reduces to $w_1(0) = 0$, while (72c) is equivalent to (45b) and (45c). With $\mathbf{q} = (\alpha_1, \alpha_2, \alpha_3)^t$ we get that $C\mathbf{q} = \mathbf{0}$ where:

$$C = \begin{bmatrix} r_1 & \xi_2 & |\boldsymbol{\xi}| \\ ir_1\xi_2 & ir_2^2 & i\xi_2\,|\boldsymbol{\xi}| \\ ir_1 t^{1/2}\xi_3 & 0 & it^{-1/2}\xi_3\,|\boldsymbol{\xi}| \end{bmatrix}.$$

For nontrivial solutions $\mathbf{w}(s)$ we must have that $\det C = 0$, i.e.,

$$r_1\xi_3^3\,|\boldsymbol{\xi}|\,\frac{\beta_1(t-1)}{\beta_3 t^{1/2}} = 0. \tag{73}$$

For $\lambda < 1$, which implies $t > 1$, it follows that this determinant is never zero and thus that the only solution of the form (10) has $\mathbf{w}(s) = \mathbf{0}$ for all s. Thus the complementing condition holds for every point on the boundary $\partial\Omega_S$. The same result holds for the case $\xi_3 = 0$ but one needs to modify the eigenvectors (47).

6.2. The reduced lateral problem

We call the *reduced lateral problem* to that of looking for axisymmetric solutions of (22a) over the set (30) with boundary conditions (67). This problem has not been studied as extensively as the corresponding axial problem, but we mention here the works of [4], [16], [11], and [12].

From Lemma 4.4 in [12], we get that for the Green–Hadamard material (32), the linearized problem (69) has nontrivial axisymmetric solutions for those values of λ for which either:

$$\begin{aligned} f_{bar}(\lambda, \omega(\lambda), k_n) \;\equiv\; & 4\rho_2 \tanh(k_n \rho_2 L/2) \\ & -\frac{(1+t)^2}{t} \tanh(k_n L/2) = 0, \end{aligned} \tag{74}$$

or

$$\begin{aligned} f_{buc}(\lambda, \omega(\lambda), k_n) \;\equiv\; & 4\rho_2 \tanh(k_n L/2) \\ & -\frac{(1+t)^2}{t} \tanh(k_n \rho_2 L/2) = 0, \end{aligned} \tag{75}$$

where (k_n) is the sequence of positive zeroes of the Bessel function J_1. The solutions λ of (74) correspond to barreling type solutions of (69) while those of (75) are of buckling type. In [11] it is shown that both (74) and (75) have at least one solution for each n.

Let us recall from (71) that the complementing condition for the problem (69) fails for those values of λ for which:

$$h(\lambda, \omega(\lambda)) \equiv 4\rho_2 - \frac{(1+t)^2}{t} = 0. \tag{76}$$

It follows from (74), (75), and the continuity of $\omega(\cdot)$, that for λ in any compact interval not containing zero we have:

$$\lim_{n\to\infty} f_{bar}(\lambda, \omega(\lambda), k_n) = h(\lambda, \omega(\lambda)) = \lim_{n\to\infty} f_{buc}(\lambda, \omega(\lambda), k_n).$$

Thus we have the following result which we state for equation (74), but it holds as well for equation (75).

Theorem 6.2. *Let λ_n be a solution of* (74) *for each n. If λ^* is an accumulation point of (λ_n), then the complementing condition for the reduced lateral problem fails for $\lambda = \lambda^*$, i.e., λ^* is a solution of* (76). *On the other hand, if λ^* is a simple solution of* (76), *then for n sufficiently large, the equation* (74) *has at least one solution λ_n and the sequence (λ_n) can be chosen such as to converge to λ^*.*

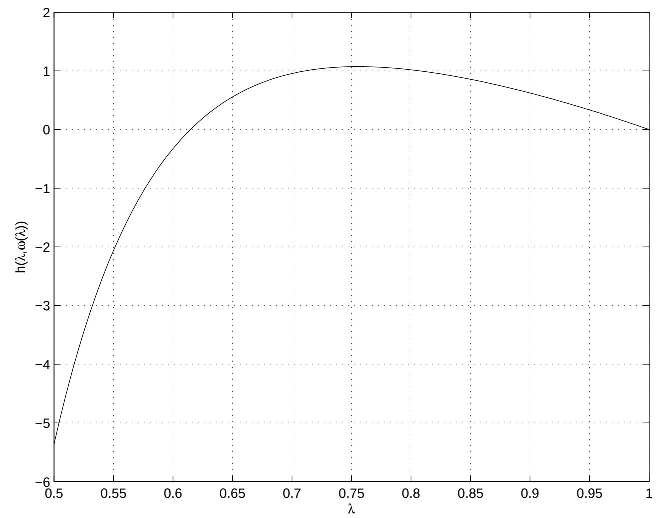

Figure 5. Sketch of the function $h(\lambda, \omega(\lambda))$ given by (76) whose roots are the values of λ at which the complementing condition fails for the lateral problem.

It follows from Lemmas 4.4 and 5.1 in [12], that this convergence holds as well for the general stored energy function (17). One would have to replace f_{bar}, f_{buc} with equations 4.55 and 4.56 respectively from [12], and h with the left hand side of equation 5.7 in the same paper. For the general stored energy function however, we do not have necessarily the existence of solutions of the corresponding equations (74) and (75).

We now consider the particular case of (32) and (34) where the coefficients and exponents in (34) are given by (65). In Figure (5) we show a sketch of the function $\lambda \to h(\lambda, \omega(\lambda))$ in (76) for the values (65). This function has only one root $\lambda_{crit} < 1$ at approximately 0.6146. In Figure (6) we show the graphs of the solution sets for equations $f_{bar}(\lambda, \omega(\lambda), k) = 0$ and $f_{buc}(\lambda, \omega(\lambda), k) = 0$ where $k = k_n L/2$ is considered as a real parameter. As for the axial problem, these graphs are generated via a continuation method applied to (68) with either $f_{bar} = 0$ or $f_{buc} = 0$. Note the resemblance of this picture with the one reported in [12] for the Blatz–Ko material. In particular we get sequences (λ_n) and (μ_n) of solutions of (74) and (75) respectively, such that $\lambda_n < \lambda_{crit} < \mu_n$ for all n, and with $\mu_n, \lambda_n \to \lambda_{crit}$.

7. Closing remarks

For the full problem for axial compression of a cylinder, we have shown that the complementing condition is always violated on a full interval

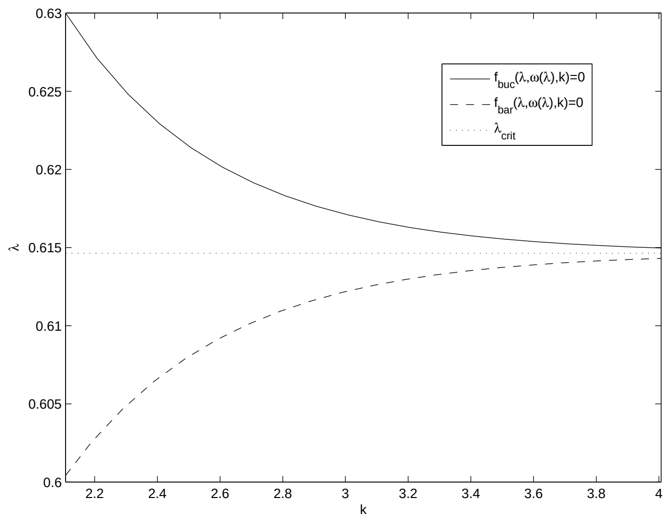

Figure 6. Sketch of the solution sets for equations (74), (75) where $k = k_n L/2$ is taken as a real parameter.

of compression ratios, while for the lateral compression (full) problem, the complementing condition is violated at least once. Both results hold for Hadamard–Green materials.

For both the axial and lateral compression reduced problems and for any isotropic material, we have shown that violation of the CC and the existence of sequences of possible bifurcation points accumulating at the point where the CC fails are equivalent. Since the eigenfunctions corresponding to each of these possible bifurcation points are more oscillatory as the mode number increases, we get that locally the corresponding nonlinear solutions develop high oscillations near the boundary. This is consistent with previous interpretations of violation of the complementing condition as a wrinkling type instability.

It still remains as an open problem whether or not the two conjectures proposed in the introduction are valid for more general BVP's. We briefly sketch some of the difficulties in obtaining such results. Let the function $\mathbf{f}_0$ in (22) be given by a function $\mathbf{g}_\lambda$ depending on some physical parameter λ. Let $\mathbf{f}_\lambda$ be a (one–parameter) of solutions of (22) corresponding to $\mathbf{f}_0 = \mathbf{g}_\lambda$. The linearization of (22) about $\mathbf{f}_\lambda$ is given by (23) using $\mathbf{f}_\lambda$ instead of $\mathbf{f}$. For the spaces (11) with $m = 1$, define the operator $L_\lambda : X \to Y$ by:

$$L_\lambda[\mathbf{u}] = (\operatorname{div} \mathbf{C}(\nabla \mathbf{f}_\lambda)[\nabla \mathbf{u}], \mathbf{C}(\nabla \mathbf{f}_\lambda)[\nabla \mathbf{u}] \cdot \mathbf{n}).$$

We assume enough regularity on the stored energy function W so as to make the operator L_λ continuous. Also we assume that L_λ is strongly elliptic for all λ.

Let (λ_k) be a sequence converging to λ^*. For simplicity, we write $L_k = L_{\lambda_k}$ and $L = L_{\lambda^*}$. We assume that L_k satisfies the complementing condition for all k. If we assume enough regularity on $\partial\Omega$, then for each k (see [2]), there exists a constant $c_k > 0$ such that for all $\mathbf{u} \in X$:

$$\|\mathbf{u}\|_X \leq c_k \left[\|L_k[\mathbf{u}]\|_Y + \|\mathbf{u}\|_Z\right]. \tag{77}$$

We now have:

Proposition 7.1. *L satisfies the complementing condition if and only if the estimates* (77) *hold with a constant independent of k.*

Proof: If L satisfies the CC, then there exists a constant $c > 0$ such that:

$$\|\mathbf{u}\|_X \leq c \left[\|L[\mathbf{u}]\|_Y + \|\mathbf{u}\|_Z\right]. \tag{78}$$

The smoothness assumptions on W and $\lambda_k \to \lambda^*$ imply that

$$\|L[\mathbf{u}] - L_k[\mathbf{u}]\|_Y \leq \varepsilon_k \|\mathbf{u}\|_X ,$$

where $\varepsilon_k \to 0$. It follows now that

$$\begin{aligned} \|\mathbf{u}\|_X &\leq c\left[\|L[\mathbf{u}] - L_k[\mathbf{u}]\|_Y + \|L_k[\mathbf{u}]\|_Y + \|\mathbf{u}\|_Z\right] \\ &\leq c\left[\varepsilon_k \|\mathbf{u}\|_X + \|L_k[\mathbf{u}]\|_Y + \|\mathbf{u}\|_Z\right]. \end{aligned}$$

Since $\varepsilon_k \to 0$ we get that for k sufficiently large,

$$\|\mathbf{u}\|_X \leq \hat{c} \left[\|L_k[\mathbf{u}]\|_Y + \|\mathbf{u}\|_Z\right],$$

and (77) holds for some constat independent of k.

Conversely, if (78) hold with L_k instead of L, and argument similar to the one above but adding and subtracting $L[\mathbf{u}]$, gives that L satisfies the estimate (78) for some constant that we denote again by c. □

It follows now from the compact embedding of $C^{2,\beta}(\Omega;\mathbb{R}^3)$ into $C^{1,\beta}(\Omega;\mathbb{R}^3)$ and the Arzelá-Ascoli's theorem, that:

Corollary 7.2. *Assume that L satisfies the CC and let $L_k[\mathbf{u}_k] = \mathbf{0}$, $\|\mathbf{u}_k\|_Z = 1$ for all k. Hence, the sequence $(\mathbf{u}_k)$ is bounded in X and it has a convergent subsequence, converging both in $C^{1,\beta}(\Omega;\mathbb{R}^3)$ and in $C^2(\Omega;\mathbb{R}^3)$.*

Remark: If the sequence $(\mathbf{u}_k)$ is such that $(\nabla\mathbf{u}_k)$ is unbounded (oscillatory behavior increasing with k), then we would get a contradiction to the result of the corollary unless L fails to satisfy the CC. Thus conjecture (i) would be true in such an scenario. This was the case in the examples that we presented for deformations of cylinders.

The analysis of conjecture (ii) seems to be closely related to the characterization of the values of λ (possible bifurcation points) for which $L_\lambda[\mathbf{u}] = \mathbf{0}$ has nontrivial solutions. These values of λ are usually characterized as the roots of a certain scalar equation called the *characteristic equation.* (See e.g. (62), (74), or (75).) This equation apart from its dependence on λ, depends on the geometry of Ω, and usually involves an index k that indicates a mode number or oscillatory behavior of the corresponding nontrivial solution. We let $\delta(\lambda, k, \Omega) = 0$ represent this characteristic equation. Also we denote the determinant in (29) by $\Delta(\boldsymbol{\xi}, \mathbf{x}_0, \lambda)$ where again, the linearization (22) is about $\mathbf{f}_\lambda$. (Recall that the CC is satisfied if and only if $\Delta(\boldsymbol{\xi}, \mathbf{x}_0, \lambda) \neq 0$.) Then conjecture (ii) is equivalent as to whether or not $\Delta(\boldsymbol{\xi}, \mathbf{x}_0, \lambda^*) = 0$ implies that $\delta(\lambda, k, \Omega) = 0$ has a solution λ_k for k sufficiently large? We shall purse this question elsewhere.

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References

1. S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I. *Comm. Pure Appl. Math.*, 12:623–727, 1959.
2. S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. *Comm. Pure Appl. Math.*, 17:35–92, 1964.
3. S. S. Antman. Nonlinear Problems of Elasticity. Springer-Verlag, New York", 1995.

4. Z. H. Guo. Vibration and stability of a cylinder subject to finite deformation. *Arch. Mech. Stos.*, 5(14), 757–768, 1962.
5. M. Gurtin. An introduction to Continuum Mechanics. Academic Press, New York, 1981.
6. T. Healey and E. Montes–Pizarro. Global Bifurcation in Nonlinear Elasticity with an Application to Barrelling States of Cylindrical Columns. *Journal of Elasticity*, 71:33–58, 2003.
7. T. J. Healey and P. Rosakis. Unbounded branches of classical injective solutions to the forced displacement problem in nonlinear elastostatics. *Journal of Elasticity*, 49:65–78, 1997.
8. T. J. Healey and H. C. Simpson. Global Continuation in Nonlinear Elasticity. *Arch. Rat. Mech. Anal.*, 143:1–28, 1998.
9. E. Montes–Pizarro and P. V. Negrón-Marrero. Local Bifurcation analysis of a Second Gradient Model for Deformations of a Rectangular Slab. *Journal of Elasticity*, Vol. 86, No. 2, 173-204, 2006.
10. C. B. Morrey. Multiple integrals in the calculus of variations. Springer–Verlag, New York, 1966.
11. P. V. Negrón–Marrero, An Analysis of the Linearized Equations for Axisymmetric Defomations of Hyperelastic Cylinders. *Mathematics and Mechanics of Solids*, 4, 109–133, 1999.
12. P. V. Negrón-Marrero and E. Montes–Pizarro. Axisymmetric Deformations of Buckling and Barrelling Type for Cylinders Under Lateral Compression–The Linear Problem. *Journal of Elasticity*, 65, 61–86, 2001.
13. C. B. Morrey. Multiple integrals in the calculus of variations. Springer–Verlag, New York, 1966.
14. P. J. Rabier and J. T. Oden. Bifurcation in Rotating Bodies. *Recherches en Mathématiques Appliquées*, Springer–Verlag, New York, 1989.
15. P. Rabinowitz. Some global results for nonlinear eigenvalue problems. *J. Funct. Anal.*, 7:487–513, 1971.
16. C. B. Sensenig. Instability of thick elastic shells. *Communications in Pure and Applied Mathematics*, 17:451–491, 1964.
17. H. C. Simpson and S. J. Spector, On Bifurcation in Finite Elasticity: Buckling of a Rectangular Rod. *J. of Elasticity*, 92:277-326, 2008.
18. H. C. Simpson and S. J. Spector. On barrelling instabilities in finite elasticity. *J. of Elasticity*, 14:103–125, 1984.
19. H. C. Simpson and S. J. Spector. On barrelling for a special material in finite elasticity. *Q. of Appl. Math.*, 14:99–111, 1984.
20. H. C. Simpson and S. J. Spector. On the positivity of the second variation in finite elasticity. *Arch. Rat. Mech. Anal.*, 98:1–30, 1987.
21. J. L. Thompson. Some existence theorems for the traction boundary value problem of linearized elastostatics. *Arch. Rat. Mech. Anal.*, 32:369–399, 1969.
22. J. Wloka. Partial Differential Equations. Cambridge University Press, NY, 1987.