An inverse problem for the rope of minimum elongation

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1 Introduction

The problem of determining the configuration of a rope or string under different types of inicial conditions is an old one. Since the 1700 related problems were considered by great mathematicians like Euler [3]. If we establish an analogy between a rope and a column, we can pose other interesting problems. For example, the one of finding the shape of a column that maximizes its height under a given axial load ([1], [2], [4]).

The problem of determining the initial configuration of the rope that minimizes its elongation while hanging by its own weight and an applied external load, while keeping either the total mass or total volume fixed, was considered in [6] for a linear constitutive equation and the nonlinear case in [5]. We call this the *direct problem*, that is, given a mass density function $\rho(\cdot)$, find the transversal area function $A(\cdot)$ that minimizes the elongation of the rope under the stated conditions. The Euler–Lagrange equations for this problem are given by nonlinear boundary value problem for the function $A(\cdot)$ depending parametrically on the mass density $\rho(\cdot)$. This problem was fully solved in [5] for the case of fixed total mass and via degree theoretic techniques for the fixed total volume case.

In this paper we study what we called the *inverse problem*. In this problem we consider the Euler–Lagrange equations for the direct problem but assuming that the function $A(\cdot)$ is given, and then determine $\rho(\cdot)$. That is, given $A(\cdot)$, find the mass density function $\rho(\cdot)$ such that the Euler–Lagrange equations for the direct problem are satisfied. In this paper we construct a numerical scheme to approximate the solution of the inverse problem for nonlinearly elastic nonhomogeneous materials.

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In Section (2) we give a derivation of the equations of equilibrium for the deformations of the rope, while in Section (3) we derive the equations describing the inverse problem. Sections (4) and (5) are devoted to the description of the numerical scheme and results.

2 The Equations of Equilibrium

2.1 Geometry of Deformation

We consider a rope or string which in its reference configuration occupies the region Ω in $\mathbb{R}^3$. We let (x, y, z) represent cartesian coordinates in Ω and assume that $[0, L] = \{x : (x, y, z) \in \Omega\}$ where the positive x axis is downward in the vertical direction. For any $x \in [0, L]$ we define the *cross–section* of Ω at x by:

$$\Omega_x = \{(y, z) : (x, y, z) \in \Omega\}, \tag{1}$$

and let $A(x)$ be the area of Ω_x. We assume that the *cross–sectional area* function $A(\cdot)$ is positive and continuous on $[0, L]$. We consider a *one–dimensional* deformation of Ω given by:

$$\mathbf{p}(x, y, z) = (u(x), y, z). \tag{2}$$

for some C^1 function $u(\cdot)$. The requirement that an (infinitesimal) volume in the reference configuration can not be reduced to zero by the deformation $\mathbf{p}$, implies that

$$u'(x) > 0 \quad , \quad \forall\, x \in [0, L]. \tag{3}$$

2.2 Mechanical Response

For any $x \in [0, L]$ we denote by $n(x)$ the force exerted by the material on $[0, x]$ on that on $[x, L]$ in a deformed configuration. We assume that the material of the rope has mass density per unit volume at x given by $\rho(x)$, where $\rho(\cdot)$ is a given positive continuously differentiable function. Hence the weight of the $[x, L]$ section of the rope is given by:

$$g \int_x^L \rho(\bar{x}) A(\bar{x})\, d\bar{x}, \tag{4}$$

where g denotes the acceleration of gravity. Assuming that a force W is applied at $x = L$, the total force exerted on the section $[x, L]$ is:

$$W + g \int_x^L \rho(\bar{x}) A(\bar{x})\, d\bar{x}. \tag{5}$$

For equilibrium, the forces must balance at each $x \in [0, L]$, i.e.,

$$n(x) = W + g \int_x^L \rho(\bar{x}) A(\bar{x})\, d\bar{x}. \tag{6}$$

We say that the material of the rope is *elastic and nonhomogeneous* if for some function $\tilde{N}(\cdot,\cdot)$ we have that

$$n(x) = \tilde{N}(u'(x), x). \tag{7}$$

The usual way to account for the lack of homogeneity is by taking

$$\tilde{N}(u'(x), x) = A(x)\hat{N}(u'(x)), \tag{8}$$

where $\hat{N} : (0,\infty) \to \mathbb{R}$ satisfies:

A1. $\hat{N}(\cdot)$ is a strictly increasing smooth function;

A2. $\hat{N}(\nu) \to \infty$ as $\nu \to \infty$;

A3. $\hat{N}(\nu) \to -\infty$ as $\nu \to 0^+$.

From properties (A1)–(A3) it follows that $\hat{N} : (0,\infty) \to \mathbb{R}$ has a smooth inverse $\hat{\nu} : \mathbb{R} \to (0,\infty)$. We further assume that

A4. $N \mapsto N^2\hat{\nu}_N(N)$ is strictly increasing on $[0,\infty)$;

A5. $N^2\hat{\nu}_N(N) \to \infty$ as $N \to \infty$.

If we combine (6), (7), and (8) we get that

$$\hat{N}(u'(x)) = A(x)^{-1}\left[W + g\int_x^L \rho(\bar{x})A(\bar{x})\,d\bar{x}\right]. \tag{9}$$

Since the top of the rope is attached to a wall we have that

$$u(0) = 0. \tag{10}$$

We consider in this paper one type of additional constraints, namely, that the total mass of the rope is a given constant M:

$$\int_0^L \rho(x)A(x)\,dx = M. \tag{11}$$

Note that we can write (9) as:

$$u'(x) = \hat{\nu}\left(A(x)^{-1}\left[W + g\int_x^L \rho(\bar{x})A(\bar{x})\,d\bar{x}\right]\right). \tag{12}$$

Integrating now over $[0,L]$ and using (10), we get the following expression for the total elongation of the rope:

$$u(L) = \int_0^L \hat{\nu}\left(A(x)^{-1}\left[W + g\int_x^L \rho(\bar{x})A(\bar{x})\,d\bar{x}\right]\right)dx. \tag{13}$$

The *direct problem* then is, given $\rho(\cdot)$, to find a function $A(\cdot)$ that minimizes the above expression for $u(L)$ subject to the constraint (11).

Let

$$B(x) = \int_x^L \rho(\bar{x}) A(\bar{x})\, d\bar{x}. \tag{14}$$

Hence $B'(x) = -\rho(x)A(x)$ and we can write (13) as

$$u(L) = \int_0^L \hat{\nu} \left(-\frac{\rho(x)(W + gB(x))}{B'(x)} \right) dx. \tag{15}$$

Note that $B(L) = 0$ and (11) is equivalent to

$$B(0) = M. \tag{16}$$

Thus our problem now is to find a function $B(\cdot)$ that minimizes (15) subject to $B(L) = 0$ and (16).

3 The Inverse Problem

The Euler–Lagrange equations for the problem of minimizing $u(L)$ are given by:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{\rho(x)(W + gB(x))}{B'(x)^2}\, \hat{\nu}_N \left(-\frac{\rho(x)(W + gB(x))}{B'(x)} \right) \right] & \\ + \frac{g\rho(x)}{B'(x)}\, \hat{\nu}_N \left(-\frac{\rho(x)(W + gB(x))}{B'(x)} \right) &= 0 \quad , \quad 0 < x < L,\end{aligned} \tag{17a}$$

$$B(0) = M \quad , \quad B(L) = 0 \quad . \tag{17b}$$

If we let

$$H(x) = -\frac{\rho(x)(W + gB(x))}{B'(x)}, \tag{18}$$

then it is shown in [5] that (17a) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x} \left[H(x)^2\, \hat{\nu}_N(H(x)) \right] - \frac{\rho'(x)}{\rho(x)}\, H(x)^2\, \hat{\nu}_N(H(x)) = 0. \tag{19}$$

This equation can be easily integrated now to get that

$$H(x)^2\, \hat{\nu}_N(H(x)) = c\rho(x), \tag{20}$$

for some constant c. The left hand side of this equation can be written as $h(H(x))$ where $h(N) = N^2 \hat{\nu}_N(N)$. Thus (20) is equivalent to

$$\frac{W + gB(x)}{B'(x)} = -\frac{1}{\rho(x)}\, h^{-1}(c\rho(x)), \tag{21}$$

where h^{-1} is the inverse function of h which exists under hypotheses (A4)–(A5). Using (14) and differentiating both sides of (21) we get that

$$-g\rho(x)A(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[A(x)h^{-1}(c\rho(x))\right] \quad , \quad 0 < x < L. \tag{22}$$

This is the equation that we solve numerically for $\rho(\cdot)$ given $A(\cdot)$.

4 The Numerical Scheme

Our problem is to solve the ordinary differential equation (22) for the function $\rho(\cdot)$. This task gets complicated because we have no initial condition and by the presence of the constant c. Note that (20) evaluated $x = 0$, with our definition of $h(\cdot)$ gives that

$$\rho(0) = \frac{1}{c}\, h(H(0)) = \frac{1}{c}\left(\frac{W + gM}{A(0)}\right)^2 \hat{\nu}_N\left(\frac{W + gM}{A(0)}\right). \tag{23}$$

Also, evaluating (20) at $x = L$ and solving for c, we get that

$$c = \frac{h(H(L))}{\rho(L)} = \frac{1}{\rho(L)}\left(\frac{W}{A(L)}\right)^2 \hat{\nu}_N\left(\frac{W}{A(L)}\right). \tag{24}$$

Using these expressions we can define a fixed point iteration to solve (22) which basically starts with an approximation to c, solves the initial value problem (22) and (23), and then updates c via (24). More formally we a sequence (ρ_i, c_i) by the following fixed point iteration:

1. Initialize c_0 and set $i = 0$.
2. Solve the initial value problem (22) and (23) using the value of c_i to get an approximation ρ_i.
3. Set
$$c_{i+1} = \frac{h(H(L))}{\rho_i(L)}.$$
4. Set $i \leftarrow i + 1$ and repeat steps (2) and (3) until $|c_{i+1} - c_i|$ is small enough.

If we denote by $\rho(\cdot, c)$ the solution of the initial value problem (22) and (23), then by virtue of (24) the above algorithm represents a fixed point iteration for the equation:

$$c = \frac{h(H(L))}{\rho(L, c)}. \tag{25}$$

5 Numerical Examples

In order to carry specific numerical computations we need to specify the constitutive function $\hat{N}(\cdot)$. This function has to satisfy conditions (A1)–(A5) and in general one has to compute numerically the functions $\hat{\nu}(\cdot)$ and $h^{-1}(\cdot)$. To keep the presentation simple we choose functions of the form:

$$\hat{N}(\nu) = E\nu^{\alpha}, \tag{26}$$

where $\alpha, E > 0$. One can show [5] that this function satisfies (A1)–(A5) except for (A3). It follows now that

$$\hat{\nu}(N) = \left(\frac{N}{E}\right)^{1/\alpha}, \tag{27a}$$

$$h^{-1}(N) = Kt^{\alpha/(1+\alpha)} \quad , \quad K = \alpha^{\alpha/(1+\alpha)} E^{1/(1+\alpha)}. \tag{27b}$$

Substituting (27b) into (22) and simplifying, we get that

$$\frac{\alpha}{1+\alpha}\,\rho'(x) = -\frac{A'(x)}{A(x)}\,\rho(x) - \frac{g}{Kc^{\alpha/(1+\alpha)}}\,\rho(x)^{(2+\alpha)/(1+\alpha)}. \tag{28}$$

This is a Bernoulli type equation which together with the initial condition (23) can be solved to get the function $\rho(\cdot, c)$ introduced at the end of Section (4). One can then assemble equation (25) to solve for c. Instead of using this approach, we employ the numerical scheme described in Section (4) because is more suitable to implement it with numerical computer packages like MATLAB and can be used for more general functions than (26). For the computations below we used the values for the parameters:

$$\alpha = 3 \quad , \quad W = 0.1 \quad , \quad g = 9.8 \quad , \quad M = 0.03 \quad , \quad E = 1.0 \quad , \quad L = 1.0,$$

the units of which are in the metric system.

5.1 Known Mass Density Function

We consider first a case in which the mass density is known. In particular we assume that the function $\rho(\cdot)$ is a constant which we continue to denote by ρ. In [5] it is shown that in this case, the minimum elongation of the rope is obtained with the following transversal area function:

$$A(x) = \frac{W}{g\rho L}\; ln\left(1 + \frac{gM}{W}\right)\left(1 + \frac{gM}{W}\right)^{1-x/L}. \tag{29}$$

We tested the numerical scheme of Section (4) by solving (28) with $A(\cdot)$ given by (29) with $\rho = 0.1$. The idea was to see if the method gives back a constant mass density function with prescribed value. This is indeed the case as the results in Figure (1) show.

5.2 Transversal Area Function Constant

We consider now the case in which $A(x) = \text{constant}$. In this case (28) is actually separable and one can easily see that $\rho(\cdot)$ must be a decreasing function of x. However the process of solving (28), (23) to assemble (25), to then solve for c, is still too cumbersome and we prefer to use the proposed numerical scheme. We show the results in Figure (2) for the case $A(x) = 0.1$ although the results are independent of A in this case. Note that the computed $\rho(\cdot)$ is indeed decreasing.

5.3 Transversal Area Function with Interior Maximum

We consider now the case in which the transversal area function is given by:

$$A(x) = 0.1\mathrm{e}^{-10\left(x-\frac{L}{2}\right)^2}. \tag{30}$$

The numerically computed $\rho(\cdot)$ is shown in Figure (3)¿ Note that this particular example shows that in general $\rho(\cdot)$ need not be monotone. In particular low dense sections of the rope of minimum elongation correspond in general with thicker areas of the rope.

References

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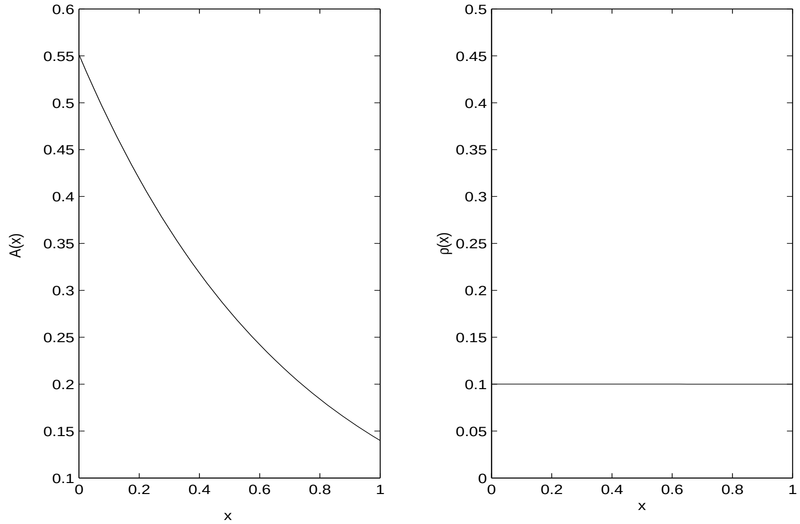

Figure 1: On the left, the function (29) which corresponds to a constant mass density function. On the right the mass density function computed by the numerical scheme with (29) as input.

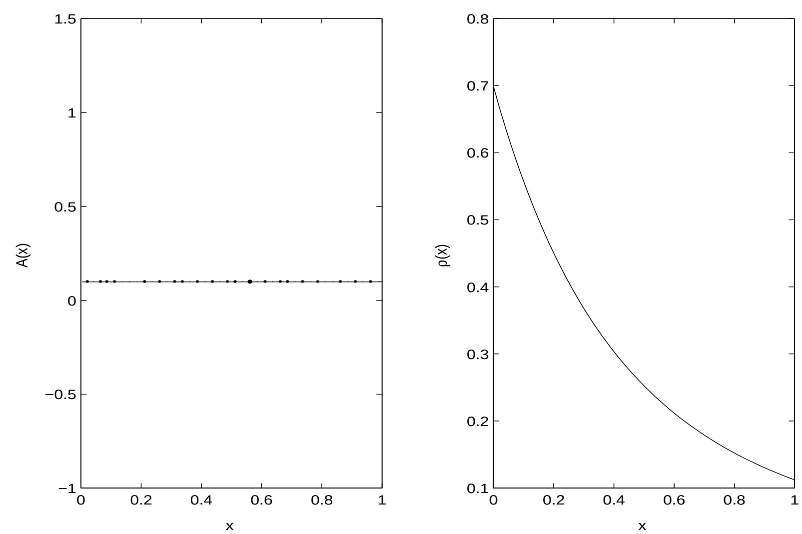

Figure 2: On the left a constant cross sectional area function. On the right the mass density function computed by the numerical scheme with $A(\cdot)$ constant.

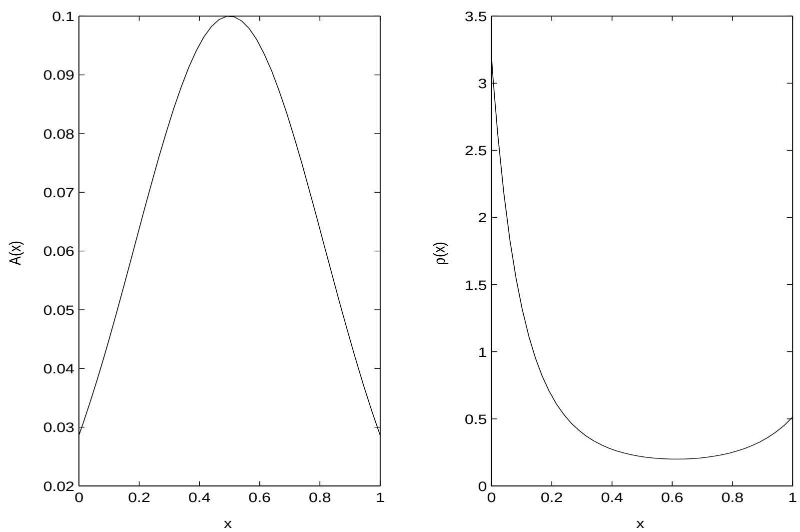

Figure 3: On the left, the function (29) which corresponds to a constant mass density function. On the right the mass density function computed by the numerical scheme with (29) as input.