THE RADIAL VOLUME DERIVATIVE AND THE CRITICAL BOUNDARY DISPLACEMENT FOR CAVITATION

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Abstract. We study the displacement boundary value problem of minimising the total energy $E(\mathbf{u})$ stored in a nonlinearly elastic body occupying a spherical domain B in its reference configuration over (possibly discontinuous) radial deformations $\mathbf{u}$ of the body subject to affine boundary data $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ for $\mathbf{x} \in \partial B$. For a given value of λ, we define what we call the *radial volume derivative* at λ, denoted $G(\lambda)$, which measures the stability or instability of the underlying homogeneous deformation $\mathbf{u}_\lambda^h(\mathbf{x}) \equiv \lambda\mathbf{x}$ to the formation of holes. We give conditions under which the critical boundary displacement $\lambda_{\rm crit}$ for radial cavitation is the unique solution of $G(\lambda) = 0$. Moreover, we prove that the radial volume derivative $G(\lambda)$ can be approximated by the corresponding volume derivative for a punctured ball B_ϵ, containing a pre–existing cavity of radius $\epsilon > 0$ in its reference state, in the limit $\epsilon \to 0$ and we use this to propose a numerical scheme to determine $\lambda_{\rm crit}$. We illustrate these general results with analytical and numerical examples.

Key words. nonlinear elasticity, radial cavitation, volume derivative, singular minimisers, quasiconvexity, critical boundary displacement.

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1. Introduction.

Cavitation (i.e., the formation of holes) is a commonly observed phenomenon in the fracture of polymers and metals (see [2]). In his seminal paper [1], Ball formulated a variational problem, in the setting of nonlinear elasticity, for which the energy minimising radial deformations of (an initially solid) ball formed a cavity at the centre of the deformed ball when the imposed boundary loads or displacements were sufficiently large. Following this paper there have been numerous studies of aspects of the problem of radial cavitation: some on analytical properties (see, e.g., [21], [16], [11]) and others relating to specific stored energies (a helpful overview is contained in [5]). Subsequent studies, e.g., of [12], [17], [10], [3] have addressed general analytic questions of existence of cavitating energy minimisers in the non–symmetric case. Furthermore, works such as Abeyaratne and Hou [6], Lopez-Pamies et al. [8, 9], Huang et al. [7], Tvergaard et al. [22] have proposed methods to approximate the onset of cavitation in non-symmetric situations. A new approach for determining the onset of cavitation for non-symmetric boundary conditions, which is based on the method presented in the current paper, is given in [14].

A central problem in the general nonsymmetric case is to identify the set of affine displacement boundary conditions for which a corresponding energy minimiser produces holes. In [14], the authors introduce a notion of a derivative of the energy functional with respect to hole formation and conjecture that the zero set of this derivative corresponds to the boundary of the set of matrices which, when used to define affine displacement boundary data, result in a discontinuous energy minimiser. The current paper considers a radial version of this conjecture and we prove that the conjecture is true in this case.

In common with a number of the cited works above, and to present the underlying mathematical structure, we restrict attention in the current study to changes in the bulk energy due to cavitation and do not include cavity initiation energy or effects

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due to surface energy, cavity contents etc., though of course such effects may play an important role in any given application (and may also affect the onset of cavitation if included)[1].

To introduce the results in more detail, first consider a nonlinear hyperelastic body occupying the unit ball $B = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| < 1\}$, $n = 2, 3, \dots$ in its reference state (the physically relevant cases being $n = 2, 3$).

The energy stored in the deformed body under a deformation $\mathbf{u} : B \to \mathbb{R}^n$ satisfying the local invertibility condition

$$\det \nabla\mathbf{u}(\mathbf{x}) > 0 \quad \text{a.e. } \mathbf{x} \in B, \tag{1.1}$$

is given by

$$E(\mathbf{u}) = \int_B W(\nabla\mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}, \tag{1.2}$$

where $W : M_+^{n\times n} \to \mathbb{R}$ is the stored energy function of the material and $M_+^{n\times n}$ denotes the set of $n \times n$ matrices with positive determinant.

If W is frame-indifferent and isotropic, then it is well known that there is a symmetric function Φ such that

$$W(\mathbf{F}) = \Phi(v_1, \dots, v_n), \tag{1.3}$$

where $v_1, \dots, v_n$ are the singular values of the matrix $\mathbf{F}$.

A radial deformation of the body is a mapping $\mathbf{u}_{\text{rad}} : B \to \mathbb{R}^n$ of the form

$$\mathbf{u}_{\text{rad}}(\mathbf{x}) = r(R)\,\frac{\mathbf{x}}{R}, \quad R = \|\mathbf{x}\|, \tag{1.4}$$

where $r : [0, 1] \to \mathbb{R}$. In this case the stored energy (1.2) can be expressed in the form

$$E(\mathbf{u}_{\text{rad}}) = \omega_n I(r) = \omega_n \int_0^1 R^{n-1}\Phi\left(r'(R), \frac{r(R)}{R}, \dots, \frac{r(R)}{R}\right) \mathrm{d}R, \tag{1.5}$$

where Φ is as in (1.3) and ω_n denotes the area of the unit sphere in $\mathbb{R}^n$ ($\omega_2 = 2\pi$, $\omega_3 = 4\pi$).

The Euler-Lagrange equation associated with the energy functional (1.5) is the radial equilibrium equation

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[R^{n-1}\Phi_{,1}(r(R))\right] = (n-1)R^{n-2}\Phi_{,2}(r(R)), \tag{1.6}$$

where

$$\Phi_{,1}(r(R)) = \Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \dots, \frac{r(R)}{R}\right), \quad \text{etc.,}$$

and $\Phi_{,i}$ denotes the partial derivative of Φ with respect to its i^{th} argument.

EXAMPLE 1.1. *A simple class of a polyconvex isotropic stored energy function to which the results in this paper can be applied is given by*

$$\Phi(v_1, \dots, v_n) = \frac{\mu}{p}(v_1^p + \cdots + v_n^p) + h(v_1 \cdots v_n) \ \textit{for} \ \ v_1, \dots, v_n > 0, \tag{1.7}$$

[1]See, e.g., [1, page 608] for comments on including surface energy effects.

where $\mu > 0$, $p \in (1, n)$ and $h : (0, \infty) \to (0, \infty)$ is a convex compressibility term which satisfies $h(d) \to \infty$ and $\frac{h(d)}{d} \to \infty$ as $d \to 0, \infty$ respectively. (However, we note that the methods in this paper apply to more general polyconvex stored energy functions under varied hypotheses.)

In this paper we consider the displacement boundary value problem in which we impose the condition

$$\mathbf{u}_{\mathrm{rad}}(\mathbf{x}) = \lambda \mathbf{x} \text{ for } \mathbf{x} \in \partial B,$$

where $\lambda > 0$, and we correspondingly define the admissible set of deformations by

$$\mathcal{A}_\lambda = \{r \in W^{1,p}((0,1)) \mid r'(R) > 0, \quad r(1) = \lambda, \quad r(0) \geq 0\}, \tag{1.8}$$

where $1 < p < n$. If $r \in \mathcal{A}_\lambda$ satisfies $r(0) > 0$, then the corresponding deformation (1.4) produces a hole of radius $r(0)$ in the deformed ball.

We define the homogeneous deformation in $\mathcal{A}_\lambda$ (corresponding to the homogeneous deformation $\mathbf{u}^h_\lambda(\mathbf{x}) \equiv \lambda \mathbf{x}$ of the ball) by

$$r^h_\lambda(R) \equiv \lambda R,$$

and we seek to identify the set of all λ for which this homogeneous deformation is no longer the global minimiser of the energy on $\mathcal{A}_\lambda$. We correspondingly define the *unstable set* of boundary strains by

$$\mathcal{U} = \{\lambda \mid I(r) < I(r^h_\lambda) \text{ for some } r \in \mathcal{A}_\lambda\}. \tag{1.9}$$

There has been much previous work on radial cavitation from which it is known (see, e.g, [1], [21], [16]) that there exists a critical value λ_{crit} such that

$$\mathcal{U} = (\lambda_{\mathrm{crit}}, \infty).$$

In this paper we give an alternative characterisation of the unstable set of boundary strains $\mathcal{U}$ which we then use as the basis for a numerical method to evaluate $\partial\mathcal{U}$ for a number of energy functions. This characterisation is based on the approach given in [14] and is motivated in part by calculations of Varvaruca [23]. We first define the set of deformations

$$\mathcal{A}_{\lambda,c} = \{r \in \mathcal{A}_\lambda \mid r(0) = c\}, \tag{1.10}$$

which produce a hole of a fixed radius $c > 0$ and volume $V = \frac{\omega_n}{n} c^n$. Next define the intermediate functional[2]

$$F(\lambda, c) = \inf_{r \in \mathcal{A}_{\lambda,c}} \frac{E(\mathbf{u}_{\mathrm{rad}}) - E(\mathbf{u}^h_\lambda)}{V} = \inf_{r \in \mathcal{A}_{\lambda,c}} \frac{I(r) - I(r^h_\lambda)}{c^n/n}. \tag{1.11}$$

A straightforward scaling argument shows that $F(\lambda, c)$ is a monotone increasing function of c and thus the following limit

$$G(\lambda) = \inf_{c>0} F(\lambda, c) = \lim_{c \searrow 0} F(\lambda, c), \tag{1.12}$$

[2]Note that $V = \frac{\omega_n c^n}{n}$ is the volume of the cavity produced by the radial deformation $\mathbf{u}_{\mathrm{rad}}$ when $r(0) = c$.

exists. We call $G(\lambda)$ the *radial volume derivative* of the energy functional (1.5) at $\lambda\mathbf{I}$. This expression first appears in [23] where it is calculated in the class of deformations of a ball for $\Phi(v_1, v_2, v_3) = \left(v_1^2 + v_2^2 + v_3^2\right)^{p/2}$, $p \in [2, 3)$.

Since the units of the volume derivative are energy per unit volume, if positive it measures the amount of energy required to force open a hole of unit volume in the given material. If negative, the volume derivative measures the corresponding amount of energy liberated by opening up such a hole.

We will show in Section 5 (see Remark 5.3) that $-G(\lambda)$ corresponds to the limiting value of the radial Cauchy stress on the cavity surface for an extended, non-homogeneous, solution $r(R)$ of (1.6) on $(0, \infty)$ and satisfying $\frac{r(R)}{R} \to \lambda$ as $R \to \infty$.

Remark 1.2. *The results of [16] are easily adapted to prove that a minimiser of I on $\mathcal{A}_{\lambda,c}$ exists for each $c > 0$ and we denote such a minimiser by r_c. In this case, the expression (1.12) is then given by*

$$G(\lambda) = \lim_{c \searrow 0} \frac{I(r_c) - I(r_\lambda^h)}{c^n/n}. \tag{1.13}$$

We define the *stable set* of boundary strains by

$$\mathcal{S} = \{\lambda \mid G(\lambda) > 0\},$$

and it follows from previous work on cavitation that

$$\mathcal{S} = (0, \lambda_{\text{crit}}).$$

We show, under suitable assumptions on Φ, that $\{\lambda_{\text{crit}}\} = \partial\mathcal{U} = \{\lambda \mid G(\lambda) = 0\}$. It turns out that in this radial problem, the criterion $G(\lambda) = 0$ exactly coincides with a condition to determine λ_{crit} introduced by Stuart in [21], derived using a shooting argument. In particular, Stuart shows that for a cavitating solution $r(R)$ of (1.6) on $(0, 1]$ with $r(1) = \lambda$ and $0 < \alpha := r'(1) < \lambda$, the radial Cauchy stress on the cavity surface is a monotone increasing function of the radial boundary stretch α. In contrast, our approach using the volume derivative yields the same criterion by using the minimum change in energy (per unit volume of cavity formed) due to cavity formation (see (1.11), (1.12)).

The structure of the paper is as follows. We first gather in Section 3 some basic results concerning the properties of cavitating or non–homogenous solutions of the radial equilibrium equation (1.6). In Section 4 we derive results on cavitating solutions with the inner cavity radius c prescribed. In particular, we study their limiting behavior as $c \to 0$. These results are used in Section 5 to actually compute the volume derivative (1.12) and to show that the vanishing of the volume derivative coincides with the condition introduced in [21] for λ_{crit}.

In Section 6 we regularise the problem of minimizing (1.5) over $\mathcal{A}_{\lambda,c}$ in the volume derivative (1.12) or (1.13), by replacing the solid ball B by a punctured ball B_ϵ with a pre-existing hole of radius $\epsilon > 0$ in its reference configuration. Specifically, we prove that if we correspondingly write

$$I^\epsilon(r) = \int_\epsilon^1 R^{n-1} \Phi\left(r'(R), \frac{r(R)}{R}, \ldots, \frac{r(R)}{R}\right) \mathrm{d}R,$$

and replace the admissible sets (1.8), (1.10) by

$$\mathcal{A}_\lambda^\epsilon = \{r \in W^{1,p}((\epsilon, 1)) \mid r'(R) > 0, \quad r(1) = \lambda, \quad r(\epsilon) \geq 0\},$$

$$\mathcal{A}^{\epsilon}_{\lambda,c} = \{r \in \mathcal{A}^{\epsilon}_{\lambda} \mid r(\epsilon) = c\},$$

and the expression (1.11) by

$$F^{\epsilon}(\lambda, c) = \inf_{r \in \mathcal{A}^{\epsilon}_{\lambda,c}} \frac{I^{\epsilon}(r) - I^{\epsilon}(r^h_{\lambda})}{c^n/n},$$

then, for each $\lambda > 0$, $F^{\epsilon}(\lambda, c)$ converges to $G(\lambda)$ as $\varepsilon, c \to 0$ provided that we choose $\epsilon = o(c)$ (so that $\frac{c}{\epsilon} \to \infty$ as $c \to 0$). Finally, in Section 7 we show how the volume derivative can be used as the basis for a numerical method for approximating λ_{crit} and we give some numerical examples of the use of this method.

2. Hypotheses on the stored energy function. We assume throughout this paper that the stored energy function $\Phi \in C^3(\mathbb{R}^n_{++})$, where $\mathbb{R}^n_{++} = \{(v_1, \dots, v_n) \mid v_i > 0,\ i = 1, \dots, n\}$ and will refer to the following hypotheses on Φ:

(H1) (**Tension-extension inequality**) For each $i = 1, \dots, n$,

$$\Phi_{,ii}(v_1, \dots, v_n) > 0, \text{ for all } (v_1, \dots, v_n) \in \mathbb{R}^n_{++}. \tag{2.1}$$

(H2) (**Baker-Ericksen Inequalities**) For each $(v_1, \dots, v_n) \in \mathbb{R}^n_{++}$, $i, j = 1, \dots, n$, $i \neq j$ and $v_i \neq v_j$:

$$\frac{v_i \Phi_{,i}(v_1, \dots, v_n) - v_j \Phi_{,j}(v_1, \dots, v_n)}{v_i - v_j} > 0. \tag{2.2}$$

(H3) For each $v > 0$ and $i = 1, \dots, n$,

$$\Phi_{,1}(a, v, \dots, v) \to +\infty \text{ as } a \to \infty, \tag{2.3a}$$

and

$$\Phi_{,1}(a, v, \dots, v) \to -\infty \text{ as } a \to 0^+. \tag{2.3b}$$

(H4) For each $(v_1, \dots, v_n) \in \mathbb{R}^n_{++}$, $i, j = 1, \dots, n$, $i \neq j$ and $v_i \neq v_j$:

$$\frac{\Phi_{,i}(v_1, \dots, v_n) - \Phi_{,j}(v_1, \dots, v_n)}{v_i - v_j} + \Phi_{,ij}(v_1, \dots, v_n) \geq 0.$$

(H5) Define

$$R(q, v) = \frac{q\Phi_{,1}(q, v, \dots, v) - v\Phi_{,2}(q, v, \dots, v)}{q - v}, \quad q \neq v.$$

There exist constants $A, B > 0$ and $0 < \beta < n - 1$ such that:

$$\text{(i)}\ \ 0 \leq R(q, v) \leq A + Bv^{\beta}, \text{ for all } \quad 0 < q \leq v, \tag{2.4a}$$

and

$$\text{(ii)}\ \ \frac{\partial R}{\partial q}(q, v) \geq 0, \text{ for all } \quad 0 < q \leq v. \tag{2.4b}$$

(H6) For all $(v_1, \dots, v_n) \in \mathbb{R}^n_{++}$:

(i) $\Phi(v_1, \dots, v_n) \geq h(v_1 \cdots v_n)$ where $h(d) \to \infty$ and $\frac{h(d)}{d} \to \infty$ as $d \to 0, \infty$ respectively.

(ii) there exists $\epsilon_0 > 0$ and $K > 0$ such that

$$|v_i \Phi_{,i}(v_1, tv_2, \ldots, tv_n)| \leq K[\Phi(v_1, \ldots, v_n) + 1],$$

whenever $|t-1| < \epsilon_0$, and $i = 2, \ldots, n$.

EXAMPLE 2.1. *Consider the class of stored energy functions*

$$\Phi(v_1, \ldots, v_n) = \sum_{i=1}^{n} \phi(v_i) + \sum_{\substack{i,j=1 \\ i<j}}^{n} \psi(v_i v_j) + h(v_1 \cdots v_n). \tag{2.5}$$

It follows now that:

(i) Φ *satisfies (H1) if all of* $\phi, \psi, h \in C^2((0,\infty))$ *are convex and at least one of* ϕ'', ψ'', h'' *is strictly positive on* $(0, \infty)$.

(ii) Φ *satisfies (H2) if* $v\phi'(v)$, $t\psi'(t)$ *are both increasing functions at least one of which is strictly increasing (e.g. if* $\phi'(v)$, $\phi''(v) > 0$ for all v *and* $\psi'(t)$, $\psi''(t) > 0$ *for all* t*).*

(iii) If $h(d) \to \infty$ *as* $d \to \infty$ *and* $h(d) \to \infty$ *as* $d \to 0$*, then* Φ *satisfies (H3). Moreover, if* ϕ *and* ψ *are nonnegative, then (i) in (H6) holds.*

(iv) If $\phi, \psi, h \in C^2((0,\infty))$ *are convex, then* Φ *satisfies (H4).*

(v) Condition (H5) is satisfied, for example, by choosing $\phi(t) = t^p$ *and* $\psi(t) = t^\alpha$*, with* $p \in [1, n)$, $\alpha \in [1, \frac{n}{2})$ *and any choice of the function* h*. If in addition,* $h(d) = Cd^\gamma + Dd^{-\delta}$ *with* $C, D > 0$ *and* $\gamma, \delta \geq 1$*, then (ii) in (H6) holds.*

3. Properties of solutions of the radial equilibrium equation. In this section we recall some basic properties of solutions of the radial equilibrium equation (1.6). Note that by the symmetry of Φ on its arguments, we have that for each $\lambda > 0$, the homogeneous deformation $r_\lambda^h(R) \equiv \lambda R$ is always a solution of (1.6).

DEFINITION 3.1. *We say that* $r \in C^2((0, R_0)), R_0 > 0$ *is a* cavitating solution *of (1.6) if* $r(0) = \lim_{R\to 0} r(R) = c > 0$.

The next two results present a monotonicity property of the tangential strain and a convexity/concavity property of solutions of (1.6).

PROPOSITION 3.2 ([1], [16]). *Assume that (H1) holds. Any non–homogeneous solution* $r(R)$ *of the radial equilibrium equation satisfies either*

i) $r'(R) < \frac{r(R)}{R}$ *, or,*

ii) $r'(R) > \frac{r(R)}{R}$,

on any maximal interval of existence. Hence $\frac{r(R)}{R}$ *is a monotonic function on any interval of existence. In particular, if* $r(R)$ *is a cavitation solution on* $(0, 1]$*, so that* $r(0) = c > 0$*, it then follows that (i) holds for all* $R \in (0, 1]$.

COROLLARY 3.3 ([1], [16]). *Assume that (H1), (H4) hold. For any non–homogeneous solution* r *of the radial equilibrium equation:*

i) $r''(R) \geq 0$ *if* $r'(R) < \frac{r(R)}{R}$*, or,*

ii) $r''(R) \leq 0$ *if* $r'(R) > \frac{r(R)}{R}$,

on any interval of existence.

The following monotonicity property of the radial Cauchy stress for solutions of (1.6), is a straightforward consequence of Proposition 3.2 and the hypotheses on Φ.

PROPOSITION 3.4. *Assume that (H1), (H2) hold. Let* $r(R)$ *be any non–homogeneous solution of the radial equation* (1.6)*. Then the radial Cauchy stress*

$$T(r(R)) = \left(\frac{R}{r(R)}\right)^{n-1} \Phi_{,1}(r(R)), \tag{3.1}$$

is monotone increasing (respectively decreasing) if r *satisfies* $r'(R) < \frac{r(R)}{R}$ *(respectively* $r'(R) > \frac{r(R)}{R}$*) on any interval of existence. Hence* $\lim_{R\to 0} T(r(R))$ *exists (as an element of* $\mathbb{R}\cup\{\infty,-\infty\}$*) for any solution* r *of the radial equation on* $(0,1)$.

The next result (cf. [16, Proposition 1.6]) demonstrates in particular that every extended solution of the radial equilibrium equation converges asymptotically to a homogeneous solution.

Proposition 3.5 (Asymptotic behaviour of solutions). *Assume that (H1), (H2) and (H4) hold. Then every cavitating solution* $r \in C^2((0,1])$ *of the radial equilibrium equation (1.6) extends to a solution of the equation on* $(0,\infty)$ *and there exists* $\mu \in (0,\infty)$ *such that* $\frac{r(R)}{R} \searrow \mu$ *as* $R \to \infty$.

Conversely, the next result shows that every homogeneous deformation is achievable as the asymptotic limit of a non-trivial solution of the radial equation.

Proposition 3.6. *Assume that (H1)-(H4) hold. For each* $\lambda \in (0,\infty)$*, there exists a nonhomogeneous solution of the radial equation (1.6) on* $(0,\infty)$ *with* $\frac{r(R)}{R} \to \lambda$ *as* $R \to \infty$.

Proof. The proof of this result follows on first writing the radial equilibrium equation in the form (3.6) and solving it with initial condition $g(\lambda) = \lambda$ to obtain a solution $g(v)$ defined for $v \in [\lambda, \lambda+\delta), \delta > 0$. Now, take a point $\mu \in (\lambda, \lambda+\delta)$ and solve the initial value problem for the radial equation, in the form (1.6), for $R \in (0,1]$ with initial conditions $r(1) = \mu$, $r'(1) = g(\mu)$. (It follows, by adapting the continuation arguments of [21], that such a solution exists for $R \in (0,1]$.) Hence, by construction, this solution also extends to a solution of (1.6) for $R \in (0,\infty)$ with the required asymptotic behaviour. □

The next few results pertain to the existence of cavitating solutions of the radial equilibrium equations with specific properties. For illustration and for the purposes of this paper we state specific hypotheses, though the conclusions of the propositions hold under a variety of hypotheses and we refer to the works [1], [16], [10], [21] for examples of other such hypotheses.

Proposition 3.7 (Existence of cavitating solutions). *Assume that (H1)-(H4) and (H6) hold. Then, for each choice of* $\lambda > c > 0$*, there exists a minimiser of (1.5) on (1.10) which corresponds to a solution* $r_c \in C^2((0,1])$ *of (1.6) satisfying*

$$r_c(1) = \lambda \text{ and } r_c(0) = c. \tag{3.2}$$

Remark 3.8. *The minimizers from the proposition have in general nonzero radial Cauchy stress (cf.* (3.1)*) on the cavity surface.*

The energy of radial cavitating solutions. The following identity (cf. [1, Page 585]) which is satisfied by solutions of (1.6), will be central to the arguments that we employ in calculating the volume derivative:

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[R^n\left(\Phi + \left(\frac{r(R)}{R} - r'(R)\right)\Phi_1\right)\right] = nR^{n-1}\Phi, \tag{3.3}$$

where the arguments of Φ and Φ_1 are $(r'(R), r(R)/R, \dots, r(R)/R)$.

Proposition 3.9. *Suppose that (H1) holds and that* r_c *is a cavitating solution satisfying*

$$r_c(1) = \lambda, \quad r_c(0) = c > 0, \tag{3.4}$$

with limiting radial Cauchy stress on the cavity surface given by

$$\lim_{R\to 0^+} T(r_c(R)) = K_c \text{ for some } K_c \in \mathbb{R}.$$

Then it follows from (3.3) that the energy of such a solution is finite and given by

$$\begin{aligned} I(r_c) &= \frac{1}{n}\lim_{\delta\to 0}\left[R^n\left(\Phi + \left(\frac{r_c(R)}{R} - r_c'(R)\right)\Phi_1\right)\right]\Bigg|_{R=\delta}^{R=1}, \\ &= \frac{1}{n}\left[\Phi(r_c'(1),\lambda,\ldots,\lambda) + (\lambda - r_c'(1))\Phi_1(r_c'(1),\lambda,\ldots,\lambda) - K_c c^n\right]. \end{aligned}$$

Proof. The result follows from slight modifications of the proof of [16, Proposition 1.13] to allow for the fact that the limiting value of the radial Cauchy stress on the cavity surface can be non-zero. □

The results of [1], [16] (see also [10]) give a variety of conditions under which (for sufficiently large λ) there exist energy minimising radial deformations which are cavitating solutions with zero radial Cauchy stress on the cavity surface. Given $P \in \mathbb{R}$, if we replace the stored energy function Φ by $\tilde{\Phi}(v_1,\ldots,v_n) = \Phi(v_1,\ldots,v_n) - Pv_1\cdots v_n$, then the same arguments applied to $\tilde{\Phi}$ yield a cavitating deformation $\tilde{r}$ for the original stored energy function Φ which satisfies $T(\tilde{r}(0)) = P$. (This follows since $\tilde{T}(r(R)) = T(r(R)) - P$ and since the extra term added to Φ to form $\tilde{\Phi}$ is a null lagrangian and hence does not change the radial equilibrium equation.) Thus we have the following:

COROLLARY 3.10. *For each $P \in \mathbb{R}$ there exists a cavitating solution of the radial equation* (1.6) *with radial Cauchy stress equal to P on the cavity surface.*

Alternative form of the radial equilibrium equation. We discuss now a change of variables employed in [21] to recast the radial equation (1.6) for non homogeneous solutions as a first order equation using the tangential stretch $\frac{r}{R}$ as an independent variable. For any nonhomogeneous solution $r(R)$ of (1.6), define the new independent variable $v = \frac{r}{R}$ and write $r'(R)$ as a new dependent variable $g(v)$. (Note that Proposition 3.2 ensures that this change of independent variable is one-to-one.) In this case equation (1.6) takes the form

$$\frac{d}{dv}\Phi_{,1}(g(v),v,\ldots,v) = (n-1)\left[\frac{\Phi_{,2}(g(v),v,\ldots,v) - \Phi_{,1}(g(v),v,\ldots,v)}{g(v)-v}\right]. \tag{3.5}$$

Expanding (3.5) we obtain

$$\begin{aligned} \frac{dg}{dv} = \frac{-(n-1)}{\Phi_{,11}(g(v),v,\ldots,v)}\Bigg[&\Phi_{,12}(g(v),v,\ldots,v) \\ &+ \frac{[\Phi_{,1}(g(v),v,\ldots,v) - \Phi_{,2}(g(v),v,\ldots,v)]}{g(v)-v}\Bigg]. \end{aligned} \tag{3.6}$$

Hence, in particular, by (H1) and (H4), it follows that $\frac{dg}{dv} < 0$ along solutions. For later use we note that the radial Cauchy stress in the new variable is given by

$$\tilde{T}(v) = \frac{1}{v^{n-1}}\Phi_{,1}(g(v),v,\ldots,v), \tag{3.7}$$

and satisfies

$$\frac{d\tilde{T}}{dv} = -(n-1)\frac{1}{v^n}\left(\frac{v\Phi_{,2}(g(v),v,\ldots,v) - g(v)\Phi_{,1}(g(v),v,\ldots,v)}{v-g(v)}\right). \tag{3.8}$$

Remark 3.11. *It is interesting to note that the radial equation* (3.6) *(which appears in [1], [21]) converts the second order radial equilibrium equation* (1.6) *into a first order differential equation with respect to the independent variable* v. *This observation can be useful in plotting the corresponding phase portrait of solutions.*

Finally we close this section by stating some properties of the curves of constant Cauchy stress in the plane (v, g) of (3.6).

Proposition 3.12. *For each* $P \in \mathbb{R}$, *define the curve of constant radial Cauchy stress* $\sigma_P : (0,\infty) \to (0,\infty)$ *by*

$$\frac{\Phi_{,1}(\sigma_P(v), v, \ldots, v)}{v^{n-1}} = P \quad \text{for all } v > 0. \tag{3.9}$$

Then $\sigma_P \in C^1((0,\infty))$ *and is defined for all* $v > 0$.

Proof. Fix P, then for each $v_0 \in (0,\infty)$, hypothesis (H3) guarantees that there exists $\sigma > 0$ satisfying $v^{1-n}\Phi_{,1}(\sigma, v, \ldots, v) = P$ (and this σ is unique by (H1)). By (H1) and the Implicit Function Theorem, a local solution $\sigma(v)$ of (3.9) exists for $v \in (v_0 - \epsilon, v_0 + \epsilon)$, for some $\epsilon > 0$. The global existence of σ_p satisfying (3.9) now follows from this local result by (H1) and the arbitrariness of v_0. □

Example 3.13. *Suppose that the stored energy function* Φ *is as in* (2.5) *but with* $\psi \equiv 0$ *and* $\phi(t) = \mu t^p$ *for some* $p \in [1, n)$ *so that*

$$\Phi(v_1, \ldots, v_n) = \mu \sum_{i=1}^{n} v_i^p + h(v_1 \cdots v_n). \tag{3.10}$$

Suppose further that Φ *satisfies (H1)–(H5). (See Example 2.1.) Then any constant Cauchy stress curve* $\sigma_P(v)$ *satisfies*

$$\sigma_P(v) \to 0 \text{ and } \sigma_P(v)v^{n-1} \to d \text{ as } v \to \infty \text{ where } h'(d) = P. \tag{3.11}$$

Finally, we note for later use that

$$\lim_{v\to\infty} \frac{1}{v^n}\left[\Phi(\sigma_P(v), v, \ldots, v) - \sigma_P(v)\Phi_{,1}(\sigma_P(v), v, \ldots, v)\right] = 0, \tag{3.12}$$

for this class of stored energy functions.

4. Properties of cavitating solutions as $c \to 0$**.** In this section we gather results on the limiting behaviour of cavitating solutions r_c satisfying

$$r_c(1) = \lambda, \quad r_c(0) = c, \tag{4.1}$$

(for fixed λ) in the limit as $c \to 0$. The first result demonstrates that these solutions converge uniformly to the corresponding homogeneous deformation. However, a perhaps at first sight surprising feature, is that the convergence of the *strains* of the cavitating deformations is to that of a non-homogeneous deformation due to a boundary layer effect near the cavity surface (see Proposition 4.4).

Proposition 4.1. *Let* Φ *satisfy (H1)-(H4). Then for each* $\lambda > 0$ *we have that* $r_c \to r_\lambda^h$ *uniformly on* $[0,1]$ *as* $c \to 0$.

Proof. Define $\phi(R) = r_c(R) - \lambda R$. Then $\phi''(R) = r''(R) \geq 0$ on $(0,1)$ by Corollary 3.3, part (i). Also

$$\phi'(1) = r_c'(1) - \lambda = r_c'(1) - r_c(1) < 0,$$

by Proposition 3.2, part (i). Hence $\phi'(R) < 0$ for $R \in (0,1)$ which implies that ϕ is monotone decreasing on $[0,1]$. Since $\phi(1) = 0$, we get that $\phi(R) > 0$ on $[0,1)$ and that $\phi(0) = c = \max_{R\in[0,1]} \phi(R)$. Hence

$$0 \le \phi(R) = r_c(R) - \lambda R \le c \text{ for } R \in [0,1],$$

and the result follows. ∎

The next result notes a monotonicity property of the cavitating solutions and follows from a uniqueness theorem for radial solutions on shells [16, Theorem 2.5] which implies in particular that the graphs of radial cavitating solutions $r \in C^2((0,1))$ satisfying $r(1) = \lambda$ cannot intersect on $(0,1)$.

Proposition 4.2. *Let (H1) hold and let $0 < c_1 < c_2$. If $r_{c_1}, r_{c_2} \in C^2((0,1])$ are cavitating solutions satisfying $r_{c_i}(0) = c_i$, $r_{c_i}(1) = \lambda$, for $i = 1,2$, then*

$$r_{c_1}(R) < r_{c_2}(R) \quad \text{for } R \in [0,1). \tag{4.2}$$

Suppose further that (H2) and (H4) hold. Then by Proposition 3.5, each r_{c_i} can be extended to $(0,\infty)$ as a solution of the radial equation (1.6), and the extended functions satisfy

$$r_{c_1}(R) > r_{c_2}(R), \quad \text{for } R \in (1,\infty). \tag{4.3}$$

Our next result turns out to be crucial for some of the main results in this and the next section. It establishes another monotonicity property, this time for the radial strains at $R = 1$, and their convergence and rate as $c \searrow 0$.

Lemma 4.3. *Suppose that (H1), (H4) hold and assume that cavitating solutions satisfying $r_c(1) = \lambda, r_c(0) = c$ exist for all $c > 0$. If $0 < c_1 < c_2$ then*

$$r'_{c_2}(1) < r'_{c_1}(1) < \lambda, \tag{4.4}$$

and $r'_c(1) \nearrow \lambda$ as $c \searrow 0$. Moreover, if (2.4b) holds, then $\lambda - r'_c(1) = O(c^n)$ as $c \searrow 0$.

Proof. The first inequality in (4.4) follows from the ordering property given in the previous proposition, while the last one follows from Proposition 3.2, part (i). In addition, from Corollary 3.3, part (i), we have that $r''_c(R) \ge 0$. Hence r_c is convex and so

$$r_c(0) \ge r_c(1) + (0-1)r'_c(1),$$

and hence

$$c \ge \lambda - r'_c(1) > 0,$$

from which it follows that $r'_c(1) \nearrow \lambda$ as $c \searrow 0$.

For the second part of the proof we use the change of variables (see [1]) $\rho = R^n$, $u(\rho) = r_c^n(R)$ for $R \in [0,1]$. Then

$$u'(\rho) = \frac{\mathrm{d}u}{\mathrm{d}\rho} = \left(\frac{\frac{d}{dR} r_c^n(R)}{\frac{d\rho}{dR}}\right)\Bigg|_{R=\rho^{\frac{1}{n}}} = \left(r'_c(R)\left(\frac{r_c(R)}{R}\right)^{n-1}\right)\Bigg|_{R=\rho^{\frac{1}{n}}}.$$

It follows from (2.4b) (see [21]) that $u''(\rho) \ge 0$. Using the convexity of u it now follows that

$$u(0) \ge u(1) + (0-1)u'(1),$$

and hence

$$\frac{c^n}{\lambda^{n-1}} \geq \lambda - r_c'(1) > 0,$$

from which it follows that $\lambda - r_c'(1) = O(c^n)$ as $c \searrow 0$. ◻

Despite the uniform convergence of the cavitating solutions of (1.6) to the homogeneous deformation proved in Proposition 4.1, the next result demonstrates that, when using the representation of the radial equation in the form (3.6), the convergence of the corresponding solution strains is to those of a non-homogeneous deformation.

THEOREM 4.4. *Let (H1), (H4) hold. Fix $\lambda > 0$ and for each $c \in (0, \lambda)$ let r_c be a cavitating solution satisfying $r_c(1) = \lambda$, $r_c(0) = c$. Let $g_c(v)$ denote the corresponding function that solves (3.5) defined on $[\lambda, \infty)$. Then for each $v \in [\lambda, \infty)$,*

$$g_c(v) \to g_0(v) \quad \text{as } c \to 0,$$

where $g_0(v)$ is the unique solution of (3.5) satisfying $g_0(\lambda) = \lambda$.

Proof. This follows from the continuous dependence of solutions to the initial value problem for (3.5) once we prove that $g_c(\lambda) \to g_0(\lambda)$ as $c \to 0$. This fact follows from Lemma 4.3 since $g_c(\lambda) = r_c'(1)$. ◻

To close this section we state the equivalent of the monotonicity result in Lemma 4.3, but for solutions of (3.5).

THEOREM 4.5. *Let $\lambda > 0$, $0 < c_1 < c_2$ and let r_{c_1} and r_{c_2} be cavitating solutions satisfying $r_{c_i}(1) = \lambda$, $r_{c_i}(0) = c_i$, $i = 1, 2$. Then the corresponding solutions $g_1(v), g_2(v)$ of (3.5) defined on $[\lambda, \infty)$ satisfy $g_1(v) > g_2(v)$ for all $v \in [\lambda, \infty)$.*

Proof. This fact follows from Lemma 4.3 since $g_c(\lambda) = r_c'(1)$, and the uniqueness of solution to the initial value problem associated to (3.5). ◻

5. The radial volume derivative. We recall that the radial volume derivative is defined by

$$G(\lambda) = \inf_{c>0} F(\lambda, c) = \lim_{c \searrow 0} F(\lambda, c), \tag{5.1}$$

where

$$F(\lambda, c) = \inf_{r \in \mathcal{A}_{\lambda,c}} \frac{I(r) - I(r_\lambda^h)}{c^n/n}, \tag{5.2}$$

and

$$\mathcal{A}_{\lambda,c} = \{r \in \mathcal{A}_\lambda \mid r(0) = c\},$$

is the set of deformations in $\mathcal{A}_\lambda$ (see (1.8)) which produce a hole of a fixed radius $c > 0$.

From the definition (5.2) and Proposition 3.9, we have that

$$F(\lambda, c) = \frac{\Phi(r_c'(1), \lambda, \ldots, \lambda) + (\lambda - r_c'(1))\Phi_1(r_c'(1), \lambda, \ldots, \lambda) - \Phi(\lambda, \ldots, \lambda)}{c^n} - K_c\,.$$

Next note that

$$\begin{aligned}\Phi(\lambda, \ldots, \lambda) = {} & \Phi(r_c'(1), \lambda, \ldots, \lambda) + (\lambda - r_c'(1))\Phi_1(r_c'(1), \lambda, \ldots, \lambda) \\ & + \frac{1}{2}\Phi_{,11}(\xi, \lambda, \ldots, \lambda)(\lambda - r_c'(1))^2,\end{aligned}$$

where $\xi \in (\lambda, r_c'(1))$. Thus we can write

$$F(\lambda, c) = -\frac{1}{2c^n}\Phi_{,11}(\xi, \lambda, \dots, \lambda)(\lambda - r_c'(1))^2 - K_c\,.$$

It now follows from Lemma 4.3 that the first term on the right hand side converges to zero as $c \to 0$ and so the volume derivative (5.1) is given by

$$G(\lambda) = -\lim_{c\to 0} K_c. \tag{5.3}$$

The next result evaluates the above limit.

Lemma 5.1. *Let (H1), (H2) and (H5)(i) (see* (2.4a)*) hold and suppose that for each $c \in (0,1)$, $r_c \in C^2((0,1))$ is a radial cavitating solution satisfying $r_c(1) = \lambda$, $r_c(0) = c$. Then*

$$K_c \to K_0 \quad as \quad c \searrow 0, \tag{5.4}$$

where

$$\begin{aligned} K_0 = {} & \frac{\Phi_{,1}(\lambda, \dots, \lambda)}{\lambda^{n-1}} \\ & -(n-1)\int_\lambda^\infty \frac{g_0(v)\Phi_{,1}(g_0(v), v, \dots, v) - v\Phi_{,2}(g_0(v), v, \dots, v)}{v^n(g_0(v) - v)}\, \mathrm{d}v, \end{aligned} \tag{5.5}$$

and $g_0(v)$ is the unique solution of (3.5) satisfying $g_0(\lambda) = \lambda$.

Proof. We recall from Proposition 3.9 that:

$$K_c = \lim_{R\to 0^+} T(r_c(R)).$$

Using $v = r_c(R)/R$ as the independent variable and writing r_c' as a function $g_c(v)$, we get that

$$K_c = \lim_{v\to\infty} \tilde{T}(v),$$

where $\tilde{T}$ is given by (3.7). It follows now from (3.7), (3.8) that

$$\begin{aligned} K_c &= \frac{\Phi_{,1}(g_c(\lambda), \lambda, \dots, \lambda)}{\lambda^{n-1}} + \int_\lambda^\infty \frac{d\tilde{T}}{dv}\, dv, \\ &= \frac{\Phi_{,1}(g_c(\lambda), \lambda, \dots, \lambda)}{\lambda^{n-1}} \\ &\quad -(n-1)\int_\lambda^\infty \frac{g_c(v)\Phi_{,1}(g_c(v), v, \dots, v) - v\Phi_{,2}(g_c(v), v, \dots, v)}{v^n(g_c(v) - v)}\, \mathrm{d}v. \end{aligned} \tag{5.6}$$

This together with Theorem 4.4, (2.4a), and the Dominated Convergence Theorem completes the proof. □

Combining Lemmas 4.3 and 5.1, and using equation (5.3), we obtain the next result.

Proposition 5.2. *Assume that* (2.4a) *and* (2.4b) *hold. Then for the radial functional* (1.5) *we have that*

$$G(\lambda) = -K_0.$$

Thus $G(\lambda) = 0$ precisely when

$$\frac{\Phi_{,1}(\lambda,\ldots,\lambda)}{\lambda^{n-1}} - (n-1)\int_{\lambda}^{\infty} \frac{g_0(v)\Phi_{,1}(g_0(v),v,v) - v\Phi_{,2}(g_0(v),v,v)}{v^n(g_0(v)-v)}\,\mathrm{d}v = 0, \quad (5.7)$$

which coincides with the condition for the critical boundary displacement given by Stuart in [21].

REMARK 5.3. *It follows from Lemma 5.1 and* (3.8) *that we can interpret* $-G(\lambda) = K_0$ *as the limiting value of the radial Cauchy stress on the cavity surface for a cavitating solution* $r(R)$ *on* $(0,\infty)$ *of the radial equilibrium equation* (1.6) *that satisfies* $\frac{r(R)}{R} \to \lambda$ *as* $R \to \infty$.

EXAMPLE 5.4. *We consider the special case of* (2.5) *in which:*

$$\Phi(v_1,\ldots,v_n) = c_1\sum_{i=1}^{n} v_i + c_2\sum_{i<j} v_i v_j + h(v_1\cdots v_n). \quad (5.8)$$

For this stored energy function one can solve in closed form the equilibrium equation (1.6). *(See [4].) The general solution is given by:*

$$r^n(R) = K_1R^n + K_2,$$

where $r'(R)(r(R)/R)^{n-1} = K_1$. *The constants* K_1, K_2 *are chosen to satisfy the boundary conditions* $r(0) = c$ *and* $r(1) = \lambda$, *which yields that:*

$$K_1 = \lambda^n - c^n, \quad K_2 = c^n.$$

With these values for the constants K_1, K_2, *and provided that the function* h *satisfies (iii) of Example 2.1, the function* r *above is the global minimizer of* $I(\cdot)$ *over* $\mathcal{A}_{\lambda,c}$. *The energy of* r *is given by (c.f.* (3.3)*):*

$$nI(r) = nc_1\lambda + nc_2\lambda^2 + h(K_1),$$

and that of r_λ^h *by:*

$$nI(r_\lambda^h) = nc_1\lambda + nc_2\lambda^2 + h(\lambda^n).$$

It follows now that:

$$\begin{aligned} G(\lambda) &= \lim_{c\to 0^+} \frac{I(r) - I(r_\lambda^h)}{c^n/n} \\ &= \lim_{c\to 0^+} \frac{h(\lambda^n - c^n) - h(\lambda^n)}{c^n} = -h'(\lambda^n). \end{aligned}$$

Thus $G(\lambda) = 0$ *if and only if* $h'(\lambda^n) = 0$, *which is exactly the same condition obtained in [4] for the criteria for the initiation of cavitation. This example can be generalized to the case of non–radial deformations (see [14]).*

6. Approximation by regularised problems. We now consider the regularized cavitation problem in which the reference configuration B is replaced by a punctured ball B_ϵ with a pre-existing hole of radius $\epsilon > 0$. We define the corresponding energy of a radial deformation of the punctured ball by

$$I^\epsilon(r) = \int_\epsilon^1 R^{n-1}\Phi\left(r'(R), \frac{r(R)}{R}, \ldots, \frac{r(R)}{R}\right)\mathrm{d}R, \quad (6.1)$$

and we correspondingly replace the admissible sets (1.8), (1.10) by

$$\mathcal{A}^{\epsilon}_{\lambda} = \{r \in W^{1,p}((\epsilon,1)) \mid r'(R) > 0, \quad r(1) = \lambda, \quad r(\epsilon) \geq 0\}, \tag{6.2}$$

$$\mathcal{A}^{\epsilon}_{\lambda,c} = \{r \in \mathcal{A}^{\epsilon}_{\lambda} \mid r(\epsilon) = c\}. \tag{6.3}$$

We next seek to approximate the expression (1.11) used in the definition of the radial volume derivative by

$$F^{\epsilon}(\lambda, c) = \inf_{r\in\mathcal{A}^{\epsilon}_{\lambda,c}} \frac{I^{\epsilon}(r) - I^{\epsilon}(r^h_{\lambda})}{c^n/n}. \tag{6.4}$$

Note that the Euler–Lagrange equation for (6.1) is still given by (1.6) for $R \in (\varepsilon, 1)$. The following result follows from this fact and slight modifications to the results in [16] for punctured balls, to account for the specification of $r_{\varepsilon}(\varepsilon)$. The last statement in the proposition follows from Theorem 4.5.

Proposition 6.1. *Let (H1)-(H6) hold. Then for each $\epsilon \in (0,1)$ and $0 < c < \lambda$ there exists a unique solution r_ϵ of the radial equilibrium equation on $(\epsilon,1)$ with $r_\epsilon(\epsilon) = c$ and $r_\epsilon(1) = \lambda$. Each such solution extends to a solution $r(R)$ of the radial equation on $(0,1)$ if $c \geq \lambda\epsilon$. The extended solution satisfies $r(0) \geq 0$ (with $r(0) > 0$ provided $c > \lambda\epsilon$). Moreover, $\lim_{R\to 0} T(r(R)) = P \in \mathbb{R}$ and the limiting Cauchy stress P is a monotonic function of the radial strain $r'(1) \in (0, \lambda]$ on the outer boundary.*

The next theorem is the equivalent of Theorem 4.4 but for punctured balls.

Theorem 6.2. *Fix $\lambda > 0$ and for each $\epsilon \in (0, \lambda)$ let r_ϵ be a solution of (1.6) over $(\varepsilon, 1)$ satisfying $r_\epsilon(1) = \lambda$, $r_\epsilon(\epsilon) = c > \lambda\epsilon$. By the last proposition, each such solution extends to a cavitation solution on $(0,1)$. Let $g_\epsilon(v)$ denote the corresponding function that solves (3.5) and which is defined on $[\lambda,\infty)$. Then for each $v \in [\lambda,\infty)$,*

$$g_\epsilon(v) \to g_0(v) \quad \text{as } c \to 0,$$

where $g_0(v)$ is the unique solution of (3.5) satisfying $g_0(\lambda) = \lambda$ (note that $\epsilon \to 0$ as $c \to 0$).

Proof. This follows from the continuous dependence of solutions to the initial value problem for (3.5) once we prove that $g_\epsilon(\lambda) \to g_0(\lambda)$ as $c \to 0$. This fact follows from arguments analogous to those used in the proof of Proposition 4.1 since by Corollary 3.3 it follows that $r''_\epsilon(R) \geq 0$ and hence

$$r_\epsilon(\epsilon) = c \geq r_\epsilon(1) + (\epsilon - 1)r'_\epsilon(1) \geq \lambda - r'_\epsilon(1) \geq 0.$$

The required result now follows since $g_\epsilon(\lambda) = r'_\epsilon(1)$. ∎

We now have the main result of this section, namely, the convergence of the approximations (6.4) to the exact volume derivative.

Theorem 6.3. *Suppose that the stored energy function Φ is of the form (3.10) and that it satisfies (H1)-(H5). Then the expression (6.4) converges to the volume derivative for the solid ball as $c \to 0$ provided that $\epsilon = o(c)$ as $c \to 0$.*

Proof. By the conservation law (3.3), and writing $r'_\epsilon(R)$ as a function $g_\epsilon(v)$,

$v = \frac{r_\epsilon(R)}{R}$, we obtain

$$\begin{aligned}F^\epsilon(\lambda, c) &= \frac{1}{c^n}\Bigg[R^n\Bigg(\Phi\left(r'_\epsilon, \frac{r_\epsilon}{R}, \dots, \frac{r_\epsilon}{R}\right) \\ &\quad + \left(\frac{r_\epsilon}{R} - r'_\epsilon\right)\Phi_1\left(r'_\epsilon, \frac{r_\epsilon}{R}, \dots, \frac{r_\epsilon}{R}\right) - \Phi(\lambda, \dots, \lambda)\Bigg)\Bigg]\Bigg|_{R=\epsilon}^{R=1} \\ &= \frac{1}{c^n}\left[\Phi(r'_\epsilon(1), \lambda, \dots, \lambda) + (\lambda - r'_\epsilon(1))\Phi_{,1}(r'_\epsilon(1), \lambda, \dots, \lambda) - \Phi(\lambda, \dots, \lambda)\right] \\ &\quad - \Bigg[\frac{1}{v^n}\Bigg(\Phi(g_\epsilon(v), v, \dots, v) - g_\epsilon(v)\Phi_{,1}(g_\epsilon(v), v, \dots, v) \\ &\quad - \Phi(\lambda, \dots, \lambda)\Bigg)\Bigg]\Bigg|_{v=\frac{c}{\epsilon}} - \tilde{T}(\epsilon), \qquad (6.5)\end{aligned}$$

where $\tilde{T}$ is given by (3.7) with $g = g_\epsilon$. The proof now proceeds in three steps.

Step 1. Since $\epsilon = o(c)$, we may assume without loss of generality, that $c \geq \lambda\epsilon$. We use the change of variables (see [1]) $\rho = R^n$ and set $u(\rho) = r_\epsilon^n\left(\rho^{\frac{1}{n}}\right)$ for $R \in [\epsilon, 1]$. Then

$$u'(\rho) = \frac{\mathrm{d}u}{\mathrm{d}\rho} = \left(\frac{\frac{d}{dR}r^n(R)}{\frac{d\rho}{dR}}\right)\Bigg|_{R=\rho^{\frac{1}{n}}} = \left(r'_\epsilon(R)\left(\frac{r_\epsilon(R)}{R}\right)^{n-1}\right)\Bigg|_{R=\rho^{\frac{1}{n}}},$$

and it follows from (2.4b) (see [21]) that $u''(\rho) \geq 0$. Hence

$$u(\epsilon) \geq u(1) + (\epsilon - 1)u'(1).$$

Thus,

$$\frac{c^n}{\lambda^{n-1}} \geq \lambda - r'_\epsilon(1) > 0,$$

from which it follows that $\lambda - r'_\epsilon(1) = O(c^n)$ as $c \searrow 0$ if $\epsilon = o(c)$. An exactly analogous argument to that used in Section 5 (using expression (5.3)) now shows that the first term in square brackets in expression (6.5) converges to zero as $c \to 0$.

Step 2. We next prove that the second term in square brackets in (6.5) also converges to zero as $c \to 0$. Let us recall that from (H5) and (3.8), we get that $\tilde{T}(v)$ is decreasing. Thus it follows that

$$P_2 \equiv \tilde{T}(c/\epsilon) \leq \tilde{T}(v) \leq \tilde{T}(\lambda), \quad v \in [\lambda, c/\epsilon).$$

Since $g_\epsilon(\lambda) < \lambda$, we get from (H1) that:

$$\tilde{T}(\lambda) = \frac{1}{\lambda^{n-1}}\Phi_{,1}(g_\epsilon(\lambda), \lambda, \dots, \lambda) < \frac{1}{\lambda^{n-1}}\Phi_{,1}(\lambda, \lambda, \dots, \lambda) \equiv P_1.$$

Hence

$$P_2 \leq \tilde{T}(v) \leq P_1, \quad v \in [\lambda, c/\epsilon).$$

This inequality together with (H1) again, imply that

$$\sigma_{P_2}(v) \leq g_\epsilon(v) \leq \sigma_{P_1}(v), \quad v \in [\lambda, c/\epsilon).$$

Since the function

$$H(s,t) = \Phi(s,t,\dots,t) - s\Phi_{,1}(s,t,\dots,t),$$

satisfies $\frac{\partial H(s,t)}{\partial s} = -s\Phi_{,11}(s,t,\dots,t) < 0$ for $s > 0$, the above inequality implies that

$$\frac{1}{v^n} H(\sigma_{P_1}(v), v) \le \frac{1}{v^n} H(g_\epsilon(v), v) \le \frac{1}{v^n} H(\sigma_{P_2}(v), v) \text{ for } v \in [\lambda, c/\epsilon).$$

Next note that the two outermost terms in the above inequality converge to zero as $v \to \infty$ (by (3.12)). Finally, the result follows since $\frac{c}{\epsilon} \to \infty$ as $c \to 0$ (since $\epsilon = o(c)$ by assumption).

Step 3. Finally, we prove that the remaining term in (6.5), namely $\tilde{T}(\epsilon)$ converges to (5.5) as $c \to 0$. This follows from an exactly analogous argument to that used in the proof of Lemma 5.1 on using the convergence result for punctured balls given in Theorem 6.2.

□

Remark 6.4. *The hypothesis that $\epsilon = o(c)$ in the last theorem guarantees that the tangential stretches on the deformed cavity surfaces of the punctured balls tend to infinity (which is the value of the tangential stretch for the cavitating solution on the cavity surface for the solid ball problem) as $\epsilon \to 0$.*

7. Numerical Results. In this section, we apply the approximation results of Section 6 and discuss some of the numerical aspects of computing the critical boundary displacement for cavitation using the volume derivative.

To numerically compute $G(\lambda)$, one has to first approximate the minimum energy

$$I^\varepsilon(r_\varepsilon) = \inf_{r \in \mathcal{A}^\epsilon_{\lambda,c}} I^\epsilon(r),$$

in equation (6.4) for a given small $c > 0$ and $\varepsilon = o(c)$. Next the difference quotient

$$\frac{I^\varepsilon(r_\varepsilon) - I^\varepsilon(r^h_\lambda)}{c^n/n}, \tag{7.1}$$

gives an approximation to $G(\lambda)$. For small c the computation of this difference quotient is susceptible to cancelation of significant digits. Thus, in actual computations, we do not make c extremely small to prevent such cancelations.

To approximate the minimum of

$$I^\varepsilon(r) = \int_\varepsilon^1 R^{n-1} \Phi\left(r'(R), \frac{r(R)}{R}, \cdots, \frac{r(R)}{R}\right) \mathrm{d}R, \tag{7.2}$$

subject to $r(\varepsilon) = c$, $r(1) = \lambda$, and $r'(R) \ge 0$ for $R \in (\varepsilon, 1)$, we use the following discretization. For $m \ge 1$, let $h = (1-\varepsilon)/m$ and $R_i = \varepsilon + ih$, $0 \le i \le m$. We discretize I^ε as follows

$$I^\varepsilon_h = h \sum_{i=1}^m \left(\frac{R_{i-1}+R_i}{2}\right)^{n-1} \Phi\left(\frac{r_i - r_{i-1}}{h}, \frac{r_{i-1}+r_i}{R_{i-1}+R_i}, \cdots, \frac{r_{i-1}+r_i}{R_{i-1}+R_i}\right),$$

subject to $r_0 = c$, $r_m = \lambda$, and $r_i \ge r_{i-1}$ for $i = 1, \dots, m$. Since the hessian matrix of I^ε_h is symmetric and tri–diagonal, the minimization process can be carried out very efficiently via Newton's method for large values of m.

TABLE 7.1

Approximations to λ_{crit} generated by computing for different values of c, a root of the difference quotient (7.1) in the volume derivative for the stored energy function in Example 7.1.

	$\varepsilon = c^2$			$\varepsilon = c^3$		
c	λ_c	e_c	q_c	λ_c	e_c	q_c
1.00E-01	1.0529	6.57E-03	2.03	1.0590	4.81E-04	3.52
5.00E-02	1.0562	3.24E-03	2.03	1.0593	1.37E-04	4.16
2.50E-02	1.0579	1.59E-03	2.05	1.0594	3.29E-05	5.64
1.25E-02	1.0587	7.74E-04	2.33	1.0595	5.83E-06	0.10
6.25E-03	1.0591	3.32E-04	6.84	1.0595	5.88E-05	0.24
3.13E-03	1.0595	4.85E-05	—	1.0597	2.45E-04	—

For the actual numerical computations, we used a stored energy function of the form (2.5) with $n = 3$:

$$\begin{aligned}\Phi(v_1, v_2, v_3) = {} & c_1\,(v_1^p + v_2^p + v_3^p) + c_2\left((v_1v_2)^\beta + (v_1v_3)^\beta + (v_2v_3)^\beta\right)\\ & +C(v_1v_2v_3)^\gamma + D(v_1v_2v_3)^{-\delta},\end{aligned} \tag{7.3}$$

where $p \in [1, 3)$, $c_1, c_2, C, D \geq 0$, $\beta, \gamma, \delta \geq 1$.

EXAMPLE 7.1. *We consider the special case of the function above in which $p = \beta = 1$. From Example 5.4 we have, with $h(d) = Cd^\gamma + Dd^{-\delta}$ in (5.8), that:*

$$G(\lambda) = \delta D\lambda^{-3(\delta+1)} - \gamma C\lambda^{3(\gamma-1)}. \tag{7.4}$$

Thus $G(\lambda) = 0$ for

$$\lambda_{crit} = \left[\frac{\delta D}{\gamma C}\right]^{\frac{1}{3(\gamma+\delta)}}.$$

For the values $\gamma = 3$, $\delta = 1$, $C = 1$, $D = 6$, $c_1 = c_2 = 1$, we obtain $\lambda_{crit} = 1.0595$, rounded to the indicated number of digits. In Table 7.1, λ_c denotes the computed value for a given c for which the difference quotient (7.1) is zero[3], $e_c = |\lambda_c - \lambda_{crit}|$ is the exact error in λ_c, and q_c is the quotient of successive errors, that is $q_c = e_c/e_{\frac{c}{2}}$. The table includes data for the cases $\varepsilon = c^2$ and $\varepsilon = c^3$. The computations were carried out using $m = 4000$ in (7.2) and the values of c are halved as one goes down in the table. The results show that the λ_c's are converging to λ_{crit}. From the computed difference quotients, one can deduce that the convergence is $O(c)$ for the choice $\varepsilon = c^2$, and $O(c^2)$ for the choice $\varepsilon = c^3$. Note however that these rates deteriorate as c gets small due to the loss of significant digits in computing the difference quotient (7.1). Finally, in Figure 7.1, we show a graph of the difference quotient (7.1) for $c = 3.125E - 3$. The graph of the difference quotient is essentially indistinguishable from that of (7.4) in this case.

EXAMPLE 7.2. *In Table 7.2 we show the computed values of λ_c for the stored energy function (7.3) with $c_1 = 1$, $c_2 = 0$, $C = 1$, $D = 2$, and $p = \gamma = \delta = 1.5$. Again we use $m = 4000$ in (7.2) and $\varepsilon = c^2$. Since for this example we do not have the exact value of λ_{crit} to compare our results with, we instead consider $e_c = \left|\lambda_c - \lambda_{\frac{c}{2}}\right|$, and $q_c = e_c/e_{\frac{c}{2}}$. Again the quotients q_c suggest a convergence rate of $O(c)$, with the quotients reducing as c becomes small due to loss of significant digits.*

[3] To compute λ_c, the difference quotient (7.1) is approximated for different values of λ, and then λ_c is an approximate root of the cubic spline interpolating this data.

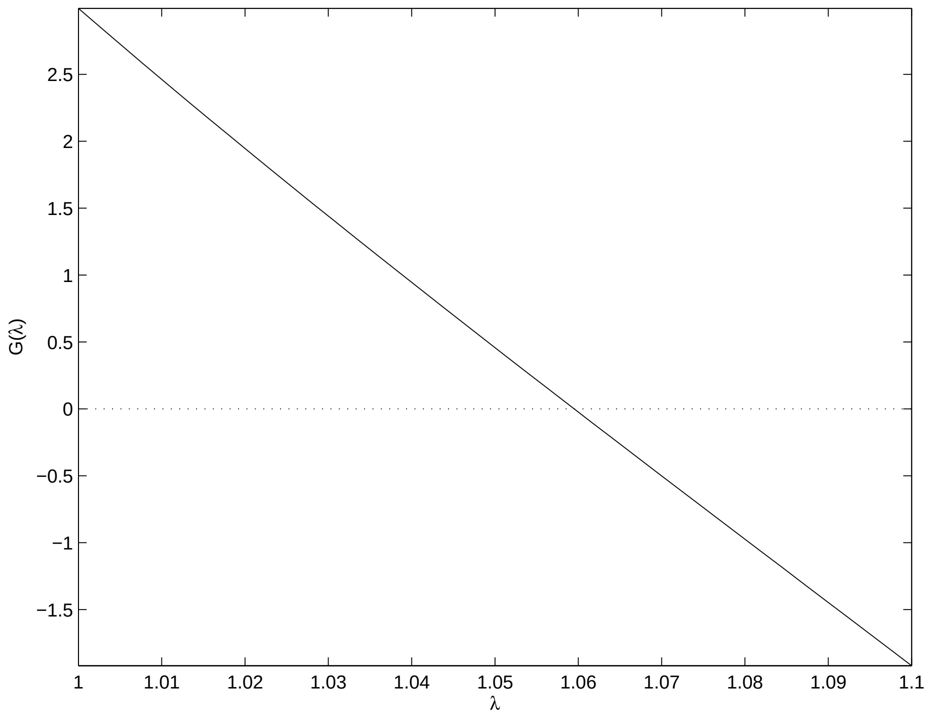

FIG. 7.1. *Graph of the difference quotient* (7.1) *for* $c = 3.125E{-}3$ *for the stored energy function in Example 7.1.*

TABLE 7.2
Approximations to λ_{crit} *for the stored energy function in Example 7.2.*

c	λ_c	e_c	q_c
1.00E-01	1.1308	6.99E-03	2.65
5.00E-02	1.1378	2.64E-03	2.60
2.50E-02	1.1404	1.02E-03	1.62
1.25E-02	1.1415	6.28E-04	0.53
6.25E-03	1.1421	1.19E-03	0.31
3.13E-03	1.1433	3.90E-03	—
1.56E-03	1.1472	—	—

EXAMPLE 7.3. *In this example we consider the stored energy function used in [15] to construct an explicit cavitating solution. The stored energy function is given by:*

$$\Phi(v_1, v_2, v_3) = \mu(v_1^2 + v_2^2 + v_3^2) + h(v_1 v_2 v_3), \tag{7.5}$$

where

$$h(d) = \mu\left(Cd^2 - 2(C+1)d + D\right), \quad d \geq 1,$$

and μ, C, D *are nonnegative constants. For* $d \in (0,1)$ *the function* $h(\cdot)$ *can be defined in any way that guarantees that* $h(\cdot)$ *is convex for* $d > 0$ *and satisfies* $h(d) \to \infty$ *as*

TABLE 7.3
Approximations to λ_{crit} for the stored energy function in Example 7.3.

c	λ_c
1.00E-01	1.000249924631959
5.00E-02	1.000104151191559
2.50E-02	1.000085929620128
1.25E-02	1.000083664002573
6.25E-03	1.000083428640763
3.13E-03	1.000083592724828

$d \to 0$. *For the values* $\mu = 0.25$, *and* $C = 10000$ *(D is chosen to make the minimum value of* $h(\cdot)$ *positive), it was reported in [15] that* λ_{crit} *is approximately 1.000083. We used* $m = 4000$ *in* (7.2) *and* $\varepsilon = 0$, *i.e., no pre–existing hole. In the table below we show the approximations to* λ_{crit} *via the volume derivative for different values of the cavity radius c. We see that the computed approximations* λ_c *are converging to the reported value of* λ_{crit}. *Again we mention that to compute* λ_c, *the difference quotient* (7.1) *is approximated for different values of* λ, *in this case for values of* λ *in* $[1, 1.001]$, *and then* λ_c *is an approximate root of the cubic spline interpolating this data.*

8. Concluding remarks. There are many alternative sets of hypotheses under which the results presented in this paper hold and we refer the interested reader, e.g., to the papers [1], [21], [10], [11] for examples of such alternatives.

We note that the volume derivative studied in the current paper is purely restricted to radial deformations and hence may not coincide with the general volume derivative as defined in [14] (the derivative in [14] is defined using the class of all, possibly non-symmetric, deformations and it is currently not known whether these energy minimisers coincide with radial minimisers in the compressible case[4]).

For the *radial problem*, a more efficient numerical method to approximate λ_{crit} than the one used in Section 7 is given in [13]. In that paper, the approach in Section 6, using regularised problems on punctured balls, is used as the basis for a very efficient numerical method for computing the critical boundary displacement λ_{crit}. Also, as previously noted, our expression (5.7) for the vanishing of the radial volume derivative, coincides with the condition for the critical boundary displacement for radial cavitation obtained, using a shooting argument, by Stuart in the interesting paper [21]. However, the methods in [13] and that in [21], do not generalise to non-symmetric problems. On the contrary, the approach of the current paper, identifying the critical boundary displacements as the zero set of the volume derivative provides a novel, and potentially very effective, criterion for the computation of the boundary of the set of linear displacement boundary conditions for which cavitation occurs in multi-dimensional, non-symmetric problems (see [14]).

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[4]See, however, the papers [18], [19], [20] in which symmetrisation arguments are used to prove that radial minimisers are minimising in the class of general non symmetric deformations.

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