Local Bifurcation Analysis of a Second Gradient Model for Deformations of a Rectangular Slab

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Abstract

In this paper we carry a derivation of the equilibrium equations of two dimensional nonlinear elasticity with an added second–gradient term proportional to a small parameter $\varepsilon > 0$. These equations are given by a fourth order semilinear system of pde's. We discuss different types of possible boundary conditions for these equations. We then specialize the equations to a rectangular slab and study the linearized problem about a homogenous deformation. We show that these equations admit solutions representable as Fourier series in one of the independent variables. Furthermore we obtain the characteristic equation for the eigenvalues (possible bifurcation points) for the linear problem and derive asymptotic representations for this equation for small ε. We used these expressions to show that in the limit as $\varepsilon \to 0$ the characteristic equation for $\varepsilon > 0$ converges uniformly (in certain regions of the parameter space) to the corresponding characteristic equation for $\varepsilon = 0$. When the base material ($\varepsilon = 0$) is that of a Blatz–Ko type, we get conditions for

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the existence of eigenvalues of the linear problem with $\varepsilon > 0$ and small. Our numerical results in this case indicate that the number of bifurcation points is finite when $\varepsilon > 0$ and that this number monotonically increases as $\varepsilon \to 0$. For the problem with $\varepsilon > 0$ we get conditions for the existence of local branches of non–trivial solutions.

1 Introduction

For many years one of the most difficult open problems of non–linear elasticity theory has been the use of global continuation methods (via degree theory) to study the governing quasilinear systems of partial differential equations of three–dimensional models, cf. Antman in [3]. Results along these lines for the displacement equations of equilibrium, together with boundary traction and displacements, were recently obtained by Healey and Simpson in [14]. Their approach is based upon the construction of a degree which has the same important properties of the classical Leray–Schauder degree. These new methods make it possible, for the first time, to study global bifurcation problems in non–linear three–dimensional elasticity that are not reducible to ordinary differential equations.

At the level of generality of [14], the behavior of the global solution branches would be characterized, in addition to the two Rabinowitz alternatives, cf. [20], by the possibility that they terminate due to loss of local injectivity; and/or ellipticity; and/or the failure of the complementing condition. The complementing condition is an algebraic compatibility requirement between the principal part of a linear elliptic differential operator and the principal part of the corresponding boundary conditions (cf. [2], [25], [26], and references therein). In the context of the linearized boundary value problems of elasticity, violations of the complementing condition have been associated to surface wrinkling. Failure of local injectivity and ellipticity can be ruled out by imposing physically reasonable constitutive assumptions. (See e.g. [13].) However, the complementing condition can not be enforced as a constitutive assumption on the stored energy function in the context of elasticity because that would rule out many interesting materials. In addition the complementing condition, being a condition on the linearized boundary value problem, also depends on the solution at which the problem is linearized. In general, we do not have an explicit linearization at a non-trivial solution, hence we cannot check the complementing condition directly along those branches. We refer here to the works in [12], [22], [18], and [19] where the complementing condition is violated at least once along the trivial solution branch for Green-Hadamard type materials and for incompressible materials in [19]. It is obvious that the trivial solution branch, which is explicitly known, does not “stop” at places where the complementing condition fails. This suggests that a branch of nontrivial solutions does not necessarily stops at places where the complementing condition fails. But, if we globally follow a nontrivial solution branch we have no way to a priori rule out failure of the complementing condition for traction boundary conditions within the context of the classical theory of elastic materials. When the complementing condition fails the global continuation method developed by Healey and Simpson cannot be applied, but that, by itself, does not necessarily implies that the global branch actually stops. It may very well continue, as the trivial solution branch does in the problems mentioned above.

One simple way to overcome the failure of the complementing condition, cf. the "Concluding Remarks" in [12], is to add to the stored energy function a term quadratic in the second order gradient of the deformation and proportional to a small parameter $\varepsilon > 0$ (cf. (2.1)). In the context of 3D nonlinear elasticity this would give us a semilinear fourth order system of equations for the equilibrium configurations in which the corresponding linearized problem never violates the complementing condition. In this paper we study the local bifurcation of equilibrium configurations for deformations of a rectangular slab but with the stored energy function modified as above. Our idea is to consider the problem with the added higher order gradient term as a singular perturbation of the problem studied by Simpson and Spector in [22].

The problem of bars under uniaxial compression have been studied among others by [8] and [9] (linearized equations for 2d and 3d problem), [22] with a local bifurcation analysis for the 2d nonlinear problem, [23] and [24] with a linear analysis including stability results for the 3d problem, for Green–Hadamard and Blatz–Ko type materials respectively. In [12] a rigorous local and global analysis is given for axisymmetric type solutions for the 3d problem of a cylindrical column under uniaxial compression.

Higher gradient models have been proposed by several authors to consider next neighbor interaction, to introduce length scale in the theory of elasticity, to study boundary phenomena, and have been studied extensively in the context of phase transitions, cf. [11], [16], [17], [27], [29], and references therein. But to the best of our knowledge the only works with a rigorous global analysis are [11] for forced phase transitions in one–dimensional shape memory models, and [16] with results on global continuation in nonlinear three dimensional elasticity.

Although the non-violation of the complementing condition simplifies the global study from a functional analytic point of view, it complicates considerably the solution of the linearized problem, and the verification of the hypothesis for the local bifurcation analysis. It seems to be impossible to verify these hypotheses in general! For example, for the two dimensional problem considered in this paper with a quadratic higher order term in the stored energy function, the corresponding characteristic equation (cf. (4.25), (4.26)) whose roots give the possible bifurcation points, is given by a determinant with 36 highly nonlinear terms to be accounted for. Even for a generalized Blatz–Ko type material, which has a relatively simple stored energy function, cf. (7.1), we are forced to verify numerically some of the hypotheses for the linear analysis.

In Section (2) we carry a derivation of the equations of two–dimensional nonlinear elasticity with an added second–gradient term. These equations are given by a fourth order semi–linear system of pde's. We discuss different types of possible boundary conditions for these equations. In Section (3) we then specialize these equations to a rectangular slab. By extending the domain periodically along the y, we are able to recast our boundary value problem (cf. (3.3)) as an operator equation between suitable Banach spaces. We then exploit some of the hidden symmetries in the Piola–Kirchhoff stress tensor to show that the resulting equation is equivalent to the original problem. We establish the Fréchet differentiability of the corresponding operator and characterize its linearization.

In Section (4) we study the linearized problem about the trivial homogenous deformation (cf. (3.7)). We show that these equations admit solutions representable as Fourier series in one of the independent variables. Furthermore we obtain the characteristic equation for the eigenvalues (possible bifurcation points) for the linear problem. The resulting eigenfunctions can be classified according to their symmetry, or lack of it, as of *barrelling or buckling type* respectively. We obtain asymptotic representations as $\varepsilon \searrow 0$ for the characteristic equations with the other variables fixed, and used these to show that in the limit as $\varepsilon \searrow 0$ both characteristic equations of buckling and barrelling type converge uniformly (in regions in which λ is bounded away from zero) to the corresponding characteristic equations in [22]. For completeness of the presentation, we show in Section (5) that the linearization of our boundary value problem about the trivial homogeneous solution satisfies the complementing condition for all values of $\lambda > 0$ whenever $\varepsilon > 0$.

In Section (6) we establish conditions that guarantee the existence of branches of nontrivial solutions to our boundary value problem bifurcating locally from the trivial branch. The presentation in this section is greatly simplified as compared to that of the usual mixed traction–displacement boundary value problem of nonlinear elasticity, because by the presence of the second order gradient term in the stored energy function, our operator automatically satisfies both the strong ellipticity and the complementing conditions. These imply that certain spectral and apriori estimates on solutions of the linearization of our boundary value problem hold, which in turn imply that the linearized operator is Fredholm of index zero. Thus we get existence of local bifurcation from a simple eigenvalue satisfying the so called crossing condition. We then show that this crossing condition is equivalent to the eigenvalue being a simple root of the corresponding characteristic equation. This result is established by generalizing the proof in [22] (see also [12]) to account for the added second order gradient term in the stored energy function.

In section (7) we consider, as an example, Blatz-Ko type materials. Even for this simple material, we are forced to do numerical studies to partially check some of the hypothesis of the local bifurcation analysis made in general in the previous sections. Our analysis suggests that when the higher order gradient term is present, i.e. $\varepsilon > 0$, for this material the problem admits only a finite number of possible bifurcation points for $\lambda \in (0, 1]$, and that this number of possible bifurcation points monotonically increases as ε approaches zero, accumulating precisely at the value of λ for which the linearized problem for the case $\varepsilon = 0$ fails to satisfy the complementing condition.

1.1 Notation

The Einstein summation convention is used for repeated latin indices. The dyadic product of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ is denoted by $\mathbf{ab}$ and is defined by $\mathbf{ab} = a_i b_j \mathbf{e}_i \mathbf{e}_j$, where $\mathbf{a} = a_i \mathbf{e}_i$, etc., with respect to a fixed (constant) orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. For any second order tensor $\mathbf{A}$ and vector $\mathbf{a}$ we write $\mathbf{A} \cdot \mathbf{a} = A_{ij} a_j \mathbf{e}_i$ and $a \cdot \mathbf{A} = A_{ij} a_i \mathbf{e}_j$ where $\mathbf{A} = A_{ij} \mathbf{e}_i \mathbf{e}_j$. For any given two second order tensors we write $\mathbf{A} \cdot \mathbf{B} = A_{ik} B_{kj} \mathbf{e}_i \mathbf{e}_j$ for the product or composition of the tensors. The inner product of two vectors is defined by $\mathbf{a} \cdot \mathbf{b} = a_i b_i$ and that of two second order tensors by $\mathbf{A} : \mathbf{B} = A_{ij} B_{ij} = \mathrm{trace}\,(\mathbf{A}^t \cdot \mathbf{B})$. If $\mathbf{A}$ and $\mathbf{B}$ are second and third order tensors respectively, then $\mathbf{A} \overset{k}{:} \mathbf{B}$ denotes the result of contracting the dyadic product $\mathbf{AB}$ on all indexes in $\mathbf{B}$ except the k–th. For example

$$\mathbf{A} \overset{2}{:} \mathbf{B} = A_{ij} B_{ikj} \mathbf{e}_k .$$

We use the following notation for partial derivatives of scalar valued functions:

$$f_{,j} = \frac{\partial f}{\partial x_j} \quad , \quad f_{,ij} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad , \quad \text{etc..}$$

Now with $\mathbf{f} = (f_1, \ldots, f_n)$ we have that

$$\nabla \mathbf{f} = f_{i,j} \mathbf{e}_i \mathbf{e}_j, \quad \mathrm{div}\, \mathbf{f} = \nabla \cdot \mathbf{f} = f_{i,i}.$$

For a second order tensor $\mathbf{A}$ we have that

$$\nabla \mathbf{A} = A_{ij,k} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k, \quad \mathrm{div}\, \mathbf{A} = \nabla \cdot \mathbf{A} = A_{ij,j} \mathbf{e}_i.$$

For a third order tensor $\mathbf{B}$ we have that

$$\nabla \mathbf{B} = B_{ijk,l} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l, \quad \mathrm{div}\, \mathbf{B} = \nabla \cdot \mathbf{B} = B_{ijk,k} \mathbf{e}_i \mathbf{e}_j.$$

It follows now that

$$\nabla^2 \mathbf{f} = f_{i,jk} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \quad , \quad \nabla^3 \mathbf{f} = f_{i,jkl} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l, \quad \nabla \cdot \nabla^2 \mathbf{f} = \Delta(f_{i,j}) \mathbf{e}_i \mathbf{e}_j.$$

If $\mathbf{n}$ is the outer unit normal to the surface S, we define the *surface gradient* and *surface divergence* respectively of the vector field $\mathbf{f}$ by (see [6]):

$$\nabla_s \mathbf{f} = (\mathbf{I} - \mathbf{nn}) \cdot \nabla \mathbf{f}^t \quad , \quad \nabla_s \cdot \mathbf{f} = (\mathbf{I} - \mathbf{nn}) : \nabla \mathbf{f}. \tag{1.1}$$

We let Lin denote the space of all linear transformations from $\mathbb{R}^n$ into $\mathbb{R}^n$ and write

$$\mathrm{Lin}^+ = \{\mathbf{H} \in \mathrm{Lin} \, : \, \det \mathbf{H} > 0\},$$

where det denotes the determinant.

The Schauder space $C^{m,\alpha}(\overline{\Omega})$ denotes the Banach space of functions with up to m continuous derivatives in $\overline{\Omega}$ with the derivatives of order m satisfying a Hölder condition with exponent α. The norm in $C^{m,\alpha}(\overline{\Omega})$ is denoted by $\|\cdot\|_{m,\alpha,\overline{\Omega}}$.

If $G : (\mathbf{x}, \mathbf{y}) \ni \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is a mapping between the Banach spaces $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, then $G_{\mathbf{x}}, G_{\mathbf{y}}, G_{\mathbf{xy}}$. etc., denote the corresponding (partial) Fréchet derivatives of G.

2 Formulation of the Governing Equations

In this section we out carry a derivation of the equations of equilibrium for nonlinear elasticity with an added second order gradient term to the stored energy function. For more general derivations see [16] and [17]. In [16] there are as well results concerning global continuation for these problems in the context of three dimensional elasticity.

We consider a body that, for convenience, we identify with the region $\overline{\mathcal{B}}$ that it occupies in a fixed reference configuration in $\mathbb{R}^n$. A *deformation* $\mathbf{f}$ of the body is a member of the space

$$\mathrm{Def} = \left\{ \mathbf{f} \in C^4(\overline{\mathcal{B}}; \mathbb{R}^n) \ : \ \det \nabla \mathbf{f} > 0 \right\}.$$

Let

$$\hat{W}(\mathbf{F}, \mathbf{G}) = W(\mathbf{F}) + \frac{\varepsilon}{2} \mathbf{G} \vdots \mathbf{G}, \tag{2.1}$$

where $\mathbf{F}$, $\mathbf{G}$ are second and third order tensors respectively, the triple dots $\vdots$ denote the inner product of third order tensors, and $W : \mathrm{Lin}^+ \to \mathbb{R}$. Now the total energy due to the deformation $\mathbf{f} : \mathcal{B} \to \mathbb{R}^n$ is given by:

$$E(\mathbf{f}) = \int_{\mathcal{B}} \hat{W}(\nabla \mathbf{f}, \nabla^2 \mathbf{f}) \, \mathrm{d}x.$$

The derivatives

$$\mathbf{S}(\mathbf{F}) = \frac{\mathrm{d}}{\mathrm{d}\mathbf{F}} W(\mathbf{F}), \quad \mathbf{C}(\mathbf{F}) = \frac{\mathrm{d}^2}{\mathrm{d}\mathbf{F}^2} W(\mathbf{F}), \tag{2.2}$$

are the usual (Piola–Kirchhoff) *stress* and *elasticity* tensors, respectively, when $\varepsilon = 0$. We assume that $\mathbf{C}(\mathbf{F})$ is strongly elliptic, i.e. that

$$\mathbf{ab} : \mathbf{C}(\mathbf{F})[\mathbf{ab}] > 0, \tag{2.3}$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ and all $\mathbf{F} \in \mathrm{Lin}^+$.

If $\mathbf{v}$ is any smooth admissible variation, we have that

$$\begin{aligned}
\left. \frac{\mathrm{d}}{\mathrm{d}\alpha} E(\mathbf{f} + \alpha \mathbf{v}) \right|_{\alpha=0} &= \frac{\mathrm{d}}{\mathrm{d}\alpha} \int_{\mathcal{B}} \Big(W(\nabla \mathbf{f} + \alpha \nabla \mathbf{v}) \\
&\quad \left. + \frac{\varepsilon}{2} (\nabla^2 \mathbf{f} + \alpha \nabla^2 \mathbf{v}) \vdots (\nabla^2 \mathbf{f} + \alpha \nabla^2 \mathbf{v}) \Big) \, \mathrm{d}x \right|_{\alpha=0} \\
&= \int_{\mathcal{B}} \Big(\mathbf{S}(\nabla \mathbf{f}) : \nabla \mathbf{v} + \varepsilon \nabla^2 \mathbf{f} \vdots \nabla^2 \mathbf{v} \Big) \, \mathrm{d}x.
\end{aligned}$$

Integrating by parts once we get that

$$\int_{\mathcal{B}} \mathbf{S}(\nabla \mathbf{f}) : \nabla \mathbf{v} \, \mathrm{d}x = \int_{\partial \mathcal{B}} (\mathbf{S}(\nabla \mathbf{f}) \cdot \mathbf{n}) \cdot \mathbf{v} \, \mathrm{d}s - \int_{\mathcal{B}} (\mathrm{div}\, \mathbf{S}(\nabla \mathbf{f})) \cdot \mathbf{v} \, \mathrm{d}x,$$

where $\mathbf{n}$ is the outer unit normal to $\partial\mathcal{B}$. Also integrating by parts twice we get that

$$\begin{aligned}\int_{\mathcal{B}} \nabla^2\mathbf{f} \,\vdots\, \nabla^2\mathbf{v}\,\mathrm{d}x &= \int_{\partial\mathcal{B}} (\nabla^2\mathbf{f}\cdot\mathbf{n}) : \nabla\mathbf{v}\,\mathrm{d}s \\ &\quad - \int_{\partial\mathcal{B}} (\Delta(\nabla\mathbf{f})\cdot\mathbf{n})\cdot\mathbf{v}\,\mathrm{d}s + \int_{\mathcal{B}} (\Delta^2\mathbf{f})\cdot\mathbf{v}\,\mathrm{d}x,\end{aligned}$$

where $\Delta^2\mathbf{f} = (\Delta^2 f_i)\mathbf{e}_i$ and $\Delta(\nabla\mathbf{f}) = (\Delta f_{i,j})\mathbf{e}_i\mathbf{e}_j$. Combining all of these results we get that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}E(\mathbf{f}+\alpha\mathbf{v})\bigg|_{\alpha=0} &= \int_{\mathcal{B}} \big(\varepsilon\Delta^2\mathbf{f} - \operatorname{div}\mathbf{S}(\nabla\mathbf{f})\big)\cdot\mathbf{v}\,\mathrm{d}x \\ &\quad + \int_{\partial\mathcal{B}} \varepsilon(\nabla^2\mathbf{f}\cdot\mathbf{n}) : \nabla\mathbf{v}\,\mathrm{d}s + \int_{\partial\mathcal{B}} (\mathbf{S}(\nabla\mathbf{f})\cdot\mathbf{n} - \varepsilon\Delta(\nabla\mathbf{f})\cdot\mathbf{n})\cdot\mathbf{v}\,\mathrm{d}s.\end{aligned}$$

We now work with the second term of the right hand side of this expression. We can write

$$\nabla\mathbf{v}^t = \mathbf{n}\mathbf{n}\cdot\nabla\mathbf{v}^t + (\mathbf{I}-\mathbf{n}\mathbf{n})\cdot\nabla\mathbf{v}^t.$$

Hence

$$\begin{aligned}(\nabla^2\mathbf{f}\cdot\mathbf{n}) : \nabla\mathbf{v} &= (\nabla^2\mathbf{f}\cdot\mathbf{n})^t : \nabla\mathbf{v}^t \\ &= (\nabla^2\mathbf{f}\cdot\mathbf{n})^t : (\mathbf{n}\mathbf{n}\cdot\nabla\mathbf{v}^t) + (\nabla^2\mathbf{f}\cdot\mathbf{n})^t : [(\mathbf{I}-\mathbf{n}\mathbf{n})\cdot\nabla\mathbf{v}^t] \\ &= (\nabla^2\mathbf{f}\cdot\mathbf{n})^t : (\mathbf{n}\mathbf{n}\cdot\nabla\mathbf{v}^t) + (\nabla^2\mathbf{f}^t\cdot\mathbf{n}) : [(\mathbf{I}-\mathbf{n}\mathbf{n})\cdot\nabla\mathbf{v}^t], \qquad (2.4)\end{aligned}$$

where the transposition in $\nabla^2\mathbf{f}$ is done with respect to its first two indexes. But

$$(\nabla^2\mathbf{f}\cdot\mathbf{n})^t : (\mathbf{n}\mathbf{n}\cdot\nabla\mathbf{v}^t) = (\nabla^2\mathbf{f} : \mathbf{n}\mathbf{n})\cdot\mathrm{D}\mathbf{v},$$

where

$$\mathrm{D}\mathbf{v} = \nabla\mathbf{v}\cdot\mathbf{n} \quad , \quad \nabla^2\mathbf{f} : \mathbf{n}\mathbf{n} \equiv (\nabla^2\mathbf{f}\cdot\mathbf{n})\cdot\mathbf{n} = \nabla^2\mathbf{f} \overset{1}{:} \mathbf{n}\mathbf{n}.$$

To simplify the second term in (2.4) we use the following identity.

Lemma 2.1. *For any two second order tensor fields* $\mathbf{A}, \mathbf{B}$ *and vector field* $\mathbf{v}$*, we have that*

$$\mathbf{A} : (\mathbf{B}\cdot\nabla\mathbf{v}^t) = \mathbf{B} : \nabla(\mathbf{A}\cdot\mathbf{v}) - \mathbf{v}\cdot(\mathbf{B} \overset{2}{:} \nabla\mathbf{A}).$$

Proof: With

$$\mathbf{A} = A_{ij}\mathbf{e}_i\mathbf{e}_j \quad , \quad \mathbf{B} = B_{lk}\mathbf{e}_l\mathbf{e}_k \quad , \quad \nabla\mathbf{v} = v_{p,q}\mathbf{e}_p\mathbf{e}_q,$$

we have that

$$\mathbf{A} : (\mathbf{B}\cdot\nabla\mathbf{v}^t) = (A_{ij}\mathbf{e}_i\mathbf{e}_j) : (B_{lp}v_{q,p}\,\mathbf{e}_l\mathbf{e}_q) = A_{lq}B_{lp}v_{q,p} = B_{lp}(A_{lq}v_{q,p}).$$

Using the identity

$$(A_{lq}v_q)_{,p} = A_{lq,p}v_q + A_{lq}v_{q,p},$$

we have that

$$\begin{aligned}\mathbf{A} : (\mathbf{B} \cdot \nabla \mathbf{v}^t) &= B_{lp}(A_{lq} v_q)_{,p} - B_{lp} A_{lq,p} v_q \\ &= B_{lp}(\mathbf{A} \cdot \mathbf{v})_{l,p} - v_q (B_{lp} A_{lq,p}) \\ &= \mathbf{B} : \nabla(\mathbf{A} \cdot \mathbf{v}) - \mathbf{v} \cdot (\mathbf{B} \overset{2}{\vdots} \nabla \mathbf{A}).\end{aligned}$$

□

Taking $\mathbf{A} = (\nabla^2 \mathbf{f}^t \cdot \mathbf{n})$ and $\mathbf{B} = \mathbf{I} - \mathbf{nn}$ in this lemma, we get that

$$\begin{aligned}(\nabla^2 \mathbf{f}^t \cdot \mathbf{n}) : [(\mathbf{I} - \mathbf{nn}) \cdot \nabla \mathbf{v}^t] &= (\mathbf{I} - \mathbf{nn}) : \nabla((\nabla^2 \mathbf{f}^t \cdot \mathbf{n}) \cdot \mathbf{v}) \\ &\quad - \mathbf{v} \cdot ((\mathbf{I} - \mathbf{nn}) \overset{2}{\vdots} \nabla(\nabla^2 \mathbf{f}^t \cdot \mathbf{n})) \\ &= \nabla_s \cdot ((\nabla^2 \mathbf{f}^t \cdot \mathbf{n}) \cdot \mathbf{v}) - \mathbf{v} \cdot ((\mathbf{I} - \mathbf{nn}) \overset{1}{\vdots} \nabla(\nabla^2 \mathbf{f} \cdot \mathbf{n})) \\ &= \nabla_s \cdot ((\nabla^2 \mathbf{f}^t \cdot \mathbf{n}) \cdot \mathbf{v}) - \mathbf{v} \cdot (\overset{1}{\nabla}_s \cdot (\nabla^2 \mathbf{f} \cdot \mathbf{n})),\end{aligned}$$

where we have used the operators defined by (1.1) and introduced the notation

$$\overset{1}{\nabla}_s \cdot (\nabla^2 \mathbf{f} \cdot \mathbf{n}) = (\mathbf{I} - \mathbf{nn}) \overset{1}{\vdots} \nabla(\nabla^2 \mathbf{f} \cdot \mathbf{n}).$$

Using the surface divergence theorem [6], since the surface $\partial\mathcal{B}$ is closed, we get now that

$$\begin{aligned}\int_{\partial\mathcal{B}} \nabla_s \cdot ((\nabla^2 \mathbf{f}^t \cdot \mathbf{n}) \cdot \mathbf{v}) \, \mathrm{d}s &= \int_{\partial\mathcal{B}} (\nabla_s \cdot \mathbf{n}) \mathbf{n} \cdot ((\nabla^2 \mathbf{f}^t \cdot \mathbf{n}) \cdot \mathbf{v}) \, \mathrm{d}s \\ &= \int_{\partial\mathcal{B}} (\nabla_s \cdot \mathbf{n})(\nabla^2 \mathbf{f} : \mathbf{nn}) \cdot \mathbf{v} \, \mathrm{d}s.\end{aligned}$$

We now have that

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\alpha} E(\mathbf{f} + \alpha \mathbf{v})\right|_{\alpha=0} &= \int_{\mathcal{B}} \big(\varepsilon \Delta^2 \mathbf{f} - \operatorname{div} \mathbf{S}(\nabla \mathbf{f})\big) \cdot \mathbf{v} \, \mathrm{d}x + \int_{\partial\mathcal{B}} \varepsilon (\nabla^2 \mathbf{f} : \mathbf{nn}) \cdot \mathrm{D}\mathbf{v} \, \mathrm{d}s \\ &\quad + \int_{\partial\mathcal{B}} \Big((\mathbf{S}(\nabla \mathbf{f}) \cdot \mathbf{n} - \varepsilon \Delta(\nabla \mathbf{f}) \cdot \mathbf{n} - \varepsilon \overset{1}{\nabla}_s \cdot (\nabla^2 \mathbf{f} \cdot \mathbf{n}) \\ &\quad + \varepsilon (\nabla_s \cdot \mathbf{n})(\nabla^2 \mathbf{f} : \mathbf{nn}) \Big) \cdot \mathbf{v} \, \mathrm{d}s.\end{aligned}$$

Since this has to vanish for any admissible variation $\mathbf{v}$, we have that the following must hold:

$$\varepsilon \Delta^2 \mathbf{f} - \operatorname{div} \mathbf{S}(\nabla \mathbf{f}) = \mathbf{0} \quad , \quad \text{in } \mathcal{B}. \tag{2.5}$$

As for the boundary conditions, they will depend on whether or not we specify either or both of $\mathbf{f}$ or $\mathrm{D}\mathbf{f}$ on $\partial\mathcal{B}$. For example if neither is specified on $\partial\mathcal{B}$, then

$$\begin{aligned}\mathbf{S}(\nabla \mathbf{f}) \cdot \mathbf{n} - \varepsilon \Delta(\nabla \mathbf{f}) \cdot \mathbf{n} - \varepsilon \overset{1}{\nabla}_s \cdot (\nabla^2 \mathbf{f} \cdot \mathbf{n}) & \\ + \varepsilon (\nabla_s \cdot \mathbf{n})(\nabla^2 \mathbf{f} : \mathbf{nn}) &= \mathbf{0} \quad , \quad \text{on } \partial\mathcal{B}, \\ \varepsilon \nabla^2 \mathbf{f} : \mathbf{nn} &= \mathbf{0} \quad , \quad \text{on } \partial\mathcal{B}.\end{aligned}$$

If $\mathbf{f} = \mathbf{g}$ on $\partial\mathcal{B}$ but $\mathrm{D}\mathbf{f}$ is not specified, we have that

$$\mathbf{f} = \mathbf{g} \quad , \quad \varepsilon\nabla^2\mathbf{f} : \mathbf{nn} = \mathbf{0} \quad , \quad \text{on } \partial\mathcal{B}.$$

If $\mathbf{f}$ is not specified but $\mathrm{D}\mathbf{f} = \mathbf{h}$ on $\partial\mathcal{B}$, then we have that

$$\begin{aligned} \mathbf{S}(\nabla\mathbf{f}) \cdot \mathbf{n} - \varepsilon\Delta(\nabla\mathbf{f}) \cdot \mathbf{n} - \varepsilon \overset{1}{\nabla}_s \cdot(\nabla^2\mathbf{f} \cdot \mathbf{n}) & \\ +\varepsilon(\nabla_s \cdot \mathbf{n})(\nabla^2\mathbf{f} : \mathbf{nn}) &= \mathbf{0} \quad , \quad \text{on } \partial\mathcal{B}, \\ \mathrm{D}\mathbf{f} &= \mathbf{h} \quad , \quad \text{on } \partial\mathcal{B}. \end{aligned}$$

We could as well specify components of $\mathbf{f}$ or $\mathrm{D}\mathbf{f}$ in the normal and tangent directions to $\partial\mathcal{B}$. For example, if $\mathbf{f} \cdot \mathbf{n}$ and $\mathrm{D}\mathbf{f} \cdot \mathbf{t}$ are specified, where $\mathbf{t}$ is any vector tangent to $\partial\mathcal{B}$, then the boundary conditions are given by

$$\begin{aligned} \mathbf{t} \cdot \Big(\mathbf{S}(\nabla\mathbf{f}) \cdot \mathbf{n} - \varepsilon\Delta(\nabla\mathbf{f}) \cdot \mathbf{n} - \varepsilon \overset{1}{\nabla}_s \cdot(\nabla^2\mathbf{f} \cdot \mathbf{n}) & \\ + \varepsilon(\nabla_s \cdot \mathbf{n})(\nabla^2\mathbf{f} : \mathbf{nn})\Big) &= 0 \quad , \quad \text{on } \partial\mathcal{B}, \\ \mathbf{f} \cdot \mathbf{n} = g \quad , \quad \mathrm{D}\mathbf{f} \cdot \mathbf{t} = h \quad , \quad (\nabla^2\mathbf{f} : \mathbf{nn}) \cdot \mathbf{n} &= 0 \quad , \quad \text{on } \partial\mathcal{B}. \end{aligned}$$

Further, we can have $\partial\mathcal{B}$ be the union of sub–boundaries on each of which we can specify any combination of $\mathbf{f}$, $\mathrm{D}\mathbf{f}$, or any of its normal or tangential components.

3 The Equations for a Rectangular Slab

We now specialize to the case in which $\mathcal{B} \subset \mathbb{R}^2$ is a rectangular slab. Thus we let

$$\mathcal{B} = \{(x, y) \,:\, -R < x < R \,,\; 0 < y < L\}\,. \tag{3.1}$$

We write $\partial\mathcal{B} = \mathcal{C}_t \cup \mathcal{C}_b \cup \mathcal{L}$ where

$$\mathcal{C}_t = \{(x, y) \,:\, -R \le x \le R \,,\; y = L\}\,, \tag{3.2a}$$
$$\mathcal{C}_b = \{(x, y) \,:\, -R \le x \le R \,,\; y = 0\}\,, \tag{3.2b}$$
$$\mathcal{L} = \{(x, y) \,:\, x = \pm R \,,\; 0 \le y \le L\}\,. \tag{3.2c}$$

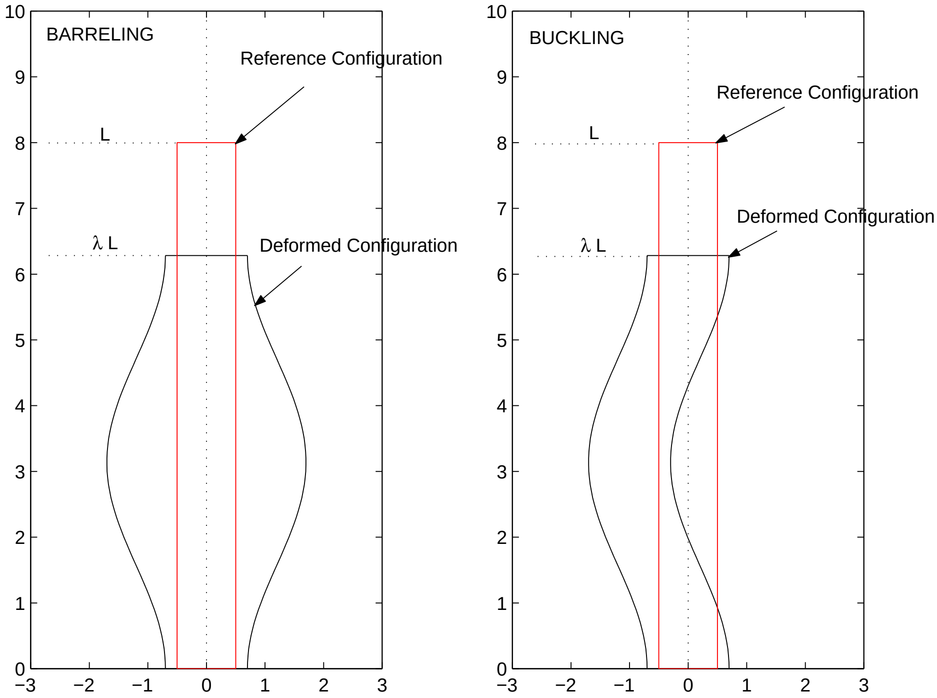

Figure 1: Possible deformations of a rectangular slab, compressed along its vertical axis, of either barrelling or buckling type.

We consider the special case of the boundary value problem of the previous section for a deformation $\mathbf{f} = (f_1, f_2)$ of $\mathcal{B}$ in which we specify[1] $\mathbf{f}\cdot\mathbf{n}$ and $\mathrm{D}\mathbf{f}\cdot\mathbf{t}$ on $\mathcal{C}_t \cup \mathcal{C}_b$:

$$\varepsilon\Delta^2\mathbf{f} - \operatorname{div}\mathbf{S}(\nabla\mathbf{f}) = \mathbf{0}\ ,\ \text{ in } \mathcal{B}, \tag{3.3a}$$

$$\mathbf{S}(\nabla\mathbf{f})\cdot\mathbf{n} - \varepsilon\Delta(\nabla\mathbf{f})\cdot\mathbf{n} - \varepsilon \overset{1}{\nabla}_s\cdot(\nabla^2\mathbf{f}\cdot\mathbf{n}) = \mathbf{0}\ ,\ \text{ on } \mathcal{L}, \tag{3.3b}$$

$$\varepsilon\nabla^2\mathbf{f} : \mathbf{nn} = \mathbf{0}\ ,\ \text{ on } \mathcal{L}, \tag{3.3c}$$

$$\mathbf{t}\cdot\big(\mathbf{S}(\nabla\mathbf{f})\cdot\mathbf{n} - \varepsilon\Delta(\nabla\mathbf{f})\cdot\mathbf{n} - \varepsilon \overset{1}{\nabla}_s\cdot(\nabla^2\mathbf{f}\cdot\mathbf{n})\big) = 0\ ,\ \text{ on } \mathcal{C}_t\cup\mathcal{C}_b, \tag{3.3d}$$

$$f_2 = 0\ ,\ \text{ on } \mathcal{C}_b\ ,\ f_2 = \lambda L\ ,\ \text{ on } \mathcal{C}_t, \tag{3.3e}$$

$$\mathrm{D}\mathbf{f}\cdot\mathbf{t} = 0\ ,\ \varepsilon\left(\nabla^2\mathbf{f} : \mathbf{nn}\right)\cdot\mathbf{n} = 0\ ,\ \text{ on } \mathcal{C}_t\cup\mathcal{C}_b, \tag{3.3f}$$

where $\mathbf{t}\cdot\mathbf{n} = 0$, and $\lambda \in [0,\infty) \subset \mathbb{R}$. Since $\mathbf{n}$ is constant on each of $\mathcal{C}_t, \mathcal{C}_b, \mathcal{L}$, we have that $\nabla_s\cdot\mathbf{n} = 0$ on each of them. (See Figure (1).)

In order to eliminate trivial nonuniqueness of solutions due to translations, we impose the following additional condition:

$$\int_{\mathcal{B}} f_1\,\mathrm{d}\mathbf{x} = 0. \tag{3.4}$$

[1] The slab is compressed along the y–axis by a factor of λ with the top and bottom free to slide along the x direction.

Note that on $\mathcal{C}_t \cup \mathcal{C}_b$ we have that $\mathbf{n} = \pm\mathbf{e}_2$, and that $\mathbf{n} = \pm\mathbf{e}_1$ on $\mathcal{L}$. Thus with $\mathbf{S} = S_{ij}\mathbf{e}_i\mathbf{e}_j$, $i,j = 1,2$, we have that the boundary value problem (3.3) reduces to:

$$\begin{aligned}
\varepsilon\Delta^2(f_i) - S_{i1,1} - S_{i2,2} &= 0 \ , \quad \text{in } \mathcal{B} \ , \ i = 1,2, && (3.5a)\\
S_{11} - \varepsilon\Delta(f_{1,1}) - \varepsilon f_{1,212} &= 0 \ , \quad \text{on } \mathcal{L} && (3.5b)\\
S_{21} - \varepsilon\Delta(f_{2,1}) - \varepsilon f_{2,212} &= 0 \ , \quad \text{on } \mathcal{L}, && (3.5c)\\
\varepsilon f_{1,11} = 0 \ , \quad \varepsilon f_{2,11} &= 0 \ , \quad \text{on } \mathcal{L}, && (3.5d)\\
S_{12} - \varepsilon f_{1,222} &= 0 \ , \quad \text{on } \mathcal{C}_t \cup \mathcal{C}_b, && (3.5e)\\
f_{1,2} = 0 \ , \quad \varepsilon f_{2,22} &= 0 \ , \quad \text{on } \mathcal{C}_t \cup \mathcal{C}_b, && (3.5f)\\
f_2 = 0 \ , \quad \text{on } \mathcal{C}_b \ , \quad f_2 &= \lambda L \ , \ \text{on } \mathcal{C}_t, && (3.5g)
\end{aligned}$$

where the argument of S_{ij} and $S_{ij,k}$ is $\nabla\mathbf{f}$.

We assume that the function $W : \mathrm{Lin}^+ \to \mathbb{R}$ which corresponds to the stored energy function (2.1) with $\varepsilon = 0$, satisfies the usual frame–indifference and isotropy conditions of nonlinearly elasticity. In that case it is well known (see e.g. [10], [28]) that there exists a function $\sigma : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$W(\mathbf{F}) = \sigma\left(\frac{1}{2}\mathbf{F} : \mathbf{F}, \det\mathbf{F}\right) \quad , \quad \mathbf{F} \in \mathrm{Lin}^+. \tag{3.6}$$

We assume σ is of class C^m, $m \geq 5$. Under the same conditions as for the case where $\varepsilon = 0$, one can check now that[2]

$$\mathbf{f}_\lambda(\mathbf{x}) = (\mu(\lambda)x, \lambda y), \tag{3.7}$$

is a solution of (3.5) where $\mu(\lambda)$ is the unique solution (see [22]) of the equation:

$$\mu(\lambda)\sigma_{,1} + \lambda\sigma_{,2} = 0, \tag{3.8}$$

with $\sigma_{,i} = \sigma_{,i}\left(\frac{1}{2}(\mu^2+\lambda^2), \mu\lambda\right)$, $i = 1,2$.

In order to recast our boundary value problem (3.3) as an operator equation between suitable spaces, we first extend the domain $\mathcal{B}$ periodically along the y direction. Then we exploit some of the hidden symmetries in the Piola–Kirchhoff stress tensor to show that the resulting equation is equivalent to the original problem. Thus we let

$$\mathcal{B}_\infty = \{(x,y) \ : \ -R < x < R, \quad -\infty < y < \infty\}. \tag{3.9}$$

For any deformation $\mathbf{f} = (f_1, f_2)$ we write

$$\mathbf{u} = \mathbf{f} - \mathbf{f}_\lambda, \tag{3.10}$$

where $\mathbf{f}_\lambda$ is given by (3.7). Thus $\mathbf{u} = (u_1, u_2)$ is the *displacement vector*. We impose the following even–odd condition on the components of $\mathbf{u}$:

$$u_1(x,y) = u_1(x,-y), \quad u_2(x,-y) = -u_2(x,y). \tag{3.11}$$

[2]The subscript λ here does not denote partial differentiation.

We define the following spaces of functions:

$$\begin{aligned}
\mathcal{Z} &= \left\{\mathbf{u} \in C^{4,\alpha}(\overline{\mathcal{B}}_\infty, \mathbb{R}^2) \,:\, \mathbf{u} \text{ is } 2L \text{ periodic in } y \text{ and satisfy (3.4), (3.11) and,} \right. \\
&\qquad \left. \varepsilon \nabla^2 \mathbf{u} : \mathbf{nn} = \mathbf{0} \text{ on } \partial\mathcal{B}_\infty \right\}, && (3.12)\\
\mathcal{Y}_0 &= \left\{\mathbf{v} \in C^{0,\alpha}(\overline{\mathcal{B}}_\infty, \mathbb{R}^2) \,:\, \mathbf{v} \text{ is } 2L \text{ periodic in } y \text{ and satisfy (3.11)}\right\}, && (3.13)\\
\mathcal{Y}_1 &= \left\{\mathbf{w} \in C^{1,\alpha}(\partial\mathcal{B}_\infty, \mathbb{R}^2) \,:\, \mathbf{w} \text{ is } 2L \text{ periodic in } y \text{ and satisfy (3.11)}\right\}, && (3.14)\\
\mathcal{Y} &= \mathcal{Y}_0 \times \mathcal{Y}_1, && (3.15)
\end{aligned}$$

with corresponding norms:

$$\|\cdot\|_{\mathcal{Z}} = \|\cdot\|_{4,\alpha,\overline{\mathcal{B}}}, \quad \|\cdot\|_{\mathcal{Y}} = \|\cdot\|_{0,\alpha,\overline{\mathcal{B}}} + \|\cdot\|_{1,\alpha,\partial\mathcal{B}}.$$

We define now

$$\mathcal{U} = \{(\lambda, \mathbf{u}) \in (0,\infty) \times \mathcal{Z} \,:\, \det(\nabla \mathbf{f}_\lambda + \nabla \mathbf{u}) > 0\}, \tag{3.16}$$

and the operator $G : \mathcal{U} \to \mathcal{Y}$ by:

$$G(\lambda, \mathbf{u}) = (\varepsilon \Delta^2 \mathbf{u} - \operatorname{div} \mathbf{S}(\nabla \mathbf{f}_\lambda + \nabla \mathbf{u}), B(\lambda, \mathbf{u})), \tag{3.17}$$

where

$$B(\lambda, \mathbf{u}) = \mathbf{S}(\nabla \mathbf{f}_\lambda + \nabla \mathbf{u}) \cdot \mathbf{n} - \varepsilon \Delta(\nabla \mathbf{u}) \cdot \mathbf{n} - \varepsilon \overset{1}{\nabla}_s \cdot (\nabla^2 \mathbf{u} \cdot \mathbf{n}). \tag{3.18}$$

We now can prove the following:

Proposition 3.1. *Let* $\mathbf{u}$ *be any solution of the operator equation*

$$G(\lambda, \mathbf{u}) = \mathbf{0}. \tag{3.19}$$

Then $\mathbf{f} = \mathbf{f}_\lambda + \mathbf{u}$ *with* $\mathbf{u}$ *restricted to* $\overline{\mathcal{B}}$*, is a solution of the boundary value problem* (3.3)*. On the other hand, if* $\mathbf{f}$ *is a solution of* (3.3)*, then* $\mathbf{u} = \mathbf{f} - \mathbf{f}_\lambda$ *can be extended periodically in* y *to a solution of* (3.19)*.*

Proof: First we observe that an easy computation using $(2.2)_1$, (3.6), and (3.7), shows that

$$S_{12}(\nabla \mathbf{f}_\lambda + \nabla \mathbf{u}) = [\mathbf{S}(\nabla \mathbf{f}_\lambda + \nabla \mathbf{u})\mathbf{e}_2] \cdot \mathbf{e}_1 = \sigma_{,1} u_{1,2} - \sigma_{,2} u_{2,1}.$$

¿From (3.5g) we have that $u_{2,1} = 0$ on $\mathcal{C}_t \cup \mathcal{C}_b$, and from strong ellipticity, cf. (2.3), it can be deduced that, cf. [22], $\sigma_{,1} > 0$. Hence, for any function $\mathbf{u}$ satisfying $u_{1,2} = 0$ on $\mathcal{C}_t \cup \mathcal{C}_b$, the condition (3.5e):

$$S_{12}(\nabla \mathbf{f}_\lambda + \nabla \mathbf{u}) - \varepsilon u_{1,222} = 0, \quad \text{on } \mathcal{C}_t \cup \mathcal{C}_b,$$

is equivalent to $u_{1,222} = 0$ on $\mathcal{C}_t \cup \mathcal{C}_b$.

Thus clearly, if $\mathbf{u}$ is a solution of (3.19), then the even–odd conditions (3.11) and the periodicity in the y direction imply that $u_2 = u_{1,222} = 0$ on $\mathcal{C}_t \cup \mathcal{C}_b$ and that the boundary condition (3.5f) is satisfied. Hence $\mathbf{f} = \mathbf{f}_\lambda + \mathbf{u}$ is a solution of (3.3)

On the other hand if $\mathbf{f}$ is a solution of (3.3) or equivalently (3.5), then with $\mathbf{u} = \mathbf{f} - \mathbf{f}_\lambda$ and by the observation made above, (3.5e) is equivalent to $u_{1,222} = 0$ on $\mathcal{C}_t \cup \mathcal{C}_b$. This together with the boundary conditions (3.5f) and (3.5g) allow us to extend $\mathbf{u}$ periodically in y according to (3.11) to a solution of (3.19). $\square$

A simple modification of the results in Valent [30], due to the unboundedness of $\overline{\mathcal{B}}_\infty$, allows us to get the following:

Proposition 3.2. *The function $G : \mathcal{U} \to \mathcal{Y}$ is of class C^2 and*

$$\begin{aligned} G_{\mathbf{u}}(\lambda, \mathbf{u})[\mathbf{h}] &= (\varepsilon \Delta^2 \mathbf{h} - \operatorname{div} \mathbf{C}(\nabla \mathbf{f}_\lambda + \nabla \mathbf{u})[\nabla \mathbf{h}], B_{\mathbf{u}}(\lambda, \mathbf{u})[\mathbf{h}]), \qquad (3.20a)\\ G_{\mathbf{u}\lambda}(\lambda, \mathbf{u})[\mathbf{h}] &= \left(-\operatorname{div} \frac{\mathrm{d}\mathbf{C}}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{f}_\lambda + \nabla \mathbf{u})[\nabla \mathbf{h}, \nabla \mathbf{f}'_\lambda], \right.\\ &\qquad \left. \left(\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{f}_\lambda + \nabla \mathbf{u})[\nabla \mathbf{h}, \nabla \mathbf{f}'_\lambda] \right) \cdot \mathbf{n} \right), \qquad (3.20b) \end{aligned}$$

where $\mathbf{f}'_\lambda = \mathrm{d}\mathbf{f}_\lambda / \mathrm{d}\lambda$ *and*

$$\begin{aligned} B_{\mathbf{u}}(\lambda, \mathbf{u})[\mathbf{h}] &= \mathbf{C}(\nabla \mathbf{f}_\lambda + \nabla \mathbf{u})[\nabla \mathbf{h}] \cdot \mathbf{n} - \varepsilon \Delta(\nabla \mathbf{h}) \cdot \mathbf{n} \\ &\quad - \varepsilon \overset{1}{\nabla}_s \cdot (\nabla^2 \mathbf{h} \cdot \mathbf{n}). \end{aligned}$$

4 The Linearized Problem

Since (3.7) is a solution of (3.5), we have that for the operator (3.17),

$$G(\lambda, \mathbf{0}) = \mathbf{0}, \quad \lambda \geq 0.$$

We look for nontrivial solutions of (3.19) bifurcating from the trivial branch $\{(\lambda, \mathbf{0}) : \lambda \geq 0\}$. For this we need to study the linearized problem about the trivial branch which by Proposition (3.2) is given by:

$$L(\lambda)[\mathbf{h}] \equiv G_{\mathbf{u}}(\lambda, \mathbf{0}) = (\varepsilon \Delta^2 \mathbf{h} - \operatorname{div} \mathbf{C}(\nabla \mathbf{f}_\lambda)[\nabla \mathbf{h}], B_{\mathbf{u}}(\lambda, \mathbf{0})[\mathbf{h}]) = \mathbf{0}, \quad \mathbf{h} \in \mathcal{Z}. \qquad (4.1)$$

In particular, we need to determine, for which values of λ, this boundary value problem has nontrivial solutions $\mathbf{h}$.

An elementary but otherwise lengthy computation shows that the boundary value problem (4.1) is equivalent to:

$$\begin{aligned} \varepsilon\left(u_{1,1111} + 2u_{1,1122} + u_{1,2222}\right) - K u_{1,11} - P u_{1,22} - M u_{2,12} = 0, &\quad \text{in } \mathcal{B}, \qquad (4.2a)\\ \varepsilon\left(u_{2,1111} + 2u_{2,1122} + u_{2,2222}\right) - P u_{2,11} - Q u_{2,22} - M u_{1,12} = 0, &\quad \text{in } \mathcal{B}, \qquad (4.2b)\\ \varepsilon u_{1,11} = 0, \quad K u_{1,1} + N u_{2,2} - \varepsilon\left(u_{1,111} + 2u_{1,122}\right) = 0, &\quad \text{on } \mathcal{L}, \qquad (4.2c)\\ \varepsilon u_{2,11} = 0, \quad P u_{2,1} + (M - N) u_{1,2} - \varepsilon\left(u_{2,111} + 2u_{2,122}\right) = 0, &\quad \text{on } \mathcal{L}, \qquad (4.2d)\\ u_{1,2} = 0, \quad \varepsilon u_{1,222} = 0, \quad u_2 = 0, \quad \varepsilon u_{2,22} = 0, &\quad \text{on } \mathcal{C}_t \cup \mathcal{C}_b, \qquad (4.2e) \end{aligned}$$

where we have written $\mathbf{h} = (u_1, u_2)$,

$$K = \sigma_{,1} + \textstyle\sum_{i,j=1}^{2} \sigma_{,ij}\, \alpha_i \alpha_j, \quad N = \sigma_{,2} + \sum_{i,j=1}^{2} \sigma_{,ij}\, \alpha_i \beta_j, \tag{4.3a}$$

$$Q = \sigma_{,1} + \textstyle\sum_{i,j=1}^{2} \sigma_{,ij}\, \beta_i \beta_j, \quad M = N - \sigma_{,2}\,, \quad P = \sigma_{,1}\,, \tag{4.3b}$$

and $\alpha_1 = \mu(\lambda)$, $\alpha_2 = \lambda$, $\beta_1 = \lambda$, $\beta_2 = \mu(\lambda)$. One can show (see [23]) that the elasticity tensor $\mathbf{C}(\nabla \mathbf{f}_\lambda)$ (c.f. (2.2)) satisfies the *strong ellipticity condition*:

$$\mathbf{ab} : \mathbf{C}(\nabla \mathbf{f}_\lambda)[\mathbf{ab}] > 0, \quad \forall\, \mathbf{a}, \mathbf{b} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}\,,$$

if and only if

$$K > 0, \quad P > 0, \quad Q > 0, \quad P + (KQ)^{1/2} > |M|\,.$$

By a proof very similar to that of Proposition (4.2) in [23], we get that any $(u_1, u_2) \in C^4([-R, R] \times [0, L])$ satisfying (4.2a)–(4.2b) and the boundary conditions (4.2e) must have a Fourier series representation of the form[3]:

$$u_1(x, y) = \sum_{k=1}^{\infty} a_k(x) \cos(q_k y), \quad u_2(x, y) = \sum_{k=1}^{\infty} b_k(x) \sin(q_k y), \tag{4.4}$$

where $q_k = k\pi/L$ and both of these series converge uniformly in $[-R, R] \times [0, L]$.

If we multiply (4.2a) and the second equation in (4.2c) by $\cos(q_k y)$, (4.2b) and the second equation in (4.2d) by $\sin(q_k y)$, and carry the required integration by parts using the remaining boundary conditions, then we get that a_k, b_k are solutions of the boundary value problem:

$$\varepsilon a_k^{(4)}(x) - (2\varepsilon q_k^2 + K) a_k''(x) + (\varepsilon q_k^4 + P q_k^2) a_k(x) - M q_k b_k'(x) = 0, \tag{4.5a}$$

$$\varepsilon b_k^{(4)}(x) - (2\varepsilon q_k^2 + P) b_k''(x) + (\varepsilon q_k^4 + Q q_k^2) b_k(x) + M q_k a_k'(x) = 0, \tag{4.5b}$$

$-R < x < R$, with

$$\varepsilon a_k''(\pm R) = 0, \quad (2\varepsilon q_k^2 + K) a_k'(\pm R) - \varepsilon a_k'''(\pm R) + N q_k b_k(\pm R) = 0, \tag{4.6a}$$

$$\varepsilon b_k''(\pm R) = 0, \quad (2\varepsilon q_k^2 + P) b_k'(\pm R) - \varepsilon b_k'''(\pm R) - (M - N) q_k a_k(\pm R) = 0, \tag{4.6b}$$

The solutions of this boundary value problem are characterized by the roots of the following polynomial equation:

$$\begin{aligned} \varepsilon^2 r^8 - \varepsilon(B_k + D_k) r^6 \;&+\; (B_k D_k - \varepsilon(A_k + C_k)) r^4 \\ &+\; (D_k A_k + E_k^2 + B_k C_k) r^2 + C_k A_k = 0, \end{aligned} \tag{4.7}$$

where

$$A_k = -(\varepsilon q_k^4 + P q_k^2), \qquad B_k = 2\varepsilon q_k^2 + K, \qquad C_k = -(\varepsilon q_k^4 + Q q_k^2), \tag{4.8}$$

$$D_k = 2\varepsilon q_k^2 + P, \qquad E_k = M q_k. \tag{4.9}$$

[3]One can show that the condition (3.4) and the second boundary condition in (4.2c) imply that $a_0(x)$ must be identically zero in the series for u_1 in (4.4).

With the substitution $\varpi = r^2$ this reduces to:

$$\begin{aligned}\varepsilon^2\varpi^4 - \varepsilon(B_k + D_k)\varpi^3 &+ (B_k D_k - \varepsilon(A_k + C_k))\varpi^2 \\ &+ (D_k A_k + E_k^2 + B_k C_k)\varpi + C_k A_k = 0. \end{aligned} \tag{4.10}$$

We have now the following:

Lemma 4.1. *For ε sufficiently small the equation* (4.10) *has four roots (counting multiplicity) with positive real part.*

Proof: When $\varepsilon = 0$ the above equation reduces to:

$$KP\varpi^2 + (M^2 - P^2 - KQ)q_k^2\varpi + PQq_k^4 = 0, \tag{4.11}$$

with roots $\varpi_1(0), \varpi_2(0)$ with positive real parts. (See Simpson and Spector [22].) We seek now solutions of (4.10) of the form

$$\varpi_j(\varepsilon) = \sum_{l=0}^{\infty} \varpi_j^{(l)}(0)\,\frac{\varepsilon^l}{l!}, \quad j = 1, 2. \tag{4.12}$$

If we substitute $\varpi_j(\varepsilon)$ into (4.10), differentiate with respect to ε once and then set $\varepsilon = 0$, we find that

$$\begin{aligned}\varpi_j'(0) &= \left[2KP\varpi_j(0) + (M^2 - P^2 - KQ)q_k^2\right]^{-1} \times \left[(K+P)\varpi_j^3(0)\right. \\ &\left. -(3P + 2K + Q)q_k^2\varpi_j^2(0) + (3P + 2Q + K)q_k^4\varpi_j(0) - (P+Q)q_k^6\right]\end{aligned} \tag{4.13}$$

for $j = 1, 2$. Similarly we can compute higher order derivatives of $\varpi_j(\varepsilon)$. If $\varpi_3(\varepsilon), \varpi_4(\varepsilon)$ are the other two roots of (4.10), then using that

$$\begin{aligned}\varpi_1(\varepsilon) + \varpi_2(\varepsilon) + \varpi_3(\varepsilon) + \varpi_4(\varepsilon) &= \frac{B_k + D_k}{\varepsilon}, \\ \varpi_1(\varepsilon)\varpi_2(\varepsilon)\varpi_3(\varepsilon)\varpi_3(\varepsilon) &= \frac{A_k C_k}{\varepsilon^2},\end{aligned}$$

we find that

$$\begin{aligned}2\varepsilon\varpi_3(\varepsilon) &= B_k + D_k - \varepsilon(\varpi_1(\varepsilon) + \varpi_2(\varepsilon)) \\ &+ \left[(B_k + D_k - \varepsilon(\varpi_1(\varepsilon) + \varpi_2(\varepsilon)))^2 - 4\,\frac{A_k C_k}{\varpi_1(\varepsilon)\varpi_2(\varepsilon)}\right]^{1/2},\end{aligned} \tag{4.14}$$

with a similar expression for $\varpi_4(\varepsilon)$ with a minus in front of the bracketed square root. Since $B_k + D_k = K + P > 0$ when $\varepsilon = 0$, we can use this and the fact that $\varpi_1(0), \varpi_2(0)$ have positive real parts, to get the result. □

Note that $A_k, B_k, \ldots$ through K, P, Q, and M, are functions of λ. Thus the roots in the previous lemma are function of λ as well. We then have:

Corollary 4.2. *The roots of equation* (4.7) *are given by* $\pm\,\omega_{1,k}(\lambda,\varepsilon)$, $\pm\,\omega_{2,k}(\lambda,\varepsilon)$, $\pm\,\omega_{3,k}(\lambda,\varepsilon)$, $\pm\,\omega_{4,k}(\lambda,\varepsilon)$ *where the* ω*'s have positive real part.*

Notation: To emphasize when some of the arguments λ, ε, or k are fixed, we will drop its dependence from $\omega_{i,k}(\lambda,\varepsilon)$. For example if we hold ε fixed while λ and k are variable, we write $\omega_{i,k}(\lambda)$. On the other hand if λ and k are fixed while ε is variable, we write $\omega_i(\varepsilon)$, etc..

If the $\{\omega_{i,k}(\lambda)\}$ are all distinct, then (4.5) has the eight linearly independent solutions:

$$\begin{bmatrix} a_{k,i}(x) \\ b_{k,i}(x) \end{bmatrix} = \begin{bmatrix} F(\omega_{i,k}(\lambda))\sinh(\omega_{i,k}(\lambda)x) \\ \cosh(\omega_{i,k}(\lambda)x) \end{bmatrix}, \quad i=1,\ldots,4, \tag{4.15}$$

$$\begin{bmatrix} a_{k,i+4}(x) \\ b_{k,i+4}(x) \end{bmatrix} = \begin{bmatrix} F(\omega_{i,k}(\lambda))\cosh(\omega_{i,k}(\lambda)x) \\ \sinh(\omega_{i,k}(\lambda)x) \end{bmatrix}, \quad i=1,\ldots,4, \tag{4.16}$$

where

$$F(r) = \frac{E_k r}{\varepsilon r^4 - B_k r^2 - A_k}.$$

If some of the $\{\omega_{i,k}(\lambda)\}$ are equal, say $\omega_{3,k}(\lambda)=\omega_{4,k}(\lambda)$ with the other two distinct, then (4.5a)–(4.5b) has the eight linearly independent solutions:

$$\begin{bmatrix} a_{k,i}(x) \\ b_{k,i}(x) \end{bmatrix} = \begin{bmatrix} F(\omega_{i,k}(\lambda))\sinh(\omega_{i,k}(\lambda)x) \\ \cosh(\omega_{i,k}(\lambda)x) \end{bmatrix}, \quad i=1,2,3, \tag{4.17}$$

$$\begin{bmatrix} a_{k,4}(x) \\ b_{k,4}(x) \end{bmatrix} = \begin{bmatrix} xF(\omega_{3,k}(\lambda))\cosh(\omega_{3,k}(\lambda)x) + F'(\omega_{3,k}(\lambda))\sinh(\omega_{3,k}(\lambda)x) \\ x\sinh(\omega_{3,k}(\lambda)x) \end{bmatrix}, \tag{4.18}$$

$$\begin{bmatrix} a_{k,i+4}(x) \\ b_{k,i+4}(x) \end{bmatrix} = \begin{bmatrix} F(\omega_{i,k}(\lambda))\cosh(\omega_{i,k}(\lambda)x) \\ \sinh(\omega_{i,k}(\lambda)x) \end{bmatrix}, \quad i=1,2,3, \tag{4.19}$$

$$\begin{bmatrix} a_{k,8}(x) \\ b_{k,8}(x) \end{bmatrix} = \begin{bmatrix} xF(\omega_{3,k}(\lambda))\sinh(\omega_{3,k}(\lambda)x) + F'(\omega_{3,k}(\lambda))\cosh(\omega_{3,k}(\lambda)x) \\ x\cosh(\omega_{3,k}(\lambda)x) \end{bmatrix}, \tag{4.20}$$

The solutions (4.15) or (4.17) and (4.18), when substituted into (4.4) represents solutions of the linearized problem of *barrelling type*, while those obtained from (4.16) or (4.19) and (4.20), are of *buckling type.*

In the case $\{\omega_{i,k}(\lambda)\}$ are all distinct, we define the matrices:

$$\begin{aligned} M_k^a(\lambda) &= \left[\mathbf{w}_a(\omega_{1,k}(\lambda)), \mathbf{w}_a(\omega_{2,k}(\lambda)), \mathbf{w}_a(\omega_{3,k}(\lambda)), \mathbf{w}_a(\omega_{4,k}(\lambda))\right], & (4.21)\\ M_k^s(\lambda) &= \left[\mathbf{w}_s(\omega_{1,k}(\lambda)), \mathbf{w}_s(\omega_{2,k}(\lambda)), \mathbf{w}_s(\omega_{3,k}(\lambda)), \mathbf{w}_s(\omega_{4,k}(\lambda))\right], & (4.22) \end{aligned}$$

where

$$\begin{aligned} \mathbf{w}_a(\omega) &= \left[F(\omega)\omega^2\cosh(\omega R), \omega^2\sinh(\omega R), \Lambda(\omega)\sinh(\omega R), \Delta(\omega)\cosh(\omega R)\right]^t, \\ \mathbf{w}_s(\omega) &= \left[F(\omega)\omega^2\sinh(\omega R), \omega^2\cosh(\omega R), \Lambda(\omega)\cosh(\omega R), \Delta(\omega)\sinh(\omega R)\right]^t, \\ \Lambda(\omega) &= (B_k\omega - \varepsilon\omega^3)F(\omega) + Nq_k, \\ \Delta(\omega) &= D_k\omega - \varepsilon\omega^3 - (M-N)q_kF(\omega). \end{aligned}$$

If $\omega_{3,k}(\lambda) = \omega_{4,k}(\lambda)$ with the other two distinct, then the corresponding matrices are given by

$$M_k^a(\lambda) = [\mathbf{w}_a(\omega_{1,k}(\lambda)), \mathbf{w}_a(\omega_{2,k}(\lambda)), \mathbf{w}_a(\omega_{3,k}(\lambda)), \mathbf{w}_a'(\omega_{3,k}(\lambda))], \tag{4.23}$$

$$M_k^s(\lambda) = [\mathbf{w}_s(\omega_{1,k}(\lambda)), \mathbf{w}_s(\omega_{2,k}(\lambda)), \mathbf{w}_s(\omega_{3,k}(\lambda)), \mathbf{w}_s'(\omega_{3,k}(\lambda))],, \tag{4.24}$$

where

$$\mathbf{w}_s'(\omega) = \frac{\mathrm{d}\mathbf{w}_s(\omega)}{\mathrm{d}\omega},$$

etc.. Since the general solution of (4.5) is given by a linear combination of (4.15)–(4.16) or (4.17)–(4.20), it follows that (4.6) implies that (4.5)–(4.6) has nontrivial solutions if and only if

$$\det M_k^s(\lambda) = 0, \tag{4.25}$$

or

$$\det M_k^a(\lambda) = 0. \tag{4.26}$$

We summarize our results in the following:

Proposition 4.3. *The boundary value problem* (4.1) *or equivalently* (4.2) *has nontrivial solutions if and only if* λ *is a root of either* (4.25) *or* (4.26). *The nontrivial solutions are given by* (4.4) *where the sum is taken over all* k*'s such that* $\det M_k^s(\lambda) = 0$ *with the coefficients given by* (4.15) *or* (4.17) *and* (4.18), *and over all* k*'s such that* $\det M_k^a(\lambda) = 0$ *with the coefficients given by* (4.16) *or* (4.19) *and* (4.20).

In the following discussion we are going to fix the values of λ and q_k and study the limiting behavior of (4.25) as $\varepsilon \to 0^+$, the analysis for (4.26) been similar. From equation (4.14) we get that

$$\omega_i(\varepsilon) \sim \frac{c_i}{\sqrt{\varepsilon}}, \quad \varepsilon \to 0^+, \quad i = 3, 4, \tag{4.27}$$

where

$$\begin{aligned} 2c_i^2 &= K + P \pm \left[(K+P)^2 - 4\frac{PQq_k^4}{\varpi_1(0)\varpi_2(0)} \right]^{1/2}, \\ &= K + P \pm |K - P|. \end{aligned}$$

Thus

$$c_3^2 = K, \quad c_4^2 = P, \qquad \text{or viceversa, if} \quad K - P \neq 0, \tag{4.28a}$$

$$c_3^2 = c_4^2 = K + P, \qquad \text{if} \quad K - P = 0. \tag{4.28b}$$

We consider only the case in which $K - P \neq 0$. That of $K - P = 0$ would correspond to a repeated root of (4.10), and (4.25) would have to be modified with the matrix in (4.24). Using (4.27) and the identity

$$(\varepsilon r^4 - B_k r^2 - A_k)(\varepsilon r^4 - D_k r^2 - C_k) + E_k^2 r^2 = 0,$$

which holds for any of the roots r in Corollary (4.2), one can show now that the following asymptotic estimates hold:

$$F(\omega_3(\varepsilon)) = O(\varepsilon^{-1/2}), \quad \Lambda(\omega_3(\varepsilon)) = O(1), \quad \Delta(\omega_3(\varepsilon)) = O(\varepsilon^{-1/2}), \tag{4.29a}$$
$$F(\omega_4(\varepsilon)) = O(\sqrt{\varepsilon}), \quad \Lambda(\omega_4(\varepsilon)) = O(1), \quad \Delta(\omega_4(\varepsilon)) = O(\sqrt{\varepsilon}), \tag{4.29b}$$

as $\varepsilon \to 0^+$. It follows now that

$$\begin{aligned}\frac{\det M_k^s(\varepsilon)}{\Pi_{j=1}^4 \cosh(\omega_j(\varepsilon)R)} &\sim \frac{c_1}{\varepsilon^{5/2}} \left(\Lambda(\omega_1(\varepsilon))\Delta(\omega_2(\varepsilon)) \tanh(\omega_2(\varepsilon)R)\right. \\ &\quad \left. -\Lambda(\omega_2(\varepsilon))\Delta(\omega_1(\varepsilon)) \tanh(\omega_1(\varepsilon)R)\right), \\ &\sim \frac{c_1}{\varepsilon^{5/2}}\, q_k^2 \left(p_1(\omega_1(0))p_2(\omega_2(0)) \tanh(\omega_2(0)R)\right. \\ &\quad \left. -p_1(\omega_2(0))p_2(\omega_1(0)) \tanh(\omega_1(0)R)\right),\end{aligned} \tag{4.30}$$

as $\varepsilon \to 0^+$, for some nonzero constant c_1, and where

$$\begin{aligned} p_1(r) &= \frac{KMr^2 + N(P - Kr^2)}{P - Kr^2}, \\ p_2(r) &= \frac{P(P - Lr^2)r - (M - N)Mr}{P - Kr^2}. \end{aligned}$$

If we let

$$f_\varepsilon^s(\lambda, q_k) = \frac{\varepsilon^{5/2} \det M_k^s(\varepsilon)}{c_1 q_k^2 \Pi_{j=1}^4 \cosh(\omega_j(\varepsilon)R)}, \tag{4.31}$$
$$\begin{aligned} f^s(\lambda, q_k) &= p_1(\omega_1(0))p_2(\omega_2(0)) \tanh(\omega_2(0)R) \\ &\quad -p_1(\omega_2(0))p_2(\omega_1(0)) \tanh(\omega_1(0)R), \end{aligned} \tag{4.32}$$

then the same analysis leading to (4.30) shows that:

$$f_\varepsilon^s(\lambda, q_k) = f^s(\lambda, q_k) + \varepsilon^\gamma g(\lambda, q_k, \varepsilon),$$

where $\gamma > 0$ and g is a continuous function over $(0,1] \times (0,\infty) \times [0,\infty)$. Thus we have:

Proposition 4.4. *Let $\lambda_0, \delta_1, \delta_2$ be such that $0 < \lambda_0 \leq 1$, $0 < \delta_1 < \delta_2 < \infty$. Then*

$$f_\varepsilon^s \to f^s, \quad \text{as } \varepsilon \to 0,$$

uniformly over $[\lambda_0, 1] \times [\delta_1, \delta_2]$.

Remark 4.5. *After multiplication by the denominators in p_1, p_2, the equation*

$$f^s(\lambda, q_k) = 0, \tag{4.33}$$

reduces to the same equation found in Simpson and Spector [22] for the critical loads of barrelling type.

Remark 4.6. *A similar result holds for the equation of buckling type* (4.26)*.*

We close this section with a result about the dimension of the kernel of the matrices $M_k^s(\varepsilon), M_k^a(\varepsilon)$. We use the notation $M_k^r(\lambda,\varepsilon)$, $r \in \{s,a\}$, to emphasize the dependence of M_k^r on both λ and ε.

Proposition 4.7. *Let $\lambda_k(\varepsilon)$ be a root of either* (4.25) *or* (4.26)*. Then for ε sufficiently small,* $\dim\ker M_k^r(\lambda_k(\varepsilon),\varepsilon) = 1$ *where $r \in \{s,a\}$.*

Proof: We do the analysis for the case $r = s$ the other one been similar. We let

$$\hat{M}_k^s(\lambda,\varepsilon) = \Theta(\lambda,\varepsilon) M_k^s(\lambda,\varepsilon), \quad \Theta(\lambda,\varepsilon) = \operatorname{diag}\left(\cosh(\omega_j(\lambda,\varepsilon)R)^{-1}\right).$$

Since $\Theta(\lambda,\varepsilon)$ is nonsingular, then

$$\dim\ker M_k^s(\lambda,\varepsilon) = \dim\ker \hat{M}_k^s(\lambda,\varepsilon).$$

If $\lambda_k(\varepsilon)$ is a root of (4.25), then $\dim\ker M_k^s(\lambda_k(\varepsilon),\varepsilon) \geq 1$. Using the asymptotic expansions (4.29), we get that as $\varepsilon \to 0^+$,

$$\hat{M}_k^s(\lambda_k(\varepsilon),\varepsilon) = \left[\begin{array}{llll} O(1) & O(1) & O(\varepsilon^{-3/2}) & O(\varepsilon^{-1/2}) \\ O(1) & O(1) & O(\varepsilon^{-1}) & O(\varepsilon^{-1}) \\ O(1) & O(1) & O(1) & O(1) \\ O(1) & O(1) & O(\varepsilon^{-1/2}) & O(\sqrt{\varepsilon}) \end{array}\right].$$

A simple inspection of the powers of ε in the last two columns of this matrix shows that these last two columns must be linearly independent as $\varepsilon \to 0^+$. Similarly neither column one or column two can be a linear combination of the last two columns for ε sufficiently small. Hence $\operatorname{rank} \hat{M}_k^s(\lambda_k(\varepsilon),\varepsilon) \geq 3$, but since $\dim\ker \hat{M}_k^s(\lambda_k(\varepsilon),\varepsilon) \geq 1$, the rank must be exactly three, i.e., $\dim\ker \hat{M}_k^s(\lambda_k(\varepsilon),\varepsilon) = 1$. □

5 The Complementing Condition

In this section we show that the linearized problem (4.1) satisfies the *complementing condition* for every value of λ. The complementing condition is an algebraic condition between the coefficients of the principal part of a differential operator and that of an associated boundary operator, that among other things guarantees certain apriory estimates on the solutions of the corresponding boundary value problem. We say that the complementing condition holds if the only exponentially decaying solution to a certain auxiliary boundary value problem on a half space, is the zero solution. Thompson in [26] made the observations that in the context of linearized elasticity the complementing condition is equivalent to the condition that all Rayleigh waves travel with nonzero velocity (see also [25]).

For the problem (4.1) the corresponding auxiliary boundary value problem on a half space is given by (see eg. [21])

$$\begin{aligned} \varepsilon\Delta^2\mathbf{h} &= \mathbf{0}, \quad \text{in } \mathcal{H}, && (5.1a)\\ -\varepsilon\Delta(\nabla\mathbf{h})\cdot\mathbf{n} - \varepsilon \overset{1}{\nabla}_s \cdot(\nabla^2\mathbf{h}\cdot\mathbf{n}) &= \mathbf{0}, \quad \varepsilon\nabla^2\mathbf{h} : \mathbf{nn} = \mathbf{0}, \quad \text{on } \partial\mathcal{H}, && (5.1b) \end{aligned}$$

where

$$\mathcal{H} = \left\{\mathbf{x}\in\mathbb{R}^2 \,:\, (\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{n} < 0\right\},$$

$\mathbf{x}_0 \in \partial\mathcal{B}_\infty$ is arbitrary but otherwise fixed, and $\mathbf{n} = \pm\mathbf{e}_1$ is the unit normal to $\partial\mathcal{B}_\infty$. We look for exponentially bounded solutions of this boundary value problem, i.e. solutions of the particular form

$$\mathbf{h}(\mathbf{x}) = \mathbf{z}(t)\mathrm{e}^{i\boldsymbol{\alpha}\cdot(\mathbf{x}-\mathbf{x}_0)}, \tag{5.2}$$

where $\boldsymbol{\alpha}\in\mathbb{R}^2$ is nonzero and perpendicular to $\mathbf{n}$, $t = -(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{n}$, and $\mathbf{z} \,:\, [0,\infty)\to\mathbb{R}^2$, with $\|\mathbf{z}(\cdot)\|$ bounded. Writing $\mathbf{z}(t) = (z_1(t), z_2(t))$, after some simplifications we get that the above boundary value problem reduces to:

$$\begin{gathered} z_1^{(4)}(t) - 2\alpha^2 z_1''(t) + \alpha^4 z_1(t) = 0, \quad z_2^{(4)}(t) - 2\alpha^2 z_2''(t) + \alpha^4 z_2(t) = 0, \quad t>0,\\ z_1''(0) = 0, \quad z_1'''(0) - 2\alpha^2 z_1'(0) = 0,\\ z_2''(0) = 0, \quad z_2'''(0) - 2\alpha^2 z_2'(0) = 0, \end{gathered}$$

where $\alpha\in\mathbb{R}$, $\alpha\neq 0$. This problem decouples to:

$$\begin{gathered} z^{(4)}(t) - 2\alpha^2 z''(t) + \alpha^4 z(t) = 0, \quad t>0,\\ z''(0) = 0, \quad z'''(0) - 2\alpha^2 z'(0) = 0. \end{gathered}$$

An easy computation shows that the bounded solutions of this problem have the form:

$$z(t) = c_1\mathrm{e}^{-|\alpha|t} + c_2 x\mathrm{e}^{-|\alpha|t}.$$

Applying the boundary conditions we get that c_1, c_2 must satisfy

$$\begin{pmatrix} \alpha^2 & -2|\alpha| \\ |\alpha|\,\alpha^2 & \alpha^2 \end{pmatrix}\begin{pmatrix} c_1\\ c_2\end{pmatrix} = \begin{pmatrix} 0\\ 0\end{pmatrix}.$$

Since the determinant of the coefficient matrix is $3\alpha^4\neq 0$, we get that $c_1 = c_2 = 0$ is the only solution, and thus that the only exponentially bounded solution of (5.1) is the zero solution. Hence (4.1) satisfies the complementing condition for any value of λ.

6 Local Bifurcation

We discuss now conditions for the existence of nontrivial solutions for the problem (3.19). The presentation in this section is greatly simplified as compared to that of the usual mixed traction–displacement boundary value problem of nonlinear elasticity ([14]), because by the presence of the second order gradient term in the stored energy function (2.1), the operator in (3.19) automatically satisfies both the strong ellipticity and the complementing conditions (see [16]).

In reference to the linear operator $L(\lambda)$ in equation (4.1), we have that standard results for elliptic systems (see [1], [2], [4], and [16]) imply that

$$\|\mathbf{h}\|_{\mathcal{Z}} \leq C \left[\|L(\lambda)[\mathbf{h}]\|_{\mathcal{Y}} + \|\mathbf{h}\|_{\mathcal{Y}_0} \right],$$

for any $\lambda \geq 0$ and for some constant $C > 0$ independent of $\mathbf{h}$ but depending on ε. By a Lemma of Peetre and a now standard homotopy argument (see [14]), we get:

Theorem 6.1. *The operator $L(\lambda) : \mathcal{Z} \to \mathcal{Y}$ is a self–adjoint Fredholm operator of index zero.*

Using the Fredholm property in this theorem, the proof of the following result is well known (see e.g. [7]):

Theorem 6.2 (Local Bifurcation). *Let the operator $G : \mathcal{U} \to \mathcal{Y}$ be given by* (3.17) *and assume that $\lambda_* \in (0, 1)$ is such that*

i) $\dim \ker L(\lambda_*) = 1$,

ii) if $\ker L(\lambda_*) = \operatorname{span}\{\mathbf{h}_*\}$, *and* $M = G_{\mathbf{u}\lambda}(\lambda_*, \mathbf{0})$, *then*

$$M\mathbf{h}_* \notin \operatorname{range} L(\lambda_*).$$

Then $(\lambda_*, \mathbf{0})$ is a bifurcation point of a local continuous branch of nontrivial solutions of (3.19).

Remark 6.3. *Remember that* $\ker L(\lambda_*) \neq \{\mathbf{0}\}$ *if and only if λ_* is a root of equations (4.25) or (4.26) for some $k \in \mathbb{N}$.*

We look now for an alternate characterization of condition (ii) in this theorem. For that we use the following identity which follows from the results in Section (2):

$$\begin{aligned}\int_{\mathcal{B}} \left(\varepsilon \nabla^2 \mathbf{f} \,\vdots\, \nabla^2 \mathbf{v} + \nabla \mathbf{v} : \mathbf{C}(\nabla \mathbf{f}_\lambda)[\nabla \mathbf{f}] \right) \mathrm{d}\mathbf{x} \quad &= \\ \int_{\mathcal{B}} \left(\varepsilon \Delta^2 \mathbf{f} - \operatorname{div} \mathbf{C}(\nabla \mathbf{f}_\lambda)[\nabla \mathbf{f}] \right) \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} \; &+ \; \int_{\mathcal{L}} (B_{\mathbf{u}}(\lambda, \mathbf{0})[\nabla \mathbf{f}]) \cdot \mathbf{v} \, \mathrm{d}s, \end{aligned} \tag{6.1}$$

for all $\mathbf{f}, \mathbf{v} \in \mathcal{Z}$, with $B_{\mathbf{u}}(\lambda, \mathbf{0})$ given in Proposition (3.2), and $\mathcal{L}$ is given in (3.2c)[4]. We now can prove the following:

[4]The boundary terms on $\mathcal{C}_b \cup \mathcal{C}_t$ are zero by the even–odd conditions (3.11) and the periodicity along the y direction.

Lemma 6.4. *Let λ_* be as in Theorem* (6.2)*. Then condition (ii) of Theorem* (6.2) *is equivalent to*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\int_{\mathcal{B}} \nabla\mathbf{h}_* : \mathbf{C}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{h}_*]\,\mathrm{d}\mathbf{x}\right]\Bigg|_{\lambda=\lambda_*} \neq 0, \tag{6.2}$$

which in turn is equivalent to λ_* being a simple root of either of the characteristic equations (4.25) *or* (4.26)*.*

Proof: Consider the linear functional $\psi : \mathcal{Y} \to \mathbb{R}$ given by:

$$\psi(\mathbf{w},\mathbf{g}) = \int_{\mathcal{B}} \mathbf{h}_* \cdot \mathbf{w}\,\mathrm{d}\mathbf{x} + \int_{\mathcal{L}} \mathbf{h}_* \cdot \mathbf{g}\,\mathrm{d}s.$$

If $(\mathbf{w},\mathbf{g}) \in \operatorname{range} L(\lambda_*)$, then there exists $\mathbf{h} \in \mathcal{Z}$ such that (c.f. (4.1)):

$$\begin{aligned} \varepsilon\Delta^2\mathbf{h} - \operatorname{div}\mathbf{C}(\nabla\mathbf{f}_{\lambda_*})[\nabla\mathbf{h}] &= \mathbf{w}, \\ B_{\mathbf{u}}(\lambda_*,\mathbf{0})[\mathbf{h}] &= \mathbf{g}. \end{aligned}$$

It follows now that

$$\begin{aligned} \psi(\mathbf{w},\mathbf{g}) &= \int_{\mathcal{B}} \mathbf{h}_* \cdot \left(\varepsilon\Delta^2\mathbf{h} - \operatorname{div}\mathbf{C}(\nabla\mathbf{f}_{\lambda_*})[\nabla\mathbf{h}]\right)\,\mathrm{d}\mathbf{x} \\ &\quad + \int_{\mathcal{L}} \mathbf{h}_* \cdot B_{\mathbf{u}}(\lambda_*,\mathbf{0})[\mathbf{h}]\,\mathrm{d}s \\ &= \int_{\mathcal{B}} \left(\varepsilon\nabla^2\mathbf{h} \,\vdots\, \nabla^2\mathbf{h}_* + \nabla\mathbf{h}_* : \mathbf{C}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{h}]\right)\mathrm{d}\mathbf{x}, \end{aligned}$$

where we used formula (6.1) with $\mathbf{v} = \mathbf{h}_*$ and $\mathbf{f} = \mathbf{h}$. Since the tensor $\mathbf{C}$ has the symmetry property:

$$\mathbf{H} : \mathbf{C}(\mathbf{F})[\mathbf{G}] = \mathbf{G} : \mathbf{C}(\mathbf{F})[\mathbf{H}],$$

for any second order tensors $\mathbf{F},\mathbf{G},\mathbf{H}$ with $\det\mathbf{F} > 0$, we can use (6.1) again to get that

$$\begin{aligned} \psi(\mathbf{w},\mathbf{g}) &= \int_{\mathcal{B}} \left(\varepsilon\nabla^2\mathbf{h} \,\vdots\, \nabla^2\mathbf{h}_* + \nabla\mathbf{h}_* : \mathbf{C}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{h}]\right)\mathrm{d}\mathbf{x} \\ &= \int_{\mathcal{B}} \left(\varepsilon\nabla^2\mathbf{h} \,\vdots\, \nabla^2\mathbf{h}_* + \nabla\mathbf{h} : \mathbf{C}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{h}_*]\right)\mathrm{d}\mathbf{x} \\ &= \int_{\mathcal{B}} \mathbf{h} \cdot \left(\varepsilon\Delta^2\mathbf{h}_* - \operatorname{div}\mathbf{C}(\nabla\mathbf{f}_{\lambda_*})[\nabla\mathbf{h}_*]\right)\,\mathrm{d}\mathbf{x} \\ &\quad + \int_{\mathcal{L}} \mathbf{h} \cdot B_{\mathbf{u}}(\lambda_*,\mathbf{0})[\mathbf{h}_*]\,\mathrm{d}s = 0, \end{aligned}$$

where for the last equality we used that $\mathbf{h}_*$ is a solution of (4.1) for $\lambda = \lambda_*$. This result together with Theorem (6.1) and condition (i) of Theorem (6.2) imply that $\operatorname{range} L(\lambda_*) =$

$\ker\psi$. It follows now that condition (ii) in Theorem (6.2) is equivalent to

$$-\int_{\mathcal{B}} \mathbf{h}_* \cdot \operatorname{div} \frac{\mathrm{d}\mathbf{C}}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{f}_{\lambda_*})[\nabla \mathbf{h}_*, \nabla \mathbf{f}'_{\lambda_*}]\,\mathrm{d}\mathbf{x} \\ + \int_{\mathcal{L}} \mathbf{h}_* \cdot \left(\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{f}_{\lambda_*})[\nabla \mathbf{h}_*, \nabla \mathbf{f}'_{\lambda_*}]\right) \cdot \mathbf{n} \neq 0 \quad ,$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[-\int_{\mathcal{B}} \mathbf{h}_* \cdot \operatorname{div} \mathbf{C}(\nabla \mathbf{f}_{\lambda})[\nabla \mathbf{h}_*]\,\mathrm{d}\mathbf{x} \right. \\ \left. + \int_{\mathcal{L}} \mathbf{h}_* \cdot (\mathbf{C}(\nabla \mathbf{f}_{\lambda})[\nabla \mathbf{h}_*]) \cdot \mathbf{n}\right]\Bigg|_{\lambda=\lambda_*} \neq 0 \quad ,$$

which after an integration by parts yields condition (6.2).

For the second part of the lemma, let $\left\{\mathbf{u}_\lambda^{(i)} \,:\, i = 1, \ldots, 4\right\}$ be a set of four linearly independent functions that satisfy (4.2a), (4.2b), (4.2e), depending continuously on λ, and such that

$$\mathbf{h}_* = \sum_{i=1}^{4} c_i^* \mathbf{u}_{\lambda_*}^{(i)},$$

with the $\{c_i^*\}$ not all zero. The functions $\left\{\mathbf{u}_\lambda^{(i)} \,:\, i = 1, \ldots, 4\right\}$ are given by

$$\mathbf{u}_\lambda^{(i)}(x, y) = (a_{k,i}(x)\cos(q_k y), b_{k,i}(x)\sin(q_k y))^t, \quad i = 1, \ldots, 4,$$

where k is the mode corresponding to $\mathbf{h}_*$ and $\{a_{k,i} \,:\, i = 1, \ldots, 4\}$ are given by (4.15) if $\mathbf{h}_*$ is of barrelling type and the roots in Corollary (4.2) are all distinct. The other possibilities, namely barrelling type with repeated roots, or buckling type with distinct roots, or buckling type with repeated roots, are handled similarly.

Let $\mathbf{c} : (\lambda_* - \delta, \lambda_* + \delta) \to \mathbb{R}^4$ be a smooth curve, to be chosen below, such that

$$\mathbf{c}(\lambda_*) = (c_1^*, c_2^*, c_3^*, c_4^*)^t = \mathbf{c}^*,$$

and define

$$\mathbf{h}_\lambda = \sum_{i=1}^{4} c_i(\lambda)\mathbf{u}_\lambda^{(i)}.$$

Using formula (6.1), the symmetry property of $\mathbf{C}(\nabla \mathbf{f}_\lambda]$, and that $\mathbf{h}_*$ is a solution of (4.1) for $\lambda = \lambda_*$, we get that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\int_{\mathcal{B}} \nabla \mathbf{h}_* : \mathbf{C}(\nabla \mathbf{f}_\lambda)[\nabla \mathbf{h}_*]\,\mathrm{d}\mathbf{x}\right]\Bigg|_{\lambda=\lambda_*} \\ = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\int_{\mathcal{B}} \left(\varepsilon \nabla^2 \mathbf{h}_\lambda \vdots \nabla^2 \mathbf{h}_\lambda + \nabla \mathbf{h}_\lambda : \mathbf{C}(\nabla \mathbf{f}_\lambda)[\nabla \mathbf{h}_\lambda]\right)\,\mathrm{d}\mathbf{x}\right]\Bigg|_{\lambda=\lambda_*}$$

The function $\mathbf{h}_\lambda$ satisfy the even–odd conditions (3.11), is $2L$ periodic in y but need not satisfy the boundary condition $\varepsilon\nabla^2\mathbf{u} : \mathbf{nn} = \mathbf{0}$ on $\mathcal{L}$. Thus a modification of formula (6.1) taking this into consideration yields that

$$\begin{aligned}
\int_{\mathcal{B}} \Big(\varepsilon\nabla^2\mathbf{h}_\lambda \vdots \nabla^2\mathbf{h}_\lambda \; + \nabla\mathbf{h}_\lambda : \mathbf{C}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{h}_\lambda]\Big)\,\mathrm{d}\mathbf{x} & \\
= & \int_{\mathcal{B}} \left(\varepsilon\Delta^2\mathbf{h}_\lambda - \operatorname{div}\mathbf{C}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{h}_\lambda]\right)\cdot\mathbf{h}_\lambda\,\mathrm{d}\mathbf{x} \\
& + \int_{\mathcal{L}} \left[(B_{\mathbf{u}}(\lambda,\mathbf{0})[\mathbf{h}_\lambda])\cdot\mathbf{h}_\lambda + \varepsilon(\nabla^2\mathbf{h}_\lambda : \mathbf{nn})\cdot \mathrm{D}\mathbf{h}_\lambda\right]\,\mathrm{d}s \\
= & \int_{\mathcal{L}} \left[(B_{\mathbf{u}}(\lambda,\mathbf{0})[\mathbf{h}_\lambda])\cdot\mathbf{h}_\lambda + \varepsilon(\nabla^2\mathbf{h}_\lambda : \mathbf{nn})\cdot \mathrm{D}\mathbf{h}_\lambda\right]\,\mathrm{d}s,
\end{aligned}$$

where for the last equality we have used that since $\mathbf{h}_\lambda$ satisfies (4.2a), (4.2b), (4.2e), then

$$\varepsilon\Delta^2\mathbf{h}_\lambda - \operatorname{div}\mathbf{C}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{h}_\lambda] = \mathbf{0}, \quad \text{in } \mathcal{B}.$$

Hence

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\int_{\mathcal{B}} \nabla\mathbf{h}_* : \mathbf{C}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{h}_*]\,\mathrm{d}\mathbf{x}\right]\Bigg|_{\lambda=\lambda_*} \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\int_{\mathcal{L}} \left[(B_{\mathbf{u}}(\lambda,\mathbf{0})[\mathbf{h}_\lambda])\cdot\mathbf{h}_\lambda + \varepsilon(\nabla^2\mathbf{h}_\lambda : \mathbf{nn})\cdot \mathrm{D}\mathbf{h}_\lambda\right]\,\mathrm{d}s\right]\Bigg|_{\lambda=\lambda_*}
\end{aligned}$$

An easy computation now based on formulas (4.15) and (4.23) gives that:

$$\begin{aligned}
\int_{\mathcal{L}} [(B_{\mathbf{u}}(\lambda,\mathbf{0})[\mathbf{h}_\lambda])\cdot\mathbf{h}_\lambda \; + \; \varepsilon(\nabla^2\mathbf{h}_\lambda : \mathbf{nn})\cdot \mathrm{D}\mathbf{h}_\lambda]\,\mathrm{d}s & \\
= 2\int_0^L & [(B_{\mathbf{u}}(\lambda,\mathbf{0})[\mathbf{h}_\lambda])\cdot\mathbf{h}_\lambda + \\
& \varepsilon(\nabla^2\mathbf{h}_\lambda : \mathbf{nn})\cdot \mathrm{D}\mathbf{h}_\lambda]\,|_{x=R}\,\mathrm{d}y \\
= L\mathbf{c}^t & (\lambda)A(\lambda)\mathbf{c}(\lambda),
\end{aligned} \tag{6.3}$$

where

$$A(\lambda) = \mathbf{\Theta}_k^t(\lambda)M_k^a(\lambda), \quad \mathbf{\Theta}_k(\lambda) = \begin{bmatrix} a_{k,1}(R) & a_{k,2}(R) & a_{k,3}(R) & a_{k,4}(R) \\ b_{k,1}(R) & b_{k,2}(R) & b_{k,3}(R) & b_{k,4}(R) \\ a'_{k,1}(R) & a'_{k,2}(R) & a'_{k,3}(R) & a'_{k,4}(R) \\ b'_{k,1}(R) & b'_{k,2}(R) & b'_{k,3}(R) & b'_{k,4}(R) \end{bmatrix}.$$

The matrix $\Theta_k(\lambda)$ is nonsingular by the linear independence of the $\{\mathbf{u}_\lambda\}$. Note that since $L(\lambda_*)[\mathbf{h}_*] = \mathbf{0}$, then $\mathbf{c}^*$ is an eigenvector of $A(\lambda_*)$ corresponding to the eigenvalue zero. We take the curve $\mathbf{c}(\cdot)$ to satisfy:

$$A(\lambda)\mathbf{c}(\lambda) = \mu_1(\lambda)\mathbf{c}(\lambda), \quad \lambda \in (\lambda_* - \delta, \lambda_* + \delta),$$

where $\mu_1 : (\lambda_* - \delta, \lambda_* + \delta) \to \mathbb{R}$ is smooth, $\mu_1(\lambda_*) = 0$, and $\mathbf{c}(\lambda_*) = \mathbf{c}^*$. With this selection of $\mathbf{c}(\cdot)$ and using (6.3) we get that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\int_{\mathcal{B}} \nabla \mathbf{h}_* : \mathbf{C}(\nabla \mathbf{f}_\lambda)[\nabla \mathbf{h}_*]\,\mathrm{d}\mathbf{x}\right]\Bigg|_{\lambda=\lambda_*} &= L\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\mu_1(\lambda)\mathbf{c}^t(\lambda)\mathbf{c}(\lambda)\right]_{\lambda=\lambda_*} \\ &= \mu_1'(\lambda_*)\mathbf{c}^t(\lambda_*)\mathbf{c}(\lambda_*) \end{aligned}$$

Thus (6.2) is equivalent to $\mu_1'(\lambda_*) \neq 0$. If zero is an eigenvalue of $A(\lambda_*)$ of multiplicity $m \geq 1$ then

$$\det A(\lambda) = g(\lambda)\prod_{i=1}^{m} \mu_i(\lambda), \quad g(\lambda_*) \neq 0,$$

with g and $\{\mu_i(\cdot)\}$ smooth ($\mu_1(\cdot)$ as above), and $\mu_i(\lambda_*) = 0$, $i = 1, \ldots, m$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\det A(\lambda)|_{\lambda=\lambda_*} = g(\lambda_*)\sum_{i=1}^{m} \mu_i'(\lambda_*)\prod_{j\neq i} \mu_j(\lambda_*),$$

which is nonzero if and only if $m = 1$ and $\mu_1'(\lambda_*) \neq 0$. Hence (6.2) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\det A(\lambda)|_{\lambda=\lambda_*} \neq 0,$$

which in turn is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\det M_k^a(\lambda)|_{\lambda=\lambda_*} \neq 0,$$

by the nonsingularity of $\Theta_k(\lambda)$. $\square$

7 An Example: Blatz–Ko Type Materials

As we mentioned before, the presence of the second order gradient term in the stored energy function (2.1) simplifies greatly the global analysis, because the operator in (3.19) automatically satisfies both the strong ellipticity and the complementing conditions. However the local bifurcation analysis becomes extremely difficult due to the complexity of the characteristic equations (4.25) and (4.26). (Each determinant has 36 terms to be accounted for!) This is so even for specific materials like the Blatz–Ko type, cf. [5], considered in this section making it necessary to check the local bifurcation conditions numerically.

We assume that the stored energy function W in (2.1), which corresponds to the problem with $\varepsilon = 0$, is of Blatz–Ko type, i.e., is given by:

$$W(\mathbf{F}) = \frac{1}{2}\mathbf{F} : \mathbf{F} + \frac{1}{m}(\det \mathbf{F})^{-m}, \tag{7.1}$$

where $m > 0$. In this case (4.3) reduces to:

$$K = m+2, \quad N = m\nu^{1/2}, \quad Q = 1+(m+1)\nu, \quad P = 1, \quad M = (m+1)\nu^{1/2}, \tag{7.2}$$

where

$$\nu = \lambda^{-4\left[\frac{m+1}{m+2}\right]}.$$

We have as well that (4.8) and (4.9) simplify to:

$$A_k = -(\varepsilon q_k^4 + q_k^2), \quad B_k = 2\varepsilon q_k^2 + m + 2,$$
$$C_k = -(\varepsilon q_k^4 + (1+(m+1)\nu)q_k^2), \quad D_k = 2\varepsilon q_k^2 + 1, \quad E_k = (m+1)\nu^{1/2} q_k.$$

The roots of (4.10) are given now by $\varpi_j = q_k^2 \varrho_j$, $j = 1,2,3,4$ where

$$\varrho_1 = 1, \quad \varrho_2 = 1 + \frac{m+2}{2\varepsilon q_k^2} - \frac{\left[(m+2)^2 + 4\varepsilon(m+1)(1-\nu)q_k^2\right]^{1/2}}{2\varepsilon q_k^2},$$
$$\varrho_3 = 1 + \frac{m+2}{2\varepsilon q_k^2} + \frac{\left[(m+2)^2 + 4\varepsilon(m+1)(1-\nu)q_k^2\right]^{1/2}}{2\varepsilon q_k^2},$$
$$\varrho_4 = 1 + \frac{1}{\varepsilon q_k^2}.$$

Note that $\varrho_4 > \varrho_1$ and

1. $\varrho_2 = \varrho_1$ at $\lambda = 1$,
2. $\varrho_2 = \varrho_4$ at
$$\lambda_u = \left[1 + \frac{1}{\varepsilon q_k^2}\right]^{-\frac{(m+2)}{4(m+1)}},$$
3. $\varrho_3 = \varrho_2$ at
$$\lambda_d = \left[1 + \frac{(m+2)^2}{4\varepsilon(m+1)q_k^2}\right]^{-\frac{(m+2)}{4(m+1)}}, \tag{7.3}$$

with $\lambda_d < \lambda_u$.

For ε and q_k fixed, we have that ϱ_1, ϱ_4 are constant. As we let λ decrease from one to λ_u, we get that ϱ_2 increases from ϱ_1 to ϱ_4. As we further decrease λ from λ_u to λ_d, ϱ_2 increases from ϱ_4 to $\varrho_3 = 1 + (m+2)/(2\varepsilon q_k^2)$. As λ decrease from one to λ_d, ϱ_3 decreases from its maximum value down to $1 + (m+2)/(2\varepsilon q_k^2)$. As we further decrease λ from λ_d to zero, both ϱ_2, ϱ_3 become complex conjugates, with constant real part given by $1 + (m+2)/(2\varepsilon q_k^2)$, and going to infinity in modulus as $\lambda \searrow 0$. We summarize these observations in the diagram in Figure (2). In fact with

$$a = 1 + \frac{m+2}{2\varepsilon q_k^2}, \quad b = \frac{\left[|(m+2)^2 + 4\varepsilon(m+1)(1-\nu)q_k^2|\right]^{1/2}}{2\varepsilon q_k^2},$$

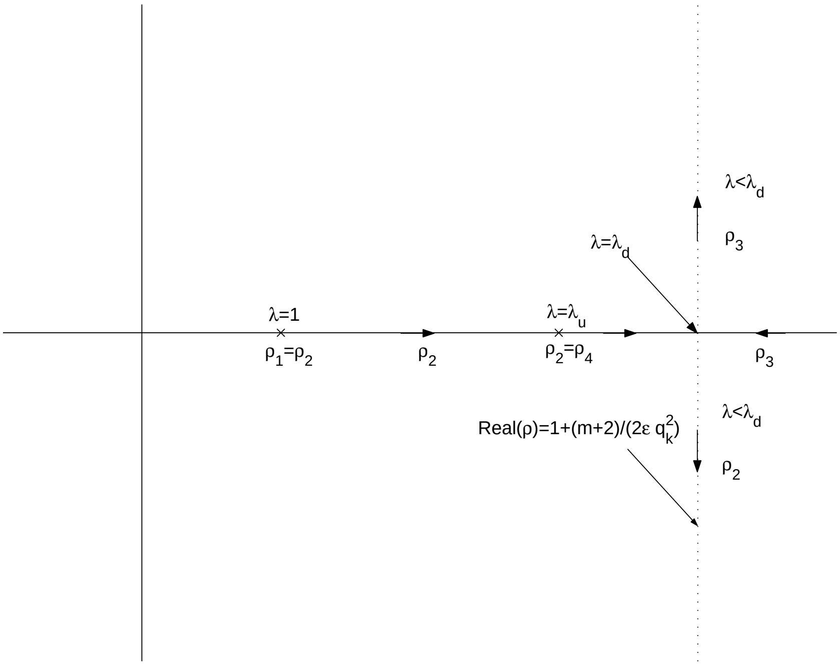

Figure 2: Diagram of ϱ_j, $j = 1, 2, 3, 4$ as $\lambda \searrow 0$ from $\lambda = 1$.

then for $\lambda < \lambda_d$ we can write $\varrho_2 = a - b\imath, \varrho_3 = a + b\imath$. Thus with $r = \sqrt{a^2 + b^2}$ we have that

$$\varrho_2^{1/2} = \sqrt{\frac{r+a}{2}} - \imath\sqrt{\frac{r-a}{2}}, \quad \varrho_3^{1/2} = \sqrt{\frac{r+a}{2}} + \imath\sqrt{\frac{r-a}{2}}.$$

It follows now that

$$\varrho_2^{1/2} \sim \sqrt{\frac{b}{2}}\,(1-\imath)\,, \quad \varrho_3^{1/2} \sim \sqrt{\frac{b}{2}}\,(1+\imath)\,, \quad \text{as } \lambda \searrow 0,$$

where we have used the principal part of the square root function.

If we view λ_d as a function of q_k, this curve divides the (q_k, λ) plane in two regions: one in which ϱ_3, ϱ_4 are real (to the left of the curve), and another in which they are complex (to right of the curve). (See Figure (3).) Thus above this curve the determinants in (4.25) and (4.26) are real–valued functions and below the curve they are purely imaginary valued functions. Thus from the numerical point of view, when looking for the roots of (4.25) or (4.26), we are essentially dealing with real–valued functions.

Let $M_k^s(\lambda)$ be given by the matrix (4.22) for those values of $\lambda \neq \lambda_d, \lambda_u, 1$, and by the corresponding formula (4.24) if $\lambda = \lambda_d$ or $\lambda = \lambda_u$ or $\lambda = 1$. A similar definition is given for $M_k^a(\lambda)$ using (4.21) and (4.23). The functions $\det M_k^s(\cdot), \det M_k^a(\cdot)$ need not be continuous. However if we let

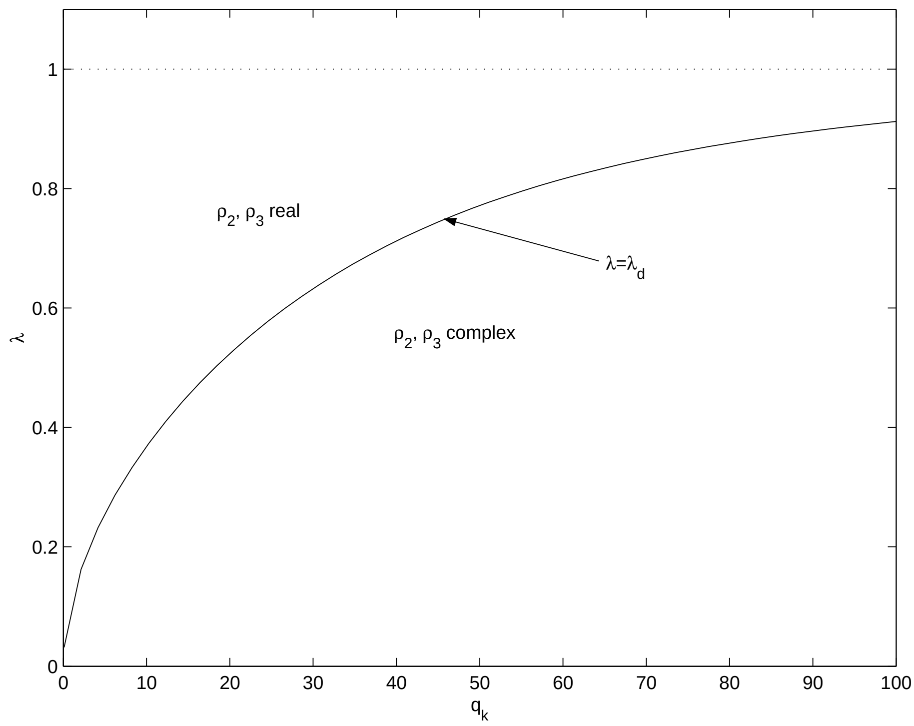

Figure 3: Curve $\lambda = \lambda_d$ as a function of q_k.

$$\hat{M}_k^s(\lambda) = \begin{cases} \dfrac{\det M_k^s(\lambda)}{\Pi_{j\neq 2}(\omega_{j,k}(\lambda) - \omega_{2,k}(\lambda))} & , \quad \lambda \neq \lambda_d, \lambda_u, 1, \\[2ex] \dfrac{\det M_k^s(\lambda)}{(\omega_{1,k}(\lambda) - \omega_{2,k}(\lambda))(\omega_{4,k}(\lambda) - \omega_{2,k}(\lambda))} & , \quad \lambda = \lambda_d, \\[2ex] \dfrac{\det M_k^s(\lambda)}{(\omega_{1,k}(\lambda) - \omega_{2,k}(\lambda))(\omega_{3,k}(\lambda) - \omega_{2,k}(\lambda))} & , \quad \lambda = \lambda_u, \\[2ex] \dfrac{\det M_k^s(\lambda)}{(\omega_{4,k}(\lambda) - \omega_{2,k}(\lambda))(\omega_{3,k}(\lambda) - \omega_{2,k}(\lambda))} & , \quad \lambda = 1, \end{cases} \tag{7.4}$$

with a similar definition for $\hat{M}_k^a(\cdot)$, then these functions are continuous functions of λ and the critical loads or possible bifurcation points for the problem (3.5) for Blatz–Ko type materials are given by the roots of $\hat{M}_k^s(\cdot)$ (barrelling type) or $\hat{M}_k^a(\cdot)$ (buckling type).

Note that the dependence of $\hat{M}_k^a(\cdot)$, $\hat{M}_k^s(\cdot)$ on the mode index k comes through q_k. Thus we let q_k to vary continuously on $(0, \infty)$ and let

$$\hat{M}_a(\lambda, q_k) = \hat{M}_k^a(\lambda), \quad \hat{M}_s(\lambda, q_k) = \hat{M}_k^s(\lambda), \quad (\lambda, q_k) \in (0, 1] \times (0, \infty).$$

We show in Figure (4) the zero contour plots for the surfaces $\hat{M}_a(\lambda, q_k)$ (solid curve) and $\hat{M}_s(\lambda, q_k)$ (dotted curve), and the curve $\lambda = \lambda_\infty$ (dashed curve) in the (λ, q_k) plane for the case $m = 13.3$, $R = 1$ and values of $\varepsilon = 10^{-j}$, $j = 4, 5, 6, 7$. Based on these figures we can conjecture the following:

i) For any given $\varepsilon > 0$ and $L > 0$, there are only a finite number of barrelling or buckling type critical loads given by the intersections of the vertical lines $q_k = k\pi/L$, $k = 1, 2, \ldots$, with the contour curves.

ii) As $\varepsilon \searrow 0$, the number of barrelling or buckling type critical loads increases.

iii) As $\varepsilon \searrow 0$, both contour curves of buckling and barrelling type, become horizontally asymptotical to the line $\lambda = \lambda_\infty$, where λ_∞ is the value at which the complementing condition for the problem with $\varepsilon = 0$ is violated[5], cf. [24].

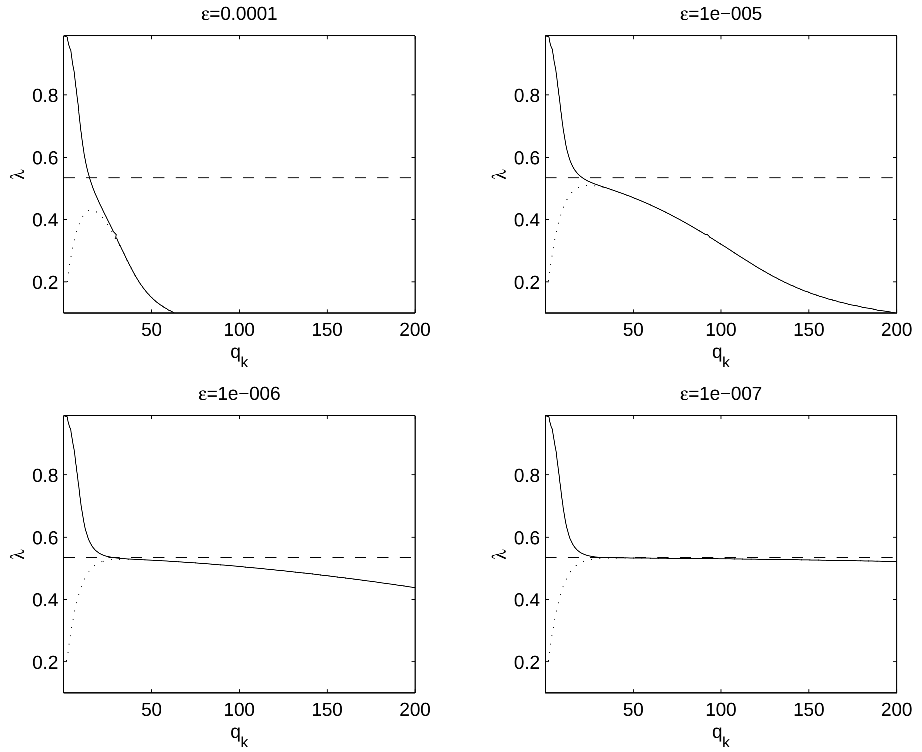

Figure 4: Zero contour plots for the surfaces $\hat{M}_a(\lambda, q_k)$ (solid curve) and $\hat{M}_s(\lambda, q_k)$ (dotted curve), and the curve $\lambda = \lambda_\infty$ (dashed curve) as functions of q_k for the case $m = 13.3$, $R = 1$ and values of $\varepsilon = 10^{-j}$, $j = 4, 5, 6, 7$.

In Simpson and Spector [24] it is shown that equation (4.33) and the corresponding one for buckling type deformations, has a unique solution which is simple for each mode

[5] $\lambda_\infty = 0.5339$ approximately for the Blatz–Ko type material with $m = 13.3$.

$q_k = k\pi/L$, $k = 1, 2, \ldots$. We let λ_k^s and λ_k^a be the corresponding solutions of (4.33) and the equation for buckling type solutions respectively. Using Proposition (4.4) and the results in Simpson and Spector [24] we have the following:

Theorem 7.1. *Let the material of the slab be given by* (7.1) *for* $\varepsilon = 0$. *For any integer* $k \geq 1$ *let* λ_k^s *and* λ_k^a *be as above. Then there exists* $\varepsilon_k > 0$ *such that equations* (4.25) *and* (4.26) *with* $q_k = k\pi/L$, *both have at least one solution* $\lambda_{k,\varepsilon}^s$ *and* $\lambda_{k,\varepsilon}^a$ *respectively for each* $\varepsilon \in (0, \varepsilon_k]$. *Moreover if* $\{\eta_j\}$ *is such that* $\eta_j \to 0$, $\eta_j \in (0, \varepsilon_k]$ *for all* j, *then the corresponding sequences of solutions* $\left\{\lambda_{k,\eta_j}^s\right\}$ *and* $\left\{\lambda_{k,\eta_j}^a\right\}$ *have subsequences converging to* λ_k^s *and* λ_k^a *respectively.*

Proof: Since λ_k^s is a simple root of $f^s(\lambda, q_k) = 0$, there exist $0 < \lambda_1 < \lambda_2 \leq 1$ such that $f^s(\lambda_1, q_k) f^s(\lambda_2, q_k) < 0$. It follows from (7.3) that we can choose $\varepsilon_{k,1} > 0$ such that $\lambda_d \leq \lambda_1$ for any $\varepsilon \leq \varepsilon_{k,1}$. Thus f_ε^s assumes real values over the set $\{(\lambda, q_k) \,:\, \lambda_1 \leq \lambda \leq \lambda_2\}$. Moreover, since $f_\varepsilon^s(\cdot, q_k) \to f^s(\cdot, q_k)$ uniformly as $\varepsilon \to 0$ over $[\lambda_1, \lambda_2]$, it follows that there exists $\varepsilon_k < \varepsilon_{k,1}$ such that $f_\varepsilon^s(\lambda_1, q_k) f_\varepsilon^s(\lambda_2, q_k) < 0$ for $0 < \varepsilon \leq \varepsilon_k$. Hence $f_\varepsilon^s(\lambda, q_k) = 0$ has at least one solution $\lambda_{k,\varepsilon}^s \in (0, 1]$ for any $\varepsilon \in (0, \varepsilon_k]$. The result on the convergence to λ_k^s of the subsequence of solutions, follows as well from the uniform convergence of $\{f_\varepsilon^s(\cdot, q_k)\}$ to $f^s(\cdot, q_k)$ and the fact that λ_k^s is the only solution of $f^s(\lambda, q_k) = 0$. The corresponding result for (4.26) can be shown similarly. $\square$

Remark 7.2. *In general we can not conclude that the root* $\lambda_{k,\varepsilon}^s$ *of* $f_\varepsilon^s(\lambda, q_k) = 0$ *predicted by the theorem, is simple (or even unique). This is an important condition for the analysis of local bifurcation. Our numerical results for the case* $m = 13.3$ *of* (7.1) *together with Proposition* (4.7) *show that for this particular case, the buckling–type roots are indeed simple and those of barrelling–type are generically simple, for* ε *small enough.*

Remark 7.3. *The numerical results described in (i)–(iii) above indicate that the sequence* $\{\varepsilon_k\}$ *in the theorem tends to zero as* $k \to \infty$.

8 Final Remarks

We have noticed that in the context of elasticity there is a recurrent relation between violation of the complementing condition and bifurcation that has not yet been study in depth. In elasticity and many other areas of applications, the problems under consideration often can be written abstractly as

$$G(\lambda, \mathbf{u}) = \mathbf{0}, \quad \lambda \in (0, \infty),$$

where $\mathbf{u}$ denotes the displacement from the corresponding trivial solution, λ is some physical parameter, and G is a differentiable nonlinear operator between appropriate Banach spaces with $G(\lambda, \mathbf{0}) = \mathbf{0}$. Recall that a necessary condition for bifurcation at

$\lambda = \lambda_*$ is that the linearized problem $G_{\mathbf{u}}(\lambda_*, \mathbf{0}) \cdot \mathbf{v} = \mathbf{0}$ has nontrivial solutions $\mathbf{v}$. We have observed for some boundary value problems (eg. [24], [18], [12], and [19]) that if λ_c is an accumulation point of $\{\lambda : G_{\mathbf{u}}(\lambda, \mathbf{0}) \cdot \mathbf{v} = \mathbf{0}$ has nontrivial solutions$\}$, then the linearized boundary value problem $G_{\mathbf{u}}(\lambda_c, \mathbf{0}) \cdot \mathbf{v} = \mathbf{0}$ fails to satisfy the complementing condition, cf. Section 5 above. This implies that if (λ_n) is a sequence of values of the parameter λ in a compact interval, such that for each n, $G(\lambda, \mathbf{u}) = \mathbf{0}$ has a branch of nontrivial solutions bifurcating from $(\lambda_n, \mathbf{0})$, then those branches locally accumulate at points where the linearized problem fails to satisfy the complementing condition. This is actually consistent with previous physical interpretations of the complementing condition as associated with oscillatory instabilities at the boundary, but it may also suggest a limitation in the theory of elasticity based on first order gradients to model such phenomena.

We showed, in Section 5 above, that the corresponding linearization along the trivial solution of the problem studied in this paper satisfies the complementing condition for all values of $\lambda \geq 0$. Hence, we expect that there exists only a finite number of possible bifurcation points, $(\lambda, \mathbf{0})$, with $\lambda \in [0, 1]$. Indeed, our numerical results for Blatz–Ko type materials, cf. Section 7, indicate that when a quadratic second–gradient term is added to the stored–energy function, there are only a finite number of possible bifurcation points in the interval $\lambda \in [0, 1]$. Furthermore, we observed that for this example the number of possible bifurcation points, $(\lambda, \mathbf{0})$, were $\lambda \in [0, 1]$, increases monotonically as $\varepsilon \to 0$ and they accumulate precisely at a point $(\lambda_c, \mathbf{0})$ at which the complementing condition condition for the problem with $\varepsilon = 0$ fails along the trivial solution branch, cf. [22]. Therefore, our analysis provides more evidence that suggests that failure of the complementing condition induces the existence of an infinite number of bifurcating branches accumulating at the value of the parameter λ at which the complementing condition fails.

In general it would be interesting to study and clarify the relationship between bifurcating branches of nontrivial solutions and violation of the complementing condition in the context of more general boundary value problems. For example, it would be interesting to study the following: if λ_c is a value of λ at which the linearized boundary value problem fails to satisfy the complementing condition, is it true that there exists an infinite sequence of bifurcation points that accumulates at λ_c? We shall pursue these questions in a future work.

Our analysis of this problem also indicates that for higher order gradient models the functional analytic aspects are greatly simplified due to the non–violation of the complementing condition. This has implications for a global analysis that we shall explore in a forthcoming paper. However the verification of the conditions for local bifurcation becomes extremely difficult due to the complexity of the corresponding characteristic equations (4.25) and (4.26).

The problem of the convergence of the local branches of nontrivial solutions as $\varepsilon \to 0$ remains as a major open problem. The solution of this problems would require certain apriori estimates on the solutions of (3.3) uniform in ε. However there are serious technical difficulties in obtaining such estimates due to the singular limit in equations (3.3) as $\varepsilon \to 0$, where the operator for $\varepsilon > 0$ is of fourth order while that for the problem with $\varepsilon = 0$ is of second order.

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