Nonlinear Problems Depending on a Parameter: Continuation Methods, Limit Points, Simple Bifurcation

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1. Introduction

We will study problems of the following general form:

$$G(u,\lambda)=0 \quad , \quad G: B\times\Re \to B \tag{1.1}$$

where B is some Banach space. An example of this type of problem are the equations describing the deformations (in-extensible and obeying Hook's Law) fixed on the ends (see Antman (1980)):

$$\theta''(s)+\lambda\sin\theta(s)=0 \quad , \quad 0<s<1 \tag{1.2a}$$

$$\theta(0)=0=\theta(1) \tag{1.2b,c}$$

In this case one can use Green's functions to show that (1.2) is equivalent to (1.1) where G an integral operator and $B=C[0,1]$.

Usually to solve (1.1) numerically, after some appropriate discretization (e.g., a finite deference approximation in (1.2)), one obtains the following special case of (1.1):

$$F(x,\lambda)=0 \quad , \quad F:\Re^n\times\Re\to\Re^n \tag{1.3}$$

i.e., $B=\Re^n$. In this paper we study conditions for the existence of solution curves $x(\lambda)$ of (1.3) and when they cease to exist, and describe limit and bifurcation points.

2. Regular Points

A point (x_0,λ_0) is a *regular point* of F in (1.3) if

$$F(x_0,\lambda_0)=0 \quad , \quad \det D_xF(x_0,\lambda_0)\neq 0 \tag{2.1}$$

Under these conditions the Implicit Function Theorem implies that there exists $\delta>0$ and a smooth function $x:(\lambda_0-\delta,\lambda_0+\delta)\to\Re^n$ such that

$$F(x(\lambda),\lambda)=0 \quad , \quad \lambda \in (\lambda_0-\delta,\lambda_0+\delta) \tag{2.2a}$$

$$x(\lambda_0)=x_0 \tag{2.2b}$$

$$D_x F(x(\lambda),\lambda)x'(\lambda)+D_\lambda F(x(\lambda),\lambda)=0 \tag{2.2c}$$

Thus in a neighborhood of a regular point, the solutions of (1.3) consist of smooth curves parameterized by λ. The equation (2.2c) can be used to compute $x(\cdot)$ numerically. In particular, if we know $x(t_0)$ where $t_0 \in (\lambda_0-\delta,\lambda_0+\delta)$, then we can approximate $x(t_0+h)$ where $t_0+h \in (\lambda_0-\delta,\lambda_0+\delta)$ using the following iterations:

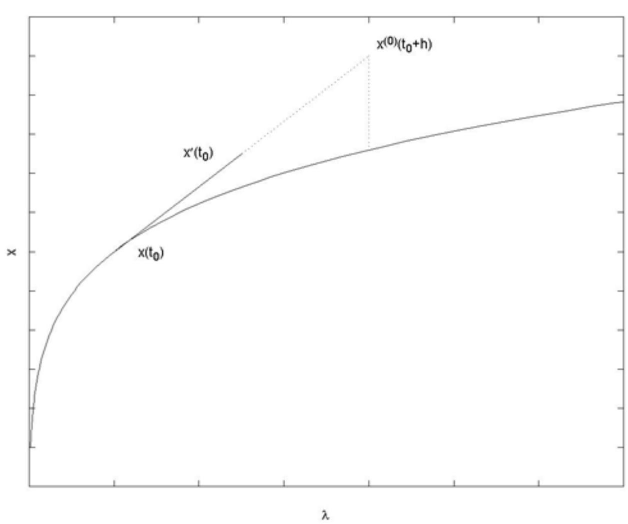
$$D_x F(x(t_0),t_0)x'(t_0)+D_\lambda F(x(t_0),t_0)=0 \tag{2.3a}$$

$$x^{(0)}(t_0+h)=x(t_0)+hx'(t_0) \tag{2.3b}$$

$$\begin{aligned} D_x F(x^{(k)}(t_0+h),t_0+h)(x^{(k+1)}(t_0+h)-x^{(k)}(t_0+h)) \\ +F(x^{(k)}(t_0+h),t_0+h)=0 \quad , \quad k=0,1,2,\ldots \end{aligned} \tag{3.3c}$$

That is, we use (2.2c) to make a *prediction* of $x(t_0+h)$ by Euler's Method (2.3a,b) and then we *correct* using Newton's Method on the x variables only applied to (1.3) in equation (3.3c). (See Figure 1). This method is effective as long as λ can be used as the continuation parameter in (1.3), i.e., whenever (2.1) is satisfied (see Rheinboldt (1986)). We now study the case when (2.1) is not satisfied.

Figure 1: Schematic diagram of a predictor-corrector continuation method.



3. Singular Points

In this section we used the so called *Liapunov-Schmidt method* to reduce problem (1.3) to a single equation in two variables when (2.1) is not satisfied.

We say that the point (x_0, λ_0) is a *(simple) singular point* of F if

$$F(x_0, \lambda_0) = 0 \quad , \quad \det D_x F(x_0, \lambda_0) = 0 \tag{3.1a,b}$$

$$\text{rank } D_x F(x_0, \lambda_0) = n - 1 \tag{3.1c}$$

Henceforth we employ the notation $D_x F^0 = D_x F(x_0, \lambda_0)$, etc.. From (3.3c) it follows that there exists a unique (up to a minus sign) $\phi \in \Re^n$ such that

$$\ker D_x F^0 = \text{span}\{\phi\} \quad , \quad \|\phi\| = 1 \tag{3.2}$$

Also there exists a unique $\phi^* \in \Re^n$ such that

$$\ker (D_x F^0)^t = \text{span}\{\phi^*\} \quad , \quad \langle \phi^*, \phi \rangle = 1 \tag{3.3a,b}$$

since $\text{rank } D_x F^0 = \text{rank } (D_x F^0)^t$. (We will show below that indeed ϕ^* can be chosen to satisfy (3.3b)). By the Fredholm Alternative Theorem we have that

$$R(D_x F^0) = (\ker (D_x F^0)^t)^\perp = \{x \in \Re^n : \langle \phi^*, x \rangle = 0\} \tag{3.4}$$

where $R(A) = \{Ax : x \in \Re^n\}$ is the range of A. We also have that

$$\Re^n = \ker (D_x F^0) \oplus R(D_x F^0) = \{\alpha\phi + v : \langle \phi^*, v \rangle = 0\} \tag{3.5}$$

This follows form $\dim \ker(D_x F^0) = 1$, $\dim R(D_x F^0) = n - 1$ and the following Lemma.

Lemma (3.1): $(\ker D_x F^0) \cap R(D_x F^0) = \{0\}$

Proof: It is enough to show that $\phi \notin R(D_x F^0)$. Let $A = D_x F^0$ and assume that $\phi \in R(A)$. Hence there exists $z \neq 0$ such that $Az = \phi$. Note that z and ϕ must linearly independent because if $a_1 z + a_2 \phi = 0$, then $0 = A(a_1 z + a_2 \phi) = a_1 Az + a_2 A\phi = a_1 Az = a_1 \phi$, which implies that $a_1 = 0$. But then $a_2 \phi = 0$, which implies $a_2 = 0$. Now define

$$X = \text{span}\{u : \exists\, p \geq 1 \ni A^p u = 0,\ A^{p-1} u \neq 0\}$$

Note that $AX \subseteq X$ because if $u \in X$, then $A^p u = 0, A^{p-1}u \neq 0$ for some $p \geq 1$. Let $w = Au$. If $w = 0$, then $Au \in X$. On the other hand if $w \neq 0$, since $A^p w = 0$, there must be an r, $1 \leq r \leq p$ such that $A^r w = 0, A^{r-1} w \neq 0$, i.e., $w \in X$. Note also that if $Au = \lambda u$ with $u \in X$, we must have that $\lambda = 0$, i.e., $A|_X$ has zero as its only eigenvalue. Since $\lambda = 0$ is a simple eigenvalue of A (by (3.1)), then the characteristic polynomial of A restricted to X must be $p(\lambda) = \pm\lambda$. It follows that $\dim X = 1$. But z and ϕ both belong to X and are linearly independent. Thus we have a contradiction to $\phi \in R(A)$. //

This lemma implies that ϕ^* can always be chosen such that (3.3b) is satisfied. In fact if $\langle \phi^*, \phi \rangle = 0$, then ϕ would have to belong to $R(A)$ of the lemma and we saw that this is impossible.

Since $(\ker D_x F^0) \cap R(D_x F^0) = \{0\}$, we have that

$$L \equiv D_x F^0 \big|_{R(D_x F^0)} \text{ is nonsingular} \tag{3.6}$$

If (x, λ) is a solution of (1.3) we can write

$$\lambda = \lambda_0 + \varepsilon \quad , \quad \varepsilon \in \Re \tag{3.7a}$$

$$x = x_0 + \alpha\phi + v \quad , \quad \alpha \in \Re, \, v \in R(D_x F^0) \tag{3.7b}$$

We now define the projection $Q : \Re^n \to R(D_x F^0)$ by

$$Q(\alpha\phi + v) = v \tag{3.8}$$

It follows now that (1.3) is equivalent to

$$\Phi(\alpha, \varepsilon, v) \equiv QF(x_0 + \alpha\phi + v, \lambda_0 + \varepsilon) = 0 \tag{3.9a}$$

$$(I - Q)F(x_0 + \alpha\phi + v, \lambda_0 + \varepsilon) = 0 \tag{3.9b}$$

Note that

$$\Phi(0,0,0) = 0 \quad , \quad D_v\Phi(0,0,0) = QD_x F^0 = L \tag{3.10a,b}$$

Since L is nonsingular, we can invoke the Implicit Function Theorem to get that there exist $\alpha_0, \varepsilon_0 > 0$, a function $v : [-\alpha_0, \alpha_0] \times [-\varepsilon_0, \varepsilon_0] \to R(D_x F^0)$ such that

$$\Phi(\alpha, \varepsilon, v(\alpha, \varepsilon)) = 0 \quad , \quad (\alpha, \varepsilon) \in [-\alpha_0, \alpha_0] \times [-\varepsilon_0, \varepsilon_0] \tag{3.11a}$$

$$v(0,0)=0 \tag{3.11b}$$

Since $I-Q$ projects onto $\ker D_x F^0 = \text{span}\{\phi\}$, equation (3.9b) reduces to

$$f(\alpha,\varepsilon) \equiv \langle \phi^*, F(x_0+\alpha\phi+v(\alpha,\varepsilon),\lambda_0+\varepsilon)\rangle = 0 \tag{3.12}$$

which is called the *bifurcation equation*. The process just described to reduce (1.3) to (3.12) is called the *Liapunov-Schmidt method*.

Note that in (3.12) we have that

$$f(0,0)=0 \tag{3.13a}$$

$$\frac{\partial f}{\partial \alpha} = \langle \phi^*, D_x F(\phi + \frac{\partial v}{\partial \alpha})\rangle \quad , \quad \frac{\partial f}{\partial \varepsilon} = \langle \phi^*, D_x F \frac{\partial v}{\partial \varepsilon} + D_\lambda F\rangle \tag{3.13b,c}$$

$$\frac{\partial^2 f}{\partial \alpha^2} = \langle \phi^*, D_{xx}F(\phi+\frac{\partial v}{\partial \alpha})(\phi+\frac{\partial v}{\partial \alpha}) + D_x F\frac{\partial^2 v}{\partial \alpha^2}\rangle \tag{3.13d}$$

$$\frac{\partial^2 f}{\partial \alpha \partial \varepsilon} = \langle \phi^*, (D_{xx}F\frac{\partial v}{\partial \varepsilon} + D_{x\lambda}F)(\phi+\frac{\partial v}{\partial \alpha}) + D_x F\frac{\partial^2 v}{\partial \alpha \partial \varepsilon}\rangle \tag{3.13e}$$

$$\frac{\partial^2 f}{\partial \varepsilon^2} = \langle \phi^*, (D_{xx}F\frac{\partial v}{\partial \varepsilon} + D_{x\lambda}F)\frac{\partial v}{\partial \varepsilon} + D_x F\frac{\partial^2 v}{\partial \varepsilon^2} + D_{\lambda\lambda}F\rangle \tag{3.13f}$$

From (3.9a) we get that

$$QD_x F(\phi + \frac{\partial v}{\partial \alpha}) = 0 \quad , \quad QD_x F\frac{\partial v}{\partial \varepsilon} + QD_\lambda F = 0 \tag{3.14a,b}$$

If we set $(\alpha,\varepsilon)=(0,0)$ in (3.14) and use that $D_x F^0 \phi = 0$, and that $QD_x F^0$ is nonsingular, we get that

$$\frac{\partial v}{\partial \alpha}(0,0) = 0 \quad , \quad \frac{\partial v}{\partial \varepsilon}(0,0) = -L^{-1}QD_\lambda F^0 \tag{3.15a,b}$$

Now since $\langle \phi^*, D_x F^0 z\rangle = 0$ for any $z \in \Re^n$, it follows from (3.15) that (3.13b-f) reduce to

$$\frac{\partial f}{\partial \alpha}(0,0) = 0 \quad , \quad \frac{\partial f}{\partial \varepsilon}(0,0) = \langle \phi^*, D_\lambda F^0\rangle \tag{3.16a,b}$$

$$\frac{\partial^2 f}{\partial \alpha^2}(0,0) = \langle \phi^*, D_{xx}F^0\, \phi\phi \rangle \qquad (3.16c)$$

$$\frac{\partial^2 f}{\partial \alpha \partial \varepsilon}(0,0) = \langle \phi^*, (D_{xx}F^0\, \frac{\partial v}{\partial \varepsilon}(0,0) + D_{x\lambda}F^0)\phi \rangle \qquad (3.16d)$$

$$\frac{\partial^2 f}{\partial \varepsilon^2} = \langle \phi^*, (D_{xx}F^0\, \frac{\partial v}{\partial \varepsilon}(0,0) + D_{x\lambda}F^0)\frac{\partial v}{\partial \varepsilon}(0,0) + D_{\lambda\lambda}F^0 \rangle \qquad (3.16e)$$

In the special case in which

$$F(0,\lambda) = 0 \quad , \quad \lambda \in \Re \qquad (3.17)$$

we get that $\partial v/\partial \varepsilon\,(0,0) = 0$ and the following further simplifications:

$$\frac{\partial f}{\partial \varepsilon}(0,0) = 0 \quad , \quad \frac{\partial^2 f}{\partial \varepsilon^2}(0,0) = 0 \qquad (3.18a,b)$$

$$\frac{\partial^2 f}{\partial \alpha \partial \varepsilon}(0,0) = \langle \phi^*, D_{x\lambda}F^0 \phi \rangle \qquad (3.18c)$$

4. Regular Limit Points

We assume that $\langle \phi^*, D_\lambda F^0 \rangle \neq 0$, i.e., that

$$D_\lambda F^0 \notin R(D_x F^0) \qquad (4.1)$$

It follows now from (3.13a), (3.16a,b) and the Implicit Function Theorem that there exists $\overline{\alpha}_0 \leq \alpha_0$ and a function $\overline{\varepsilon} : [-\overline{\alpha}_0, \overline{\alpha}_0] \rightarrow \Re$ such that

$$f(\alpha, \overline{\varepsilon}(\alpha)) = 0 \quad , \quad |\alpha| \leq \overline{\alpha}_0 \qquad (4.2a)$$

$$\overline{\varepsilon}(0) = 0 \qquad (4.2b)$$

Combining (3.11), (4.1) and (3.7) we get that

$$(x_0 + \alpha\phi + v(\alpha, \overline{\varepsilon}(\alpha)), \lambda_0 + \overline{\varepsilon}(\alpha)) \quad , \quad |\alpha| \leq \overline{\alpha}_0 \qquad (4.3)$$

represents a curve of solutions to (1.3) parameterized by α. If we differentiate (4.2a) twice with respect to α, we get that

$$\frac{\partial f}{\partial \alpha}(\alpha,\bar{\varepsilon}(\alpha)) + \bar{\varepsilon}'(\alpha)\frac{\partial f}{\partial \varepsilon}(\alpha,\bar{\varepsilon}(\alpha)) = 0 \tag{4.4a}$$

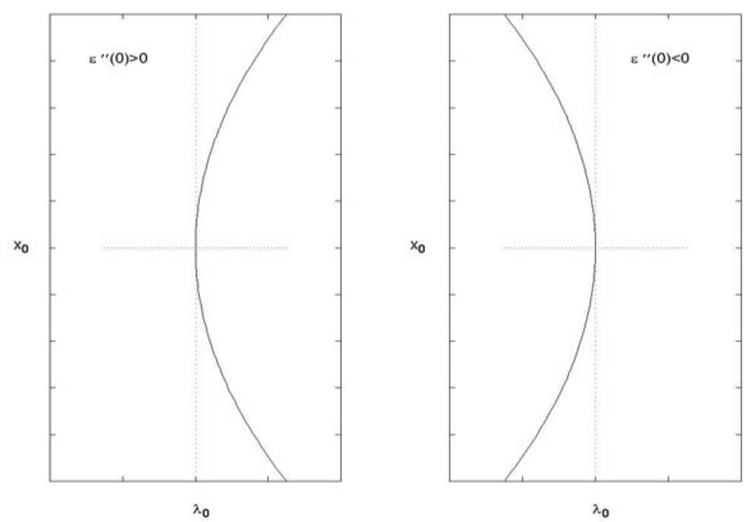
$$\begin{aligned}\frac{\partial^2 f}{\partial \alpha^2}(\alpha,\bar{\varepsilon}(\alpha)) + 2\bar{\varepsilon}'(\alpha)\frac{\partial^2 f}{\partial \varepsilon \partial \alpha}(\alpha,\bar{\varepsilon}(\alpha)) + \\ \bar{\varepsilon}''(\alpha)\frac{\partial f}{\partial \varepsilon}(\alpha,\bar{\varepsilon}(\alpha)) + \bar{\varepsilon}'(\alpha)^2\frac{\partial^2 f}{\partial \varepsilon^2}(\alpha,\bar{\varepsilon}(\alpha)) = 0\end{aligned} \tag{4.4b}$$

It follows now from (3.16) and (4.1) that

$$\bar{\varepsilon}'(0) = 0 \quad , \quad \bar{\varepsilon}''(0) = -\frac{\langle \phi^*, D_{xx}F^0\phi\phi\rangle}{\langle \phi^*, D_\lambda F^0\rangle} \tag{4.5a,b}$$

When $\bar{\varepsilon}''(0) > 0$ or $\bar{\varepsilon}''(0) < 0$ the point (x_0,λ_0) is called a *regular limit point* of (1.3) with respect to the variable λ. These cases are illustrated in Figure 2 below. The case $\bar{\varepsilon}''(0) = 0$ requires higher order terms in the Taylor expansion of $\bar{\varepsilon}(\cdot)$ to determine the shape of the solution curve nearby (x_0,λ_0).

Figure 2: Shape of the solution curve of (1.3) near a regular limit point.



The results of this section are not surprising since conditions (3.1c) and (4.1) imply that $DF^0 = (D_x F^0, D_\lambda F^0)$ has full rank. Thus the case discussed in this section is as in Section 2 but using a component of (x,λ) different from λ as the continuation parameter.

Numerically limit points are computed by a procedure similar to the one used for regular points. Equation (1.3) is augmented with another equation of the following form

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{d\lambda}{ds}\right)^2 = 1 \tag{4.6}$$

where "s" is the arc length parameter for the solution curve. Some variants of (4.6) based on an approximate arc length are used. If we differentiate $F(x(s),\lambda(s)) = 0$ with respect to "s", we get that

$$D_x F(x(s),\lambda(s))\frac{dx}{ds} + D_\lambda F \frac{d\lambda}{ds}(x(s),\lambda(s)) = 0 \tag{4.7}$$

Now (4.6), (4.7) can be used to compute $(dx/ds, d\lambda/ds)$ which can be used in the predictor step, etc.. A more convenient selection than (4.6) is

$$\frac{dx}{ds}(s_0)^t (x(s) - x(s_0)) + \frac{d\lambda}{ds}(s_0)(\lambda(s) - \lambda(s_0)) = s - s_0 \tag{4.8}$$

where $x_0 = x(s_0)$ and $\lambda_0 = \lambda(s_0)$. One can show that (4.6), (4.7) or (4.7) and the derivative with respect to "s" of (4.8) have a unique solution in the cases of regular points or limit points. (See Keller (1977), Rheinboldt (1986)).

5. Simple Bifurcation Points

Suppose now that $\langle \phi^*, D_\lambda F^0 \rangle = 0$, i.e.,

$$D_\lambda F^0 \in R(D_x F^0) \tag{5.1}$$

It follows from (3.13a) and (3.16a,b) that

$$f(0,0) = \frac{\partial f}{\partial \alpha}(0,0) = \frac{\partial f}{\partial \varepsilon}(0,0) = 0 \tag{5.2}$$

We will show that in this case (x_0, λ_0) is a bifurcation point of (1.3), i.e., in a neighborhood of (x_0, λ_0) the solutions of (1.3) can be described by two smooth curves intersecting at (x_0, λ_0).

Suppose $(\alpha(t), \varepsilon(t))$ is a solution curve of (3.12) such that $\alpha(0) = \varepsilon(0) = 0$. If we differentiate twice $f(\alpha(t), \varepsilon(t)) = 0$, we get that

$$\frac{\partial f}{\partial \alpha}(\alpha(t),\varepsilon(t))\alpha'(t) + \frac{\partial f}{\partial \varepsilon}(\alpha(t),\varepsilon(t))\varepsilon'(t) = 0 \tag{5.3a}$$

$$\frac{\partial^2 f}{\partial \alpha^2}(\alpha(t),\varepsilon(t))\alpha'(t)^2 + 2\varepsilon'(t)\alpha'(t)\frac{\partial^2 f}{\partial\varepsilon\partial\alpha}(\alpha(t),\varepsilon(t)) + \alpha''(t)\frac{\partial f}{\partial\alpha}(\alpha(t),\varepsilon(t)) + \varepsilon''(t)\frac{\partial f}{\partial\varepsilon}(\alpha(t),\varepsilon(t)) + \varepsilon'(t)^2\frac{\partial^2 f}{\partial\varepsilon^2}(\alpha(t),\varepsilon(t)) = 0 \tag{5.3b}$$

It follows from (5.2) that at $t = 0$ (5.3a) is satisfied for any $(\alpha'(0),\varepsilon'(0))$ and equation (5.3b) reduces to

$$\frac{\partial^2 f}{\partial \alpha^2}(0,0)\alpha'(0)^2 + 2\varepsilon'(0)\alpha'(0)\frac{\partial^2 f}{\partial\varepsilon\partial\alpha}(0,0) + \varepsilon'(0)^2\frac{\partial^2 f}{\partial\varepsilon^2}(0,0) = 0 \tag{5.4}$$

In general this quadratic equation has two linearly independent solutions that represent possible tangents at $(0,0)$ of solution curves of (3.12). We assume that

$$\left(\frac{\partial^2 f}{\partial\alpha\partial\varepsilon}(0,0)\right)^2 - \frac{\partial^2 f}{\partial\alpha^2}(0,0)\frac{\partial^2 f}{\partial\varepsilon^2}(0,0) > 0 \tag{5.5}$$

This is called the transversality condition and as we will show it guarantees bifurcation from (x_0,λ_0). If $\partial^2 f/\partial\alpha^2(0,0) \neq 0$, equation (5.4) can be written as

$$\frac{\partial^2 f}{\partial\alpha^2}(0,0)[\alpha'(0) - m_1\varepsilon'(0)][\alpha'(0) - m_2\varepsilon'(0)] = 0 \tag{5.6}$$

where m_1, m_2 are the roots of the quadratic

$$A^0 m^2 + 2B^0 m + C^0 = 0 \tag{5.7a}$$

where

$$A^0 = \frac{\partial^2 f}{\partial\alpha^2}(0,0) \quad , \quad B^0 = \frac{\partial^2 f}{\partial\varepsilon\partial\alpha}(0,0) \quad , \quad C^0 = \frac{\partial^2 f}{\partial\varepsilon^2}(0,0) \tag{5.7b}$$

Note that (5.5) implies that m_1, m_2 are real and different. Moreover since $d\alpha/d\varepsilon(0) = d\alpha/dt(0) / d\varepsilon/dt(0)$, we get from (5.6) that

$$\frac{d\alpha}{d\varepsilon}(0) = m_1, m_2 \tag{5.8}$$

Hence $\alpha(\varepsilon) = m_i\varepsilon + o(\varepsilon)$, $i = 1, 2$. Is reasonable then to look for a function $w(\varepsilon)$ such that $\alpha(\varepsilon) = \varepsilon w(\varepsilon)$, $w(0) = m_i$, $i = 1, 2$. We now show that such a function exists. Define

$$G(w,\varepsilon)=\begin{cases}\frac{1}{\varepsilon}f(\varepsilon w,\varepsilon) & , \quad \varepsilon\neq 0\\ \frac{1}{2}(A^0w^2+2B^0w+C^0 & , \quad \varepsilon=0\end{cases} \tag{5.9}$$

One can show now that $G\in C^1$. Note that $G(w,\varepsilon)=0$, $\varepsilon\neq 0$ if and only if $f(\varepsilon w,\varepsilon)=0$. Also

$$G(m_i,0)=0 \quad , \quad i=1,2 \tag{5.10}$$

and

$$\begin{aligned}\frac{\partial G}{\partial w}(m_i,0)&=\lim_{h\to 0}\frac{G(m_i+h,0)-G(m_i,0)}{h}\\ &=\lim_{h\to 0}\frac{1}{2h}(A^0(m_i+h)^2+2B^0(m_i+h))\\ &=A^0m_i+B^0\\ &=\pm\sqrt{(B^0)^2-A^0C^0}\end{aligned} \tag{5.11}$$

which is nonzero for $i=1,2$ by (5.5) and (5.7b). It follows now from the Implicit Function Theorem that there exist $\varepsilon_0>0$ and functions $w_i:[-\varepsilon_0,\varepsilon_0]\to\Re$, $i=1,2$ such that

$$G(w_i(\varepsilon),\varepsilon)=0 \quad , \quad |\varepsilon|\leq\varepsilon_0 \tag{5.12a}$$

$$w_i(0)=m_i \quad , \quad i=1,2 \tag{5.12b}$$

It follows then that

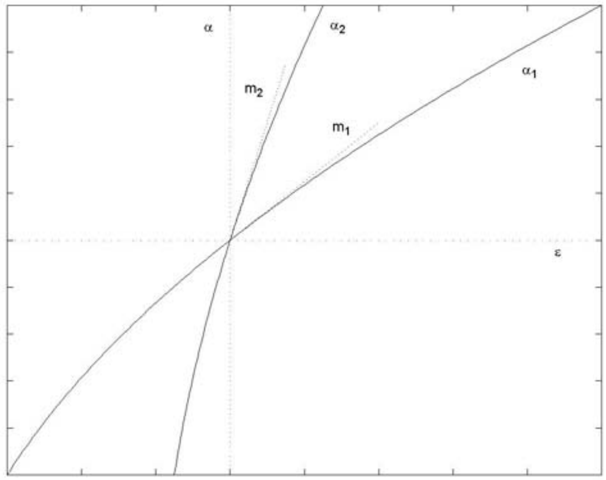
$$f(\alpha_i(\varepsilon),\varepsilon)=0 \quad , \quad |\varepsilon|\leq\varepsilon_0 \tag{5.13a}$$

$$\alpha_i(\varepsilon)=\varepsilon w_i(\varepsilon) \quad , \quad w_i(0)=m_i \quad , \quad i=1,2 \tag{5.13b}$$

Schematically the situation is as in Figure 2. A similar analysis can be done for the case $\partial^2 f/\partial\alpha^2(0,0)=0$. From (3.7), (3..11), and (5.13) it follows that the solution curves of (1.3) are given by

$$(x_0+\varepsilon w_i(\varepsilon)\phi+v(\varepsilon w_i(\varepsilon),\varepsilon),\lambda_0+\varepsilon) \quad , \quad |\varepsilon|\leq\varepsilon_0 \quad , \quad i=1,2 \tag{5.14}$$

Figure 2: Schematic situation near a simple bifurcation point.



The numerical computation of the solution curves (5.14) is usually as follows. Typically, one is computing one of the two curves, called the *primary curve*, and the points of bifurcation of the second curve are detected using the function

$$g(x,\lambda) = \det D_x F(x,\lambda) \tag{5.15}$$

More specifically, if $(x(\varepsilon),\lambda(\varepsilon))$ denotes the primary curve and $(x(0),\lambda(0)) = (x^*,\lambda^*)$ is a simple bifurcation point, then

$$h(\varepsilon) = g(x(\varepsilon),\lambda(\varepsilon)) \tag{5.16}$$

changes sign at $\varepsilon = 0$. In this a bifurcation point can be detected and can be further computed using a root finding algorithm applied to (5.16) (see Kubicek and Marek (1983)). We can use now equations (5.7), (5.8) to compute the tangent of the bifurcating curve at (x^*,λ^*). This tangent can be use now in the predictor part of the continuation method which could then continue following the bifurcating curve. This general scheme requires second derivative of f in (3.12) which in turn requires second derivatives of the function F in (1.3). Usually F comes from a discretization of a problem like (1.1) (e.g. (1.2)) which makes the dimension n in (1.3) large thus making the computation of derivatives of F an expensive computational task. Other algorithms are based on perturbation methods or difference quotients to approximate derivatives of F. (See Keller (1977), Keller and Langford (1972), and Rheinboldt (1978)).

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