

COURSE 1

MAY 2001

1. The price of an investment at the end of month n is modeled by $p_n = ne^{-bn}$ where b is a constant. The model predicts that the price at the end of the sixth month is the same as the price at the end of the fifth month.

Determine b .

(A) $\ln\left(\frac{5}{6}\right)$

(B) $\frac{5}{6}\ln\left(\frac{5}{6}\right)$

(C) $\frac{5}{6}\ln\left(\frac{6}{5}\right)$

(D) $\ln\left(\frac{7}{6}\right)$

(E) $\ln\left(\frac{6}{5}\right)$

2. A stock pays annual dividends. The first dividend is 8 and each dividend thereafter is 7% larger than the prior dividend.

Let m be the number of dividends paid by the stock when the cumulative amount paid first exceeds 500 .

Calculate m .

- (A) 23
- (B) 24
- (C) 25
- (D) 26
- (E) 27

3. The coordinates of an object moving in \mathbf{R}^2 are:

$$x = 4 \sin \frac{t}{2}$$

$$y = 2t \cos t$$

for time $t > 0$.

Calculate the length of the velocity vector of the object at time $t = \frac{\pi}{2}$.

- (A) $\sqrt{2}$
- (B) π
- (C) $\sqrt{\pi^2 + 2}$
- (D) $\sqrt{\pi^2 + 4}$
- (E) $\pi + \sqrt{2}$

4. A company agrees to accept the highest of four sealed bids on a property. The four bids are regarded as four independent random variables with common cumulative distribution function

$$F(x) = \frac{1}{2}(1 + \sin \pi x) \quad \text{for } \frac{3}{2} \leq x \leq \frac{5}{2}.$$

Which of the following represents the expected value of the accepted bid?

- (A) $\pi \int_{3/2}^{5/2} x \cos \pi x \, dx$
- (B) $\frac{1}{16} \int_{3/2}^{5/2} (1 + \sin \pi x)^4 \, dx$
- (C) $\frac{1}{16} \int_{3/2}^{5/2} x (1 + \sin \pi x)^4 \, dx$
- (D) $\frac{1}{4} \pi \int_{3/2}^{5/2} \cos \pi x (1 + \sin \pi x)^3 \, dx$
- (E) $\frac{1}{4} \pi \int_{3/2}^{5/2} x \cos \pi x (1 + \sin \pi x)^3 \, dx$

5. A company is reviewing tornado damage claims under a farm insurance policy. Let X be the portion of a claim representing damage to the house and let Y be the portion of the same claim representing damage to the rest of the property. The joint density function of X and Y is

$$f(x, y) = \begin{cases} 6[1 - (x + y)] & \text{for } x > 0, y > 0, x + y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the probability that the portion of a claim representing damage to the house is less than 0.2 .

- (A) 0.360
- (B) 0.480
- (C) 0.488
- (D) 0.512
- (E) 0.520

6. An insurance company issues life insurance policies in three separate categories: standard, preferred, and ultra-preferred. Of the company's policyholders, 50% are standard, 40% are preferred, and 10% are ultra-preferred. Each standard policyholder has probability 0.010 of dying in the next year, each preferred policyholder has probability 0.005 of dying in the next year, and each ultra-preferred policyholder has probability 0.001 of dying in the next year.

A policyholder dies in the next year.

What is the probability that the deceased policyholder was ultra-preferred?

- (A) 0.0001
- (B) 0.0010
- (C) 0.0071
- (D) 0.0141
- (E) 0.2817

7. A joint density function is given by

$$f(x, y) = \begin{cases} kx & \text{for } 0 < x < 1, \ 0 < y < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant.

What is $\text{Cov}(X, Y)$?

(A) $-\frac{1}{6}$

(B) 0

(C) $\frac{1}{9}$

(D) $\frac{1}{6}$

(E) $\frac{2}{3}$

8. The number of items produced by a manufacturer is given by $p = 100\sqrt{xy}$, where x is the amount of capital and y is the amount of labor. At a particular point in time:

- (i) the manufacturer has 2 units of capital;
- (ii) capital is increasing at a rate of 1 unit per month;
- (iii) the manufacturer has 3 units of labor; and
- (iv) labor is decreasing at a rate of 0.5 units per month.

Determine the rate of change in the number of items produced at the given time.

- (A) 41
- (B) 61
- (C) 82
- (D) 102
- (E) 245

9. An actuary is studying the prevalence of three health risk factors, denoted by A, B, and C, within a population of women. For each of the three factors, the probability is 0.1 that a woman in the population has only this risk factor (and no others). For any two of the three factors, the probability is 0.12 that she has exactly these two risk factors (but not the other). The probability that a woman has all three risk factors, given that she has A and B, is $\frac{1}{3}$.

What is the probability that a woman has none of the three risk factors, given that she does not have risk factor A ?

- (A) 0.280
- (B) 0.311
- (C) 0.467
- (D) 0.484
- (E) 0.700

10. Two life insurance policies, each with a death benefit of 10,000 and a one-time premium of 500, are sold to a couple, one for each person. The policies will expire at the end of the tenth year. The probability that only the wife will survive at least ten years is 0.025, the probability that only the husband will survive at least ten years is 0.01, and the probability that both of them will survive at least ten years is 0.96 .

What is the expected excess of premiums over claims, given that the husband survives at least ten years?

- (A) 350
- (B) 385
- (C) 397
- (D) 870
- (E) 897

11. An insurance company has 160,000 to spend on the development and marketing of a new insurance policy. If x is spent on development and y is spent on marketing,

$$\frac{x^{1/4}y^{3/4}}{1000}$$

policies will be sold during the first year.

Calculate the maximum possible number of policies the company can sell during the first year.

- (A) 40
- (B) 52
- (C) 80
- (D) 91
- (E) 120

12. You are given $P[A \cup B] = 0.7$ and $P[A \cup B'] = 0.9$.

Determine $P[A]$.

- (A) 0.2
- (B) 0.3
- (C) 0.4
- (D) 0.6
- (E) 0.8

13. A study is being conducted in which the health of two independent groups of ten policyholders is being monitored over a one-year period of time. Individual participants in the study drop out before the end of the study with probability 0.2 (independently of the other participants).

What is the probability that at least 9 participants complete the study in one of the two groups, but not in both groups?

- (A) 0.096
- (B) 0.192
- (C) 0.235
- (D) 0.376
- (E) 0.469

14. The stock prices of two companies at the end of any given year are modeled with random variables X and Y that follow a distribution with joint density function

$$f(x, y) = \begin{cases} 2x & \text{for } 0 < x < 1, \ x < y < x + 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the conditional variance of Y given that $X = x$?

- (A) $\frac{1}{12}$
- (B) $\frac{7}{6}$
- (C) $x + \frac{1}{2}$
- (D) $x^2 - \frac{1}{6}$
- (E) $x^2 + x + \frac{1}{3}$

15. Let C be the curve defined by:

$$x = 2t^2 + t - 1 \text{ and}$$

$$y = t^2 - 3t + 1$$

for $-\infty < t < \infty$.

What is the slope of the line tangent to C at $(0, 5)$?

(A) -5

(B) -1

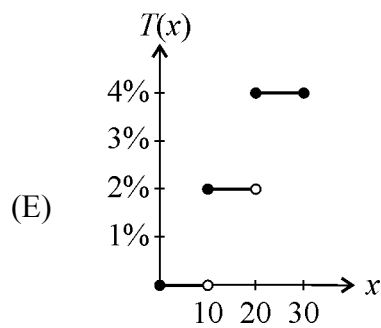
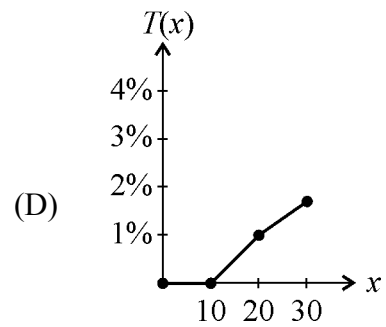
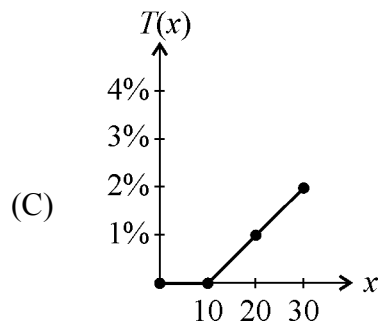
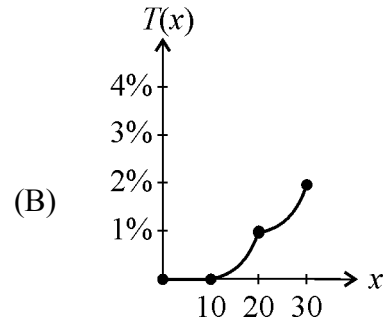
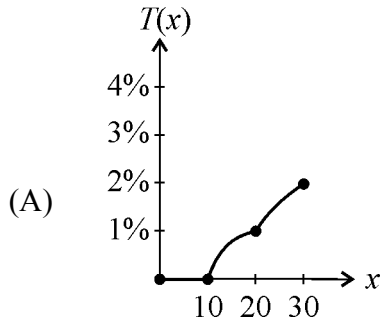
(C) $\frac{3}{5}$

(D) $\frac{5}{3}$

(E) 7

16. A certain state has an income tax rate of 0% on the first 10 of income, 2% on the next 10, and 4% on the excess over 20. Let $T(x)$ represent the ratio of total tax to total income x .

Which graph below best represents the graph of T for $0 \leq x \leq 30$?



17. An auto insurance company insures an automobile worth 15,000 for one year under a policy with a 1,000 deductible. During the policy year there is a 0.04 chance of partial damage to the car and a 0.02 chance of a total loss of the car. If there is partial damage to the car, the amount X of damage (in thousands) follows a distribution with density function

$$f(x) = \begin{cases} 0.5003 e^{-x/2} & \text{for } 0 < x < 15 \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected claim payment?

- (A) 320
- (B) 328
- (C) 352
- (D) 380
- (E) 540

18. The temperature of a particle located at the point (u, v) is $f(u, v) = e^{uv}$. The location is determined by two inputs x and y such that

$$\frac{\partial u}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = 2x$$

$$\frac{\partial v}{\partial x} = 2x$$

$$\frac{\partial v}{\partial y} = 2y.$$

Also, $(u, v) = (4, 5)$ when $(x, y) = (2, 1)$.

Calculate the rate of change of temperature as y changes, when $(x, y) = (2, 1)$.

- (A) $6e^{20}$
- (B) $12e^{20}$
- (C) $20e^{20}$
- (D) $28e^{20}$
- (E) $54e^{20}$

19. A company manufactures a brand of light bulb with a lifetime in months that is normally distributed with mean 3 and variance 1 . A consumer buys a number of these bulbs with the intention of replacing them successively as they burn out. The light bulbs have independent lifetimes.

What is the smallest number of bulbs to be purchased so that the succession of light bulbs produces light for at least 40 months with probability at least 0.9772 ?

- (A) 14
- (B) 16
- (C) 20
- (D) 40
- (E) 55

20. A device that continuously measures and records seismic activity is placed in a remote region. The time, T , to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$.

Determine $E[X]$.

- (A) $2 + \frac{1}{3}e^{-6}$
- (B) $2 - 2e^{-2/3} + 5e^{-4/3}$
- (C) 3
- (D) $2 + 3e^{-2/3}$
- (E) 5

21. The rate at which a disease spreads through a town can be modeled by the differential equation

$$\frac{dQ}{dt} = Q(N - Q)$$

where $Q(t)$ is the number of residents infected at time t and N is the total number of residents.

Which of the following is a solution for $Q(t)$?

- (A) ae^t where a is a constant
- (B) $\frac{aN e^t - 1}{ae^t}$ where a is a constant
- (C) $\frac{aN e^t + 1}{ae^t}$ where a is a constant
- (D) $\frac{aN e^{Nt}}{1 - ae^{Nt}}$ where a is a constant
- (E) $\frac{aN e^{Nt}}{1 + ae^{Nt}}$ where a is a constant

22. The waiting time for the first claim from a good driver and the waiting time for the first claim from a bad driver are independent and follow exponential distributions with means 6 years and 3 years, respectively.

What is the probability that the first claim from a good driver will be filed within 3 years and the first claim from a bad driver will be filed within 2 years?

- (A) $\frac{1}{18}(1 - e^{-2/3} - e^{-1/2} + e^{-7/6})$
- (B) $\frac{1}{18}e^{-7/6}$
- (C) $1 - e^{-2/3} - e^{-1/2} + e^{-7/6}$
- (D) $1 - e^{-2/3} - e^{-1/2} + e^{-1/3}$
- (E) $1 - \frac{1}{3}e^{-2/3} - \frac{1}{6}e^{-1/2} + \frac{1}{18}e^{-7/6}$

23. A hospital receives $\frac{1}{5}$ of its flu vaccine shipments from Company X and the remainder of its shipments from other companies. Each shipment contains a very large number of vaccine vials.

For Company X's shipments, 10% of the vials are ineffective. For every other company, 2% of the vials are ineffective. The hospital tests 30 randomly selected vials from a shipment and finds that one vial is ineffective.

What is the probability that this shipment came from Company X ?

- (A) 0.10
- (B) 0.14
- (C) 0.37
- (D) 0.63
- (E) 0.86

24. A device contains two components. The device fails if either component fails. The joint density function of the lifetimes of the components, measured in hours, is $f(s, t)$, where $0 < s < 1$ and $0 < t < 1$.

What is the probability that the device fails during the first half hour of operation?

- (A) $\int_0^{0.5} \int_0^{0.5} f(s, t) ds dt$
- (B) $\int_0^1 \int_0^{0.5} f(s, t) ds dt$
- (C) $\int_{0.5}^1 \int_{0.5}^1 f(s, t) ds dt$
- (D) $\int_0^{0.5} \int_0^1 f(s, t) ds dt + \int_0^1 \int_0^{0.5} f(s, t) ds dt$
- (E) $\int_0^{0.5} \int_{0.5}^1 f(s, t) ds dt + \int_0^1 \int_0^{0.5} f(s, t) ds dt$

25. The volume, V , and the surface area, S , of a spherical balloon with radius r are:

$$V = \frac{4}{3}\pi r^3$$
$$S = 4\pi r^2 .$$

The volume of the balloon increases at a rate of $60 \text{ cm}^3/\text{min}$ when the balloon's diameter is 6 cm.

How fast is the surface area of the balloon increasing when the balloon's diameter is 6 cm?

- (A) $20 \text{ cm}^2/\text{min}$
- (B) $40 \text{ cm}^2/\text{min}$
- (C) $80 \text{ cm}^2/\text{min}$
- (D) $113 \text{ cm}^2/\text{min}$
- (E) $120 \text{ cm}^2/\text{min}$

26. A company offers earthquake insurance. Annual premiums are modeled by an exponential random variable with mean 2. Annual claims are modeled by an exponential random variable with mean 1. Premiums and claims are independent. Let X denote the ratio of claims to premiums.

What is the density function of X ?

- (A) $\frac{1}{2x+1}$
- (B) $\frac{2}{(2x+1)^2}$
- (C) e^{-x}
- (D) $2e^{-2x}$
- (E) xe^{-x}

27. Claim amounts for wind damage to insured homes are independent random variables with common density function

$$f(x) = \begin{cases} \frac{3}{x^4} & \text{for } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

where x is the amount of a claim in thousands.

Suppose 3 such claims will be made.

What is the expected value of the largest of the three claims?

- (A) 2025
- (B) 2700
- (C) 3232
- (D) 3375
- (E) 4500

28. A function f is used to model the number of deaths each year due to a particular disease.

The function has the following properties:

$$\begin{aligned}f(k) &= \text{the number of deaths in year } k, \\f(t) &> 0 \text{ for } t \geq 0, \text{ and} \\f'(t) &< 0 \text{ for } t \geq 0.\end{aligned}$$

An actuary without access to a computer wants to estimate the number of deaths

$$N = \sum_{k=1}^{20} f(k)$$

from the disease in the first 20 years. The actuary requires that the

estimate be greater than or equal to N .

Among the following expressions, which is the best estimate that meets the actuary's requirement?

(A) $\int_0^{20} f(t) dt$

(B) $f(0) + \int_1^{20} f(t) dt$

(C) $f(1) + \int_1^{20} f(t) dt$

(D) $\int_1^{20} f(t) dt$

(E) $f(0) + f(1) + \int_2^{20} f(t) dt$

29. A large pool of adults earning their first driver's license includes 50% low-risk drivers, 30% moderate-risk drivers, and 20% high-risk drivers. Because these drivers have no prior driving record, an insurance company considers each driver to be randomly selected from the pool.

This month, the insurance company writes 4 new policies for adults earning their first driver's license.

What is the probability that these 4 will contain at least two more high-risk drivers than low-risk drivers?

- (A) 0.006
- (B) 0.012
- (C) 0.018
- (D) 0.049
- (E) 0.073

30. An ice cream vendor can sell 500 ice cream cones at a price of 2 per ice cream cone. For each 0.01 increase in the price per ice cream cone, the vendor will sell 5 fewer ice cream cones. For each 0.01 decrease in price, it will sell 5 more.

The vendor has fixed costs of 75 and variable costs of 0.10 per ice cream cone.

What price per ice cream cone should the vendor charge in order to maximize profit?

- (A) 1.50
- (B) 1.55
- (C) 2.00
- (D) 2.45
- (E) 2.50

31. An insurer offers a health plan to the employees of a large company. As part of this plan, the individual employees may choose exactly two of the supplementary coverages A, B, and C, or they may choose no supplementary coverage. The proportions of the company's employees that choose coverages A, B, and C are $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{5}{12}$, respectively.

Determine the probability that a randomly chosen employee will choose no supplementary coverage.

- (A) 0
- (B) $\frac{47}{144}$
- (C) $\frac{1}{2}$
- (D) $\frac{97}{144}$
- (E) $\frac{7}{9}$

32. A company has two electric generators. The time until failure for each generator follows an exponential distribution with mean 10. The company will begin using the second generator immediately after the first one fails.

What is the variance of the total time that the generators produce electricity?

- (A) 10
- (B) 20
- (C) 50
- (D) 100
- (E) 200

33. For Company A there is a 60% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 10,000 and standard deviation 2,000 .

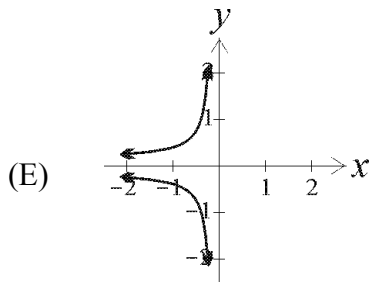
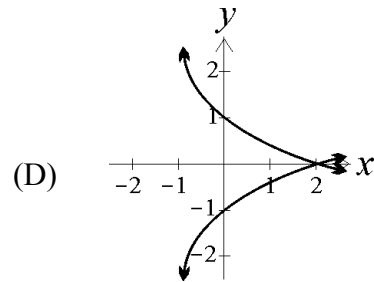
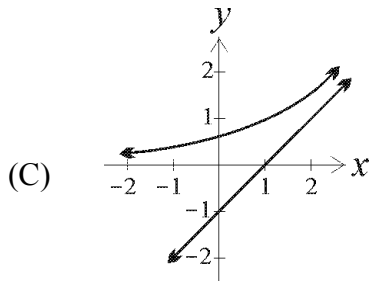
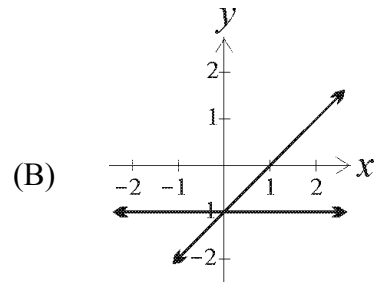
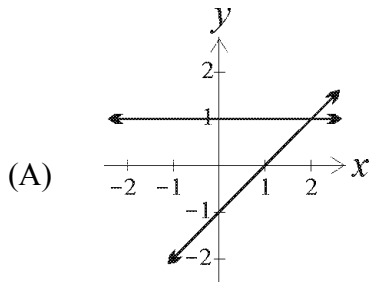
For Company B there is a 70% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 9,000 and standard deviation 2,000 .

Assume that the total claim amounts of the two companies are independent.

What is the probability that, in the coming year, Company B's total claim amount will exceed Company A's total claim amount?

- (A) 0.180
- (B) 0.185
- (C) 0.217
- (D) 0.223
- (E) 0.240

34. Which of the following represents the graph of a function and its derivative?



35. The warranty on a machine specifies that it will be replaced at failure or age 4, whichever occurs first. The machine's age at failure, X , has density function

$$f(x) = \begin{cases} \frac{1}{5} & \text{for } 0 < x < 5 \\ 0 & \text{otherwise.} \end{cases}$$

Let Y be the age of the machine at the time of replacement.

Determine the variance of Y .

- (A) 1.3
- (B) 1.4
- (C) 1.7
- (D) 2.1
- (E) 7.5

36. A town in the shape of a square with each side measuring 4 has an industrial plant at its center. The industrial plant is polluting the air such that the concentration of pollutants at each location (x,y) in the town can be modeled by the function

$$C(x,y) = 22,500(8 - x^2 - y^2) \text{ for } -2 \leq x \leq 2 \text{ and } -2 \leq y \leq 2 .$$

Calculate the average pollution concentration over the entire town.

- (A) 30,000
- (B) 120,000
- (C) 480,000
- (D) 1,920,000
- (E) 7,680,000

37. A tour operator has a bus that can accommodate 20 tourists. The operator knows that tourists may not show up, so he sells 21 tickets. The probability that an individual tourist will not show up is 0.02, independent of all other tourists.

Each ticket costs 50, and is non-refundable if a tourist fails to show up. If a tourist shows up and a seat is not available, the tour operator has to pay 100 (ticket cost + 50 penalty) to the tourist.

What is the expected revenue of the tour operator?

- (A) 935
- (B) 950
- (C) 967
- (D) 976
- (E) 985

38. Let f be a continuous function on $[0, \infty)$ such that $f(0) = 0$, $f(50) = 200$, and

$$f'(x) = \begin{cases} 4 & \text{for } 0 < x < 10 \\ k & \text{for } 10 < x < 30 \\ 3 & \text{for } x > 30, \end{cases}$$

where k is a constant.

Determine k .

- (A) $\frac{1}{3}$
- (B) 4
- (C) $\frac{9}{2}$
- (D) 5
- (E) 7

39. An insurance company insures a large number of homes. The insured value, X , of a randomly selected home is assumed to follow a distribution with density function

$$f(x) = \begin{cases} 3x^{-4} & \text{for } x > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Given that a randomly selected home is insured for at least 1.5, what is the probability that it is insured for less than 2 ?

- (A) 0.578
- (B) 0.684
- (C) 0.704
- (D) 0.829
- (E) 0.875

40. A public health researcher examines the medical records of a group of 937 men who died in 1999 and discovers that 210 of the men died from causes related to heart disease. Moreover, 312 of the 937 men had at least one parent who suffered from heart disease, and, of these 312 men, 102 died from causes related to heart disease.

Determine the probability that a man randomly selected from this group died of causes related to heart disease, given that neither of his parents suffered from heart disease.

- (A) 0.115
- (B) 0.173
- (C) 0.224
- (D) 0.327
- (E) 0.514

**Course 1
May 2001
Answer Key**

1	E	21	E
2	C	22	C
3	C	23	A
4	E	24	E
5	C	25	B
6	D	26	B
7	B	27	A
8	A	28	C
9	C	29	D
10	E	30	B
11	D	31	C
12	D	32	E
13	E	33	D
14	A	34	A
15	D	35	C
16	A	36	B
17	B	37	E
18	D	38	D
19	B	39	A
20	D	40	B

A	7
B	8
C	8
D	9
E	8

1.

E

We are given that

$$5e^{-5b} = p_5 = p_6 = 6e^{-6b}$$

It follows that

$$\frac{5}{6} = \frac{e^{-6b}}{e^{-5b}} = e^{-6b} e^{5b} = e^{-6b+5b} = e^{-b}$$

$$\ln\left(\frac{5}{6}\right) = -b$$

$$b = -\ln\left(\frac{5}{6}\right) = \ln\left(\frac{6}{5}\right)$$

2.

C

First, solve for m such that

$$500 = 8 + 8(1.07) + \dots + 8(1.07)^{m-1} = 8 \left[\frac{1 - (1.07)^m}{1 - 1.07} \right] = 8 \left[\frac{(1.07)^m - 1}{0.07} \right]$$

$$5.375 = (1.07)^m$$

$$\ln(5.375) = m \ln(1.07)$$

$$m = \frac{\ln(5.375)}{\ln(1.07)} = 24.86$$

We conclude that $m = 25$.

3.

C

Observe that

$$\frac{dx}{dt} = 2 \cos(t/2) \quad \text{and} \quad \frac{dy}{dt} = 2 \cos t - 2t \sin t$$

Therefore,

$$\left. \frac{dx}{dt} \right|_{t=\pi/2} = 2 \cos(\pi/4) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\left. \frac{dy}{dt} \right|_{t=\pi/2} = 2 \cos(\pi/2) - \pi \sin(\pi/2) = -\pi$$

It follows that the length of the velocity vector at time $t = \frac{\pi}{2}$ is given by

$$\sqrt{(\sqrt{2})^2 + (-\pi)^2} = \sqrt{\pi^2 + 2} .$$

4.

E

Let $X_1, X_2, X_3,$ and X_4 denote the four independent bids with common distribution function F . Then if we define $Y = \max(X_1, X_2, X_3, X_4)$, the distribution function G of Y is given by

$$\begin{aligned} G(y) &= \Pr[Y \leq y] \\ &= \Pr[(X_1 \leq y) \cap (X_2 \leq y) \cap (X_3 \leq y) \cap (X_4 \leq y)] \\ &= \Pr[X_1 \leq y] \Pr[X_2 \leq y] \Pr[X_3 \leq y] \Pr[X_4 \leq y] \\ &= [F(y)]^4 \\ &= \frac{1}{16}(1 + \sin \pi y)^4, \quad \frac{3}{2} \leq y \leq \frac{5}{2} \end{aligned}$$

It then follows that the density function g of Y is given by

$$\begin{aligned} g(y) &= G'(y) \\ &= \frac{1}{4}(1 + \sin \pi y)^3 (\pi \cos \pi y) \\ &= \frac{\pi}{4} \cos \pi y (1 + \sin \pi y)^3, \quad \frac{3}{2} \leq y \leq \frac{5}{2} \end{aligned}$$

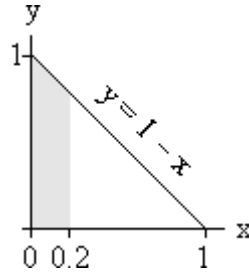
Finally,

$$\begin{aligned} E[Y] &= \int_{3/2}^{5/2} yg(y) dy \\ &= \int_{3/2}^{5/2} \frac{\pi}{4} y \cos \pi y (1 + \sin \pi y)^3 dy \end{aligned}$$

5.

C

The domain of X and Y is pictured below. The shaded region is the portion of the domain over which $X < 0.2$.



Now observe

$$\begin{aligned} \Pr[X < 0.2] &= \int_0^{0.2} \int_0^{1-x} 6[1-(x+y)] dy dx = 6 \int_0^{0.2} \left[y - xy - \frac{1}{2}y^2 \right]_0^{1-x} dx \\ &= 6 \int_0^{0.2} \left[1-x - x(1-x) - \frac{1}{2}(1-x)^2 \right] dx = 6 \int_0^{0.2} \left[(1-x)^2 - \frac{1}{2}(1-x)^2 \right] dx \\ &= 6 \int_0^{0.2} \frac{1}{2}(1-x)^2 dx = -(1-x)^3 \Big|_0^{0.2} = -(0.8)^3 + 1 \\ &= 0.488 \end{aligned}$$

6.

D

Let

S = Event of a standard policy

F = Event of a preferred policy

U = Event of an ultra-preferred policy

D = Event that a policyholder dies

Then

$$\begin{aligned} P[U|D] &= \frac{P[D|U]P[U]}{P[D|S]P[S] + P[D|F]P[F] + P[D|U]P[U]} \\ &= \frac{(0.001)(0.10)}{(0.01)(0.50) + (0.005)(0.40) + (0.001)(0.10)} \\ &= 0.0141 \end{aligned}$$

7. B

Let us first determine k :

$$1 = \int_0^1 \int_0^1 kx dx dy = \int_0^1 \frac{1}{2} kx^2 \Big|_0^1 dy = \int_0^1 \frac{k}{2} dy = \frac{k}{2}$$

$$k = 2$$

Then

$$E[X] = \int_0^1 \int_0^1 2x^2 dy dx = \int_0^1 2x^2 dx = \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3}$$

$$E[Y] = \int_0^1 \int_0^1 y \cdot 2x dx dy = \int_0^1 y dy = \frac{1}{2} y^2 \Big|_0^1 = \frac{1}{2}$$

$$\begin{aligned} E[XY] &= \int_0^1 \int_0^1 2x^2 y dx dy = \int_0^1 \frac{2}{3} x^3 y \Big|_0^1 dy = \int_0^1 \frac{2}{3} y dy \\ &= \frac{2}{6} y^2 \Big|_0^1 = \frac{2}{6} = \frac{1}{3} \end{aligned}$$

$$\text{Cov}[X, Y] = E[XY] - E[X]E[Y] = \frac{1}{3} - \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{3} - \frac{1}{3} = 0$$

(Alternative Solution)

Define $g(x) = kx$ and $h(y) = 1$. Then

$$f(x, y) = g(x)h(y)$$

In other words, $f(x, y)$ can be written as the product of a function of x alone and a function of y alone. It follows that X and Y are independent. Therefore, $\text{Cov}[X, Y] = 0$.

8. A

By the chain rule,

$$\frac{dp}{dt} = \frac{d}{dt} [100\sqrt{xy}] = 50x^{-\frac{1}{2}}y^{\frac{1}{2}} \frac{dx}{dt} + 50x^{\frac{1}{2}}y^{-\frac{1}{2}} \frac{dy}{dt}$$

At the time t_0 in question, we are told that

$$x = 2, \quad \frac{dx}{dt} = 1, \quad y = 3, \quad \text{and} \quad \frac{dy}{dt} = -\frac{1}{2}$$

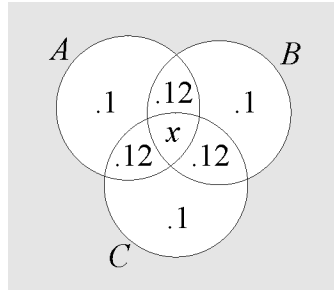
Therefore,

$$\left. \frac{dp}{dt} \right|_{t=t_0} = 50\sqrt{\frac{3}{2}}(1) + 50\sqrt{\frac{2}{3}}\left(-\frac{1}{2}\right) = 40.8$$

9.

C

The Venn diagram below summarizes the unconditional probabilities described in the problem.



In addition, we are told that

$$\frac{1}{3} = P[A \cap B \cap C | A \cap B] = \frac{P[A \cap B \cap C]}{P[A \cap B]} = \frac{x}{x + 0.12}$$

It follows that

$$x = \frac{1}{3}(x + 0.12) = \frac{1}{3}x + 0.04$$

$$\frac{2}{3}x = 0.04$$

$$x = 0.06$$

Now we want to find

$$\begin{aligned} P[(A \cup B \cup C)^c | A^c] &= \frac{P[(A \cup B \cup C)^c]}{P[A^c]} \\ &= \frac{1 - P[A \cup B \cup C]}{1 - P[A]} \\ &= \frac{1 - 3(0.10) - 3(0.12) - 0.06}{1 - 0.10 - 2(0.12) - 0.06} \\ &= \frac{0.28}{0.60} = 0.467 \end{aligned}$$

10.

E

Let

W = event that wife survives at least 10 years

H = event that husband survives at least 10 years

B = benefit paid

P = profit from selling policies

Then

$$\Pr[H] = P[H \cap W] + \Pr[H \cap W^c] = 0.96 + 0.01 = 0.97$$

and

$$\Pr[W | H] = \frac{\Pr[W \cap H]}{\Pr[H]} = \frac{0.96}{0.97} = 0.9897$$

$$\Pr[W^c | H] = \frac{\Pr[H \cap W^c]}{\Pr[H]} = \frac{0.01}{0.97} = 0.0103$$

It follows that

$$\begin{aligned} E[P] &= E[1000 - B] \\ &= 1000 - E[B] \\ &= 1000 - \{(0)\Pr[W | H] + (10,000)\Pr[W^c | H]\} \\ &= 1000 - 10,000(0.0103) \\ &= 1000 - 103 \\ &= 897 \end{aligned}$$

11. D

Observe that x and y follow the constraint equation

$$x + y = 160,000$$

$$x = 160,000 - y \quad \text{where} \quad 0 \leq y \leq 160,000$$

Now this constraint equation can be used to express policy sales $g(x, y)$ as a function $f(y)$ of marketing y alone:

$$f(y) = g(160,000 - y, y) = 0.001(160,000 - y)^{1/4} y^{3/4}$$

We can then compute $f'(y)$ as follows:

$$\begin{aligned} f'(y) &= \left\{ -\frac{1}{4}(160,000 - y)^{-3/4} y^{3/4} + \frac{3}{4}(160,000 - y)^{1/4} y^{-1/4} \right\} / 1000 \\ &= \frac{-1}{4000}(160,000 - y)^{-3/4} y^{3/4} + \frac{3}{4000}(160,000 - y)^{1/4} y^{-1/4} \\ &= \frac{-1}{4000}(160,000 - y)^{-3/4} y^{-1/4} [y - 3(160,000 - y)] \\ &= \frac{-1}{4000}(160,000 - y)^{-3/4} y^{-1/4} (4y - 480,000) \\ &= \frac{1}{1000}(160,000 - y)^{-3/4} y^{-1/4} (120,000 - y) \quad , \quad 0 \leq y \leq 160,000 \end{aligned}$$

Note that

$$f'(y) > 0 \quad \text{for} \quad 0 \leq y < 120,000 \quad ,$$

$$f'(y) = 0 \quad \text{for} \quad y = 120,000 \quad , \quad \text{and}$$

$$f'(y) < 0 \quad \text{for} \quad 120,000 < y < 160,000$$

We conclude that sales are maximized when $y = 120,000$. Therefore,

$$f(120,000) = 0.001(160,000 - 120,000)^{1/4} (120,000)^{3/4} = 91.2 \text{ maximizes } f.$$

11. Alternate solution using Lagrange multipliers:
Solve:

$$x + y - 160,000 = 0$$

$$\frac{\partial}{\partial x} \frac{x^{1/4} y^{3/4}}{1000} = \lambda \frac{\partial}{\partial x} (x + y - 160,000)$$

$$\frac{\partial}{\partial y} \frac{x^{1/4} y^{3/4}}{1000} = \lambda \frac{\partial}{\partial y} (x + y - 160,000)$$

From last two equations:

$$\frac{1}{4000} x^{-3/4} y^{3/4} = \lambda$$

$$\frac{3}{4000} x^{1/4} y^{-1/4} = \lambda$$

Eliminating λ :

$$3x^{1/4} y^{-1/4} = x^{-3/4} y^{3/4}$$

$$3x = y$$

Using first equation:

$$4x = 160,000$$

$$x = 40,000$$

$$y = 120,000$$

Extreme value (which must be a maximum) is $\frac{(40,000)^{1/4} (120,000)^{3/4}}{1000} = 91.2$

12. D

First note

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

$$P[A \cup B'] = P[A] + P[B'] - P[A \cap B']$$

Then add these two equations to get

$$P[A \cup B] + P[A \cup B'] = 2P[A] + (P[B] + P[B']) - (P[A \cap B] + P[A \cap B'])$$

$$0.7 + 0.9 = 2P[A] + 1 - P[(A \cap B) \cup (A \cap B')]$$

$$1.6 = 2P[A] + 1 - P[A]$$

$$P[A] = 0.6$$

13. E
Let

X = number of group 1 participants that complete the study.

Y = number of group 2 participants that complete the study.

Now we are given that X and Y are independent.

Therefore,

$$\begin{aligned}
 & P\{[(X \geq 9) \cap (Y < 9)] \cup [(X < 9) \cap (Y \geq 9)]\} \\
 &= P[(X \geq 9) \cap (Y < 9)] + P[(X < 9) \cap (Y \geq 9)] \\
 &= 2P[(X \geq 9) \cap (Y < 9)] \quad (\text{due to symmetry}) \\
 &= 2P[X \geq 9]P[Y < 9] \\
 &= 2P[X \geq 9]P[X < 9] \quad (\text{again due to symmetry}) \\
 &= 2P[X \geq 9](1 - P[X \geq 9]) \\
 &= 2\left[\binom{10}{9}(0.2)(0.8)^9 + \binom{10}{10}(0.8)^{10}\right] \left[1 - \binom{10}{9}(0.2)(0.8)^9 - \binom{10}{10}(0.8)^{10}\right] \\
 &= 2[0.376][1 - 0.376] = 0.469
 \end{aligned}$$

14. A

Let $f_1(x)$ denote the marginal density function of X . Then

$$f_1(x) = \int_x^{x+1} 2xy \, dy = 2xy \Big|_x^{x+1} = 2x(x+1-x) = 2x \quad , \quad 0 < x < 1$$

Consequently,

$$f(y|x) = \frac{f(x,y)}{f_1(x)} = \begin{cases} 1 & \text{if: } x < y < x+1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y|X] = \int_x^{x+1} y \, dy = \frac{1}{2}y^2 \Big|_x^{x+1} = \frac{1}{2}(x+1)^2 - \frac{1}{2}x^2 = \frac{1}{2}x^2 + x + \frac{1}{2} - \frac{1}{2}x^2 = x + \frac{1}{2}$$

$$\begin{aligned}
 E[Y^2|X] &= \int_x^{x+1} y^2 \, dy = \frac{1}{3}y^3 \Big|_x^{x+1} = \frac{1}{3}(x+1)^3 - \frac{1}{3}x^3 \\
 &= \frac{1}{3}x^3 + x^2 + x + \frac{1}{3} - \frac{1}{3}x^3 = x^2 + x + \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}[Y|X] &= E[Y^2|X] - \{E[Y|X]\}^2 = x^2 + x + \frac{1}{3} - \left(x + \frac{1}{2}\right)^2 \\
 &= x^2 + x + \frac{1}{3} - x^2 - x - \frac{1}{4} = \frac{1}{12}
 \end{aligned}$$

15. D

$$\text{At the point } (0, 5), \quad 0 = 2t^2 + t - 1 = (2t - 1)(t + 1)$$

$$\text{and} \quad 5 = t^2 - 3t + 1$$

The first equation says $t = \frac{1}{2}$ or $t = -1$ and the second says $t = -1$.

The slope of the tangent line to C at (0, 5) is then

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{(x,y)=(0,5)} &= \left. \frac{dy}{dt} \right|_{t=-1} / \left. \frac{dx}{dt} \right|_{t=-1} = \left. \frac{2t-3}{4t+1} \right|_{t=-1} \\ &= [2(-1)-3] / [4(-1)+1] = \frac{5}{3} \end{aligned}$$

16. A

We are given that

$$T(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq 10 \\ \frac{0.02(x-10)}{x} & \text{for } 10 < x \leq 20 \\ \frac{0.04(x-20)+0.02(10)}{x} & \text{for } x > 20 \end{cases} \quad T(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq 10 \\ 0.02 - \frac{1}{5x} & \text{for } 10 < x \leq 20 \\ 0.04 - \frac{3}{5x} & \text{for } x > 20 \end{cases}$$

$$\text{Therefore, } T'(x) = \begin{cases} 0 & \text{for } 0 < x < 10 \\ \frac{1}{5x^2} & \text{for } 10 < x < 20 \\ \frac{3}{5x^2} & \text{for } x > 20 \end{cases} \quad \text{and} \quad T''(x) = \begin{cases} 0 & \text{for } 0 < x < 10 \\ -\frac{2}{5x^3} & \text{for } 10 < x < 20 \\ -\frac{6}{5x^3} & \text{for } x > 20 \end{cases}$$

We can infer the following about $T(x)$:

i) $T(x) = 0$ for $0 < x \leq 10$

ii) $T(x)$ is strictly increasing for $10 < x < 20$

and $x > 20$ since $T'(x) > 0$ on both of these intervals.

iii) $T(x)$ is concave down for $10 < x < 20$

and $x > 20$ since $T''(x) < 0$ on both of these intervals.

It follows that (A) is the only graph that satisfies conditions (i)-(iii).

17.

B

Let Y denote the claim payment made by the insurance company.

Then

$$Y = \begin{cases} 0 & \text{with probability } 0.94 \\ \text{Max } (0, x - 1) & \text{with probability } 0.04 \\ 14 & \text{with probability } 0.02 \end{cases}$$

and

$$\begin{aligned} E[Y] &= (0.94)(0) + (0.04)(0.5003) \int_1^{15} (x-1)e^{-x/2} dx + (0.02)(14) \\ &= (0.020012) \left[\int_1^{15} xe^{-x/2} dx - \int_1^{15} e^{-x/2} dx \right] + 0.28 \\ &= 0.28 + (0.020012) \left[-2xe^{-x/2} \Big|_1^{15} + 2 \int_1^{15} e^{-x/2} dx - \int_1^{15} e^{-x/2} dx \right] \\ &= 0.28 + (0.020012) \left[-30e^{-7.5} + 2e^{-0.5} + \int_1^{15} e^{-x/2} dx \right] \\ &= 0.28 + (0.020012) \left[-30e^{-7.5} + 2e^{-0.5} - 2e^{-x/2} \Big|_1^{15} \right] \\ &= 0.28 + (0.020012) (-30e^{-7.5} + 2e^{-0.5} - 2e^{-7.5} + 2e^{-0.5}) \\ &= 0.28 + (0.020012) (-32e^{-7.5} + 4e^{-0.5}) \\ &= 0.28 + (0.020012)(2.408) \\ &= 0.328 \quad (\text{in thousands}) \end{aligned}$$

It follows that the expected claim payment is 328 .

18.

D

By the chain rule,

$$\frac{\partial f}{\partial y} = ve^{uv} \frac{\partial u}{\partial y} + ue^{uv} \frac{\partial v}{\partial y} = ve^{uv} 2x + ue^{uv} 2y$$

Therefore,

$$\left. \frac{\partial f}{\partial y} \right|_{(x,y)=(2,1)} = 5e^{(4)(5)} (2)(2) + 4e^{(4)(5)} (2)(1) = 28e^{20}$$

19. B

Let X_1, \dots, X_n denote the life spans of the n light bulbs purchased. Since these random variables are independent and normally distributed with mean 3 and variance 1, the random variable $S = X_1 + \dots + X_n$ is also normally distributed with mean

$$\mu = 3n$$

and standard deviation

$$\sigma = \sqrt{n}$$

Now we want to choose the smallest value for n such that

$$0.9772 \leq \Pr[S > 40] = \Pr\left[\frac{S - 3n}{\sqrt{n}} > \frac{40 - 3n}{\sqrt{n}}\right]$$

This implies that n should satisfy the following inequality:

$$-2 \geq \frac{40 - 3n}{\sqrt{n}}$$

To find such an n , let's solve the corresponding equation for n :

$$-2 = \frac{40 - 3n}{\sqrt{n}}$$

$$-2\sqrt{n} = 40 - 3n$$

$$3n - 2\sqrt{n} - 40 = 0$$

$$(3\sqrt{n} + 10)(\sqrt{n} - 4) = 0$$

$$\sqrt{n} = 4$$

$$n = 16$$

20. D

The density function of T is

$$f(t) = \frac{1}{3}e^{-t/3}, \quad 0 < t < \infty$$

Therefore,

$$\begin{aligned} E[X] &= E[\max(T, 2)] \\ &= \int_0^2 \frac{2}{3}e^{-t/3} dt + \int_2^\infty \frac{t}{3}e^{-t/3} dt \\ &= -2e^{-t/3} \Big|_0^2 - te^{-t/3} \Big|_2^\infty + \int_2^\infty e^{-t/3} dt \\ &= -2e^{-2/3} + 2 + 2e^{-2/3} - 3e^{-t/3} \Big|_2^\infty \\ &= 2 + 3e^{-2/3} \end{aligned}$$

21.

E

The differential equation that we are given is separable. As a result, the general solution is given by

$$\int \frac{1}{Q(N-Q)} dQ = \int dt = t + C$$

where C is a constant. Now in order to calculate the integral on the lefthand side of this equation, we first need to determine the partial fractions of the integrand. In other words, we need to find constants A and B such that

$$\frac{1}{Q(N-Q)} = \frac{A}{Q} + \frac{B}{N-Q}$$

$$1 = A(N-Q) + BQ$$

$$1 = AN + (B-A)Q$$

It follows that

$$AN = 1$$

$$B - A = 0$$

$$B = A = \frac{1}{N}$$

so $\frac{1}{Q(N-Q)} = \frac{1}{NQ} + \frac{1}{N(N-Q)}$ and

$$\int \frac{1}{Q(N-Q)} dQ = \frac{1}{N} \int \frac{1}{Q} dQ + \frac{1}{N} \int \frac{1}{N-Q} dQ = \frac{1}{N} \ln Q - \frac{1}{N} \ln(N-Q) + K = \frac{1}{N} \ln \left[\frac{Q}{N-Q} \right] + K$$

where K is a constant. Consequently,

$$\frac{1}{N} \ln \left[\frac{Q}{N-Q} \right] + K = t + C$$

$$\left(\frac{Q}{N-Q} \right)^{1/N} e^K = e^t e^C$$

$$\left(\frac{Q}{N-Q} \right)^{1/N} = e^t e^{C-K}$$

$$\frac{Q}{N-Q} = e^{Nt} e^{N(C-K)}$$

$$Q = ae^{Nt} (N-Q) = aNe^{Nt} - ae^{Nt} Q \quad \text{where } a = e^{N(C-K)} \text{ is a constant}$$

$$(1 + ae^{Nt})Q = aNe^{Nt}$$

$$Q(t) = \frac{aNe^{Nt}}{1 + ae^{Nt}}$$

22. C

Let X denote the waiting time for a first claim from a good driver, and let Y denote the waiting time for a first claim from a bad driver. The problem statement implies that the respective distribution functions for X and Y are

$$F(x) = 1 - e^{-x/6}, \quad x > 0 \quad \text{and}$$

$$G(y) = 1 - e^{-y/3}, \quad y > 0$$

Therefore,

$$\begin{aligned} \Pr[(X \leq 3) \cap (Y \leq 2)] &= \Pr[X \leq 3] \Pr[Y \leq 2] \\ &= F(3)G(2) \\ &= (1 - e^{-1/2})(1 - e^{-2/3}) \\ &= 1 - e^{-2/3} - e^{-1/2} + e^{-7/6} \end{aligned}$$

23. A

Let

C = Event that shipment came from Company X

I_1 = Event that one of the vaccine vials tested is ineffective

$$\text{Then by Bayes' Formula, } P[C | I_1] = \frac{P[I_1 | C]P[C]}{P[I_1 | C]P[C] + P[I_1 | C^c]P[C^c]}$$

Now

$$P[C] = \frac{1}{5}$$

$$P[C^c] = 1 - P[C] = 1 - \frac{1}{5} = \frac{4}{5}$$

$$P[I_1 | C] = \binom{30}{1}(0.10)(0.90)^{29} = 0.141$$

$$P[I_1 | C^c] = \binom{30}{1}(0.02)(0.98)^{29} = 0.334$$

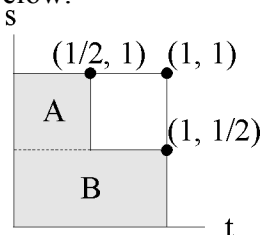
Therefore,

$$P[C | I_1] = \frac{(0.141)(1/5)}{(0.141)(1/5) + (0.334)(4/5)} = 0.096$$

24.

E

The domain of s and t is pictured below.



Note that the shaded region is the portion of the domain of s and t over which the device fails sometime during the first half hour. Therefore,

$$\Pr\left[\left(S \leq \frac{1}{2}\right) \cup \left(T \leq \frac{1}{2}\right)\right] = \int_0^{1/2} \int_{1/2}^1 f(s, t) ds dt + \int_0^1 \int_0^{1/2} f(s, t) ds dt$$

(where the first integral covers A and the second integral covers B).

25.

B

Note that V , S and r are all functions of time t . Therefore,

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

and

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

We are given that

$$\frac{dV}{dt} = 60 \quad \text{when} \quad r = \frac{6}{2} = 3.$$

It follows that

$$60 = 4\pi (3)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{5}{3\pi}$$

$$\frac{dS}{dt} = 8\pi (3) \left(\frac{5}{3\pi}\right) = 40$$

26. B
Let

u be annual claims,
 v be annual premiums,
 $g(u, v)$ be the joint density function of U and V ,
 $f(x)$ be the density function of X , and
 $F(x)$ be the distribution function of X .

Then since U and V are independent,

$$g(u, v) = (e^{-u}) \left(\frac{1}{2} e^{-v/2} \right) = \frac{1}{2} e^{-u} e^{-v/2}, \quad 0 < u < \infty, \quad 0 < v < \infty$$

and

$$\begin{aligned} F(x) &= \Pr[X \leq x] = \Pr\left[\frac{u}{v} \leq x\right] = \Pr[U \leq Vx] \\ &= \int_0^\infty \int_0^{vx} g(u, v) du dv = \int_0^\infty \int_0^{vx} \frac{1}{2} e^{-u} e^{-v/2} du dv \\ &= \int_0^\infty \left. -\frac{1}{2} e^{-u} e^{-v/2} \right|_0^{vx} dv = \int_0^\infty \left(-\frac{1}{2} e^{-vx} e^{-v/2} + \frac{1}{2} e^{-v/2} \right) dv \\ &= \int_0^\infty \left(-\frac{1}{2} e^{-v(x+1/2)} + \frac{1}{2} e^{-v/2} \right) dv \\ &= \left[\frac{1}{2x+1} e^{-v(x+1/2)} - e^{-v/2} \right]_0^\infty \\ &= -\frac{1}{2x+1} + 1 \end{aligned}$$

Finally,

$$f(x) = F'(x) = \frac{2}{(2x+1)^2}$$

27. A

First, observe that the distribution function of X is given by

$$F(x) = \int_1^x \frac{3}{t^4} dt = -\frac{1}{t^3} \Big|_1^x = 1 - \frac{1}{x^3}, \quad x > 1$$

Next, let $X_1, X_2,$ and X_3 denote the three claims made that have this distribution. Then if Y denotes the largest of these three claims, it follows that the distribution function of Y is given by

$$\begin{aligned} G(y) &= \Pr[X_1 \leq y] \Pr[X_2 \leq y] \Pr[X_3 \leq y] \\ &= \left(1 - \frac{1}{y^3}\right)^3, \quad y > 1 \end{aligned}$$

while the density function of Y is given by

$$g(y) = G'(y) = 3 \left(1 - \frac{1}{y^3}\right)^2 \left(\frac{3}{y^4}\right) = \left(\frac{9}{y^4}\right) \left(1 - \frac{1}{y^3}\right)^2, \quad y > 1$$

Therefore,

$$\begin{aligned} E[Y] &= \int_1^\infty \frac{9}{y^3} \left(1 - \frac{1}{y^3}\right)^2 dy = \int_1^\infty \frac{9}{y^3} \left(1 - \frac{2}{y^3} + \frac{1}{y^6}\right) dy \\ &= \int_1^\infty \left(\frac{9}{y^3} - \frac{18}{y^6} + \frac{9}{y^9}\right) dy = \left[-\frac{9}{2y^2} + \frac{18}{5y^5} - \frac{9}{8y^8}\right]_1^\infty \\ &= 9 \left[\frac{1}{2} - \frac{2}{5} + \frac{1}{8}\right] = 2.025 \text{ (in thousands)} \end{aligned}$$

28. C

Since $f(t) > 0$ and $f'(t) < 0$ for $t \geq 0$, the following inequalities hold:

$$(i) f(t_0) > f(t_1) \quad \text{if } 0 \leq t_0 < t_1$$

$$(ii) f(k) < \int_{k-1}^k f(t) dt \quad \text{if } k \geq 1$$

$$(iii) f(k) > \int_k^{k+1} f(t) dt \quad \text{if } k \geq 0$$

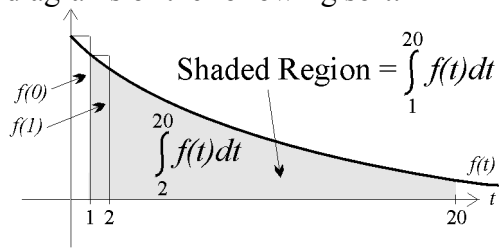
Applying these inequalities, we see that

$$\begin{aligned} \boxed{f(0) + f(1) + \int_2^{20} f(t) dt} &> f(0) + \int_1^2 f(t) dt + \int_2^{20} f(t) dt \\ &= \boxed{f(0) + \int_1^{20} f(t) dt} > \int_0^1 f(t) dt + \int_1^{20} f(t) dt \\ &= \boxed{\int_0^{20} f(t) dt} = \int_0^1 f(t) dt + \int_1^{20} f(t) dt \\ &> \boxed{f(1) + \int_1^{20} f(t) dt} = f(1) + \sum_{k=1}^{19} \int_k^{k+1} f(t) dt \\ &> f(1) + \sum_{k=1}^{19} f(k+1) = f(1) + \sum_{k=2}^{20} f(k) \\ &= \boxed{\sum_{k=1}^{20} f(k)} > \sum_{k=1}^{20} \int_k^{k+1} f(t) dt = \int_1^{21} f(t) dt \\ &> \boxed{\int_1^{20} f(t) dt} \end{aligned}$$

We conclude that $f(1) + \int_1^{20} f(t) dt$ produces the smallest number that exceeds

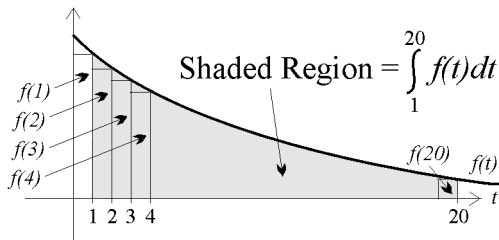
$$N = \sum_{k=1}^{20} f(k).$$

28. Note a more heuristic approach to the result that $(E) > (B) > (A) > (C) > \sum_{k=1}^{20} f(k) > (D)$ can be obtained from diagrams of the following sort:



gives $(E) = f(0) + f(1) + \int_2^{20} f(t) dt > f(0) + \int_1^{20} f(t) dt = (B)$

and



gives $f(1) + \int_1^{20} f(t) dt > \sum_{k=1}^{20} f(k)$

29. D
Let

X = number of low-risk drivers insured

Y = number of moderate-risk drivers insured

Z = number of high-risk drivers insured

$f(x, y, z)$ = probability function of X , Y , and Z

Then f is a trinomial probability function, so

$$\begin{aligned} \Pr[z \geq x + 2] &= f(0, 0, 4) + f(1, 0, 3) + f(0, 1, 3) + f(0, 2, 2) \\ &= (0.20)^4 + 4(0.50)(0.20)^3 + 4(0.30)(0.20)^3 + \frac{4!}{2!2!} (0.30)^2 (0.20)^2 \\ &= 0.0488 \end{aligned}$$

30. B
Let

x = number of ice cream cones sold

$p(x)$ = price of x ice cream cones

$C(x)$ = cost of selling x ice cream cones

$R(x)$ = revenue from selling x ice cream cones

$P(x)$ = profit from selling x ice cream cones

We are told that $p(x)$ satisfies the following relationship:

$$x = 500 - 5 \left[\frac{p(x) - 2}{0.01} \right] = 500 - 500p(x) + 1000 = 1500 - 500p(x)$$

$$500p(x) = 1500 - x$$

$$p(x) = 3 - \frac{x}{500}$$

Therefore,

$$R(x) = xp(x) = 3x - \frac{x^2}{500}$$

$$C(x) = 0.10x + 75$$

$$P(x) = R(x) - C(x) = 3x - \frac{x^2}{500} - 0.10x - 75 = 2.9x - \frac{x^2}{500} - 75$$

Now, since $P(x)$ is quadratic, it is clear that $P(x)$ will be maximized for x such that

$$0 = P'(x) = 2.9 - \frac{x}{250}$$

$$\frac{x}{250} = 2.9$$

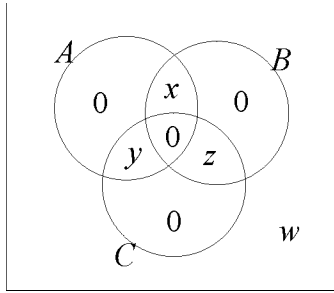
$$x = 725$$

The profit maximizing price is thus

$$p(725) = 3 - \frac{725}{500} = 1.55$$

31. C

A Venn diagram for this situation looks like:



We want to find $w = 1 - (x + y + z)$

We have $x + y = \frac{1}{4}$, $x + z = \frac{1}{3}$, $y + z = \frac{5}{12}$

Adding these three equations gives

$$(x + y) + (x + z) + (y + z) = \frac{1}{4} + \frac{1}{3} + \frac{5}{12}$$

$$2(x + y + z) = 1$$

$$x + y + z = \frac{1}{2}$$

$$w = 1 - (x + y + z) = 1 - \frac{1}{2} = \frac{1}{2}$$

Alternatively the three equations can be solved to give $x = 1/12$, $y = 1/6$, $z = 1/4$

again leading to $w = 1 - \left(\frac{1}{12} + \frac{1}{6} + \frac{1}{4}\right) = \frac{1}{2}$

32. E

Let X and Y denote the times that the two backup generators can operate. Now the variance of an exponential random variable with mean β is β^2 . Therefore,

$$\text{Var}[X] = \text{Var}[Y] = 10^2 = 100$$

Then assuming that X and Y are independent, we see

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] = 100 + 100 = 200$$

33. D
Let

I_A = Event that Company A makes a claim
 I_B = Event that Company B makes a claim
 X_A = Expense paid to Company A if claims are made
 X_B = Expense paid to Company B if claims are made

Then we want to find

$$\begin{aligned} & \Pr\left\{\left[I_A^C \cap I_B\right] \cup \left[\left(I_A \cap I_B\right) \cap \left(X_A < X_B\right)\right]\right\} \\ &= \Pr\left[I_A^C \cap I_B\right] + \Pr\left[\left(I_A \cap I_B\right) \cap \left(X_A < X_B\right)\right] \\ &= \Pr\left[I_A^C\right] \Pr\left[I_B\right] + \Pr\left[I_A\right] \Pr\left[I_B\right] \Pr\left[X_A < X_B\right] \quad (\text{independence}) \\ &= (0.60)(0.30) + (0.40)(0.30) \Pr\left[X_B - X_A \geq 0\right] \\ &= 0.18 + 0.12 \Pr\left[X_B - X_A \geq 0\right] \end{aligned}$$

Now $X_B - X_A$ is a linear combination of independent normal random variables.

Therefore, $X_B - X_A$ is also a normal random variable with mean

$$M = E\left[X_B - X_A\right] = E\left[X_B\right] - E\left[X_A\right] = 9,000 - 10,000 = -1,000$$

and standard deviation $\sigma = \sqrt{\text{Var}\left(X_B\right) + \text{Var}\left(X_A\right)} = \sqrt{\left(2000\right)^2 + \left(2000\right)^2} = 2000\sqrt{2}$

It follows that

$$\begin{aligned} \Pr\left[X_B - X_A \geq 0\right] &= \Pr\left[Z \geq \frac{1000}{2000\sqrt{2}}\right] \quad (Z \text{ is standard normal}) \\ &= \Pr\left[Z \geq \frac{1}{2\sqrt{2}}\right] \\ &= 1 - \Pr\left[Z < \frac{1}{2\sqrt{2}}\right] \\ &= 1 - \Pr\left[Z < 0.354\right] \\ &= 1 - 0.638 = 0.362 \end{aligned}$$

Finally,

$$\begin{aligned} \Pr\left\{\left[I_A^C \cap I_B\right] \cup \left[\left(I_A \cap I_B\right) \cap \left(X_A < X_B\right)\right]\right\} &= 0.18 + (0.12)(0.362) \\ &= 0.223 \end{aligned}$$

34. A

The graph (A) contains the curves $y = x - 1$ and $y = 1 = \frac{d}{dx}[x - 1]$.

(Note graph (D) can be eliminated because both curves have non-zero slopes where the other crosses the x -axis.)

35. C

$$\text{Note } Y = \begin{cases} X & \text{if } 0 \leq X \leq 4 \\ 4 & \text{if } 4 < X \leq 5 \end{cases}$$

Therefore,

$$\begin{aligned} E[Y] &= \int_0^4 \frac{1}{5} x dx + \int_4^5 \frac{4}{5} dx = \frac{1}{10} x^2 \Big|_0^4 + \frac{4}{5} x \Big|_4^5 \\ &= \frac{16}{10} + \frac{20}{5} - \frac{16}{5} = \frac{8}{5} + \frac{4}{5} = \frac{12}{5} \end{aligned}$$

$$\begin{aligned} E[Y^2] &= \int_0^4 \frac{1}{5} x^2 dx + \int_4^5 \frac{16}{5} dx = \frac{1}{15} x^3 \Big|_0^4 + \frac{16}{5} x \Big|_4^5 \\ &= \frac{64}{15} + \frac{80}{5} - \frac{64}{5} = \frac{64}{15} + \frac{16}{5} = \frac{64}{15} + \frac{48}{15} = \frac{112}{15} \end{aligned}$$

$$\text{Var}[Y] = E[Y^2] - (E[Y])^2 = \frac{112}{15} - \left(\frac{12}{5}\right)^2 = 1.71$$

36. B

Let T denote the total concentration of pollutants over the town. Then due to symmetry,

$$\begin{aligned} T &= 4 \int_0^2 \int_0^2 22,500(8 - x^2 - y^2) dx dy \\ &= (4)(7500) \int_0^2 [24x - x^3 - 3xy^2]_0^2 dy \\ &= 30,000 \int_0^2 (48 - 8 - 6y^2) dy \\ &= 30,000 \int_0^2 (40 - 6y^2) dy \\ &= 30,000 [40y - 2y^3]_0^2 = 30,000(80 - 16) \\ &= 30,000(64) = 1,920,000 \end{aligned}$$

And since the town covers 16 square miles, it follows that the average pollution concentration A is $A = T/16 = 1,920,000/16 = 120,000$

37. E

Observe that the bus driver collect $21 \times 50 = 1050$ for the 21 tickets he sells. However, he may be required to refund 100 to one passenger if all 21 ticket holders show up. Since passengers show up or do not show up independently of one another, the probability that all 21 passengers will show up is $(1 - 0.02)^{21} = (0.98)^{21} = 0.65$. Therefore, the tour operator's expected revenue is $1050 - (100)(0.65) = 985$.

38. D

From f' , observe that

$$f(x) = \begin{cases} 4x + c_1 & \text{for } 0 < x < 10 \\ kx + c_2 & \text{for } 10 < x < 30 \\ 3x + c_3 & \text{for } x > 30 \end{cases}$$

As a result, $200 = f(50) = 3(50) + c_3 = 150 + c_3$ implies $c_3 = 50$

And $0 = f(0) = 4(0) + c_1 = c_1$,

Then due to the continuity requirement,

$$10k + c_2 = f(10) = 4(10) + c_1 = 40 + 0 = 40, \text{ and}$$

$$30k + c_2 = f(30) = 3(30) + c_3 = 90 + 50 = 140$$

Solving these last two equations simultaneously, we see that

$$20k = 100 \quad \text{or} \quad k = 5 .$$

39. A

Let F denote the distribution function of f . Then

$$F(x) = \Pr[X \leq x] = \int_1^x 3t^{-4} dt = -t^{-3} \Big|_1^x = 1 - x^{-3}$$

Using this result, we see

$$\begin{aligned} \Pr[X < 2 \mid X \geq 1.5] &= \frac{\Pr[(X < 2) \cap (X \geq 1.5)]}{\Pr[X \geq 1.5]} = \frac{\Pr[X < 2] - \Pr[X \leq 1.5]}{\Pr[X \geq 1.5]} \\ &= \frac{F(2) - F(1.5)}{1 - F(1.5)} = \frac{(1.5)^{-3} - (2)^{-3}}{(1.5)^{-3}} = 1 - \left(\frac{3}{4}\right)^3 = 0.578 \end{aligned}$$

40. B

Let

H = event that a death is due to heart disease

F = event that at least one parent suffered from heart disease

Then based on the medical records,

$$P[H \cap F^c] = \frac{210 - 102}{937} = \frac{108}{937}$$

$$P[F^c] = \frac{937 - 312}{937} = \frac{625}{937}$$

$$\text{and } P[H \mid F^c] = \frac{P[H \cap F^c]}{P[F^c]} = \frac{108/937}{625/937} = \frac{108}{625} = 0.173$$