A model for the dynamics of a kite with an arbitrary lift coefficient

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Abstract

In this paper we consider a generalization of a model for kite flight studied by Adomaitis (1989). In that paper the lift coefficient C_l is taken as $b\sin 2\theta$ where θ is the angle that the string of the kite, assumed to be completely straight, makes with the horizontal. In our analysis we assumed on C_l only properties observed experimentally in wind tunnels, namely that C_l is concave downward. We show that in this general scenario there can be multiple turning points for the curve of steady states as a certain parameter (inversely proportional to the square of wind speed) changes. We show as well that there can be multiple branches of stable steady states solutions and Hopf bifurcations.

1 Introduction

In this paper we consider a generalization of a model for kite flight studied originally by Adomaitis [1]. Experiments with wind tunnels show that the lift coefficient function is in general concave downward. In [1] the lift coefficient is taken as $b\sin 2\theta$ where θ is the angle that the string of the kite, assumed to be completely straight, makes with the horizontal. This assumption on the lift coefficient although consistent with the observed behavior is rather restrictive and rules out many interesting situations that could have important physical consequences. In our analysis we assumed on only properties observed experimentally in wind tunnels, namely that the lift coefficient function C_l is concave downward. We show that in this general scenario there can be multiple turning points for the curve of steady states vs a certain parameter β which is inversely proportional to

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the square of wind speed. We show as well that there can be multiple branches of stable steady states solutions and Hopf bifurcations.

In Section (2) we present the basic equations for the equilibrium states of the kite model that we consider. We state our main hypotheses on the lift coefficient function C_l, namely conditions (4). Under these general conditions we establish the existence of turning points for the curve of θ vs β. In Section (3) we describe the dynamical system governing the motion of the kite, the equilibrium states of which are precisely those discussed in Section (2). We then show that there exists at least one value of θ for which there is a Hopf bifurcation, i.e., a branch of periodic solutions bifurcates from this point. We also show that the stability pattern of the states states, changes precisely at the points of Hopf bifurcation.

2 Equilibrium States

The forces acting on the kite during flight are the lift (F_l), drag (F_d), tension of the string (T), and the weight of the kite (mg). On equilibrium, these forces must balance along the vertical and horizontal directions, leading to the following system of equations:

$$F_l = T\sin\theta + mg, \quad F_d = T\cos\theta, \tag{1}$$

where θ is the angle between the horizontal and the kite string assumed to be straight. The drag and lift forces are given by the relations:

$$F_d = \frac{1}{2}\rho C_d(\theta)\lambda^2 A, \quad F_l = \frac{1}{2}\rho C_l(\theta)\lambda^2 A, \tag{2}$$

where C_d, C_l are the drag and lift coefficients respectively, ρ is the air density, λ is the wind speed, and A is the area of the kite.

If we assumed that string is normal to the surface of the kite, a reasonable model for C_d is given by:

$$C_d(\theta) = a\cos\theta, \tag{3}$$

for some positive constant a.

Experiments in wind tunnels give that C_l is in general a nonnegative concave downward function. Thus we only assume that

$$C_l(0) = 0, \tag{4a}$$
$$C_l'(\theta) > 0, \quad 0 \leq \theta < \theta_d, \tag{4b}$$
$$C_l'(\theta) < 0, \quad \theta_d < \theta \leq \theta_u, \tag{4c}$$
$$C_l''(\theta) < 0, \quad 0 < \theta < \theta_u, \tag{4d}$$

for some $\theta_d < \theta_u < \pi/2$.

If we use (2) and (3) to eliminate T from (1), we arrive at the following expression:

$$\beta = \frac{C_l(\theta)}{a} - \sin\theta, \tag{5}$$

where

$$\beta = \frac{2mg}{a\rho\lambda^2 A}. \tag{6}$$

Equation (5) describes the equilibrium states of the kite (values of θ) given a value of β. In general there can be multiple values of θ satisfying (5) given a value of β. Since β can be view as a function of θ, the values of θ at which $\mathrm{d}\beta/\mathrm{d}\theta = 0$ represent turning points for the graph of θ vs β. These turning points are the solutions of the equation:

$$C_l'(\theta) = a\cos\theta. \tag{7}$$

We now have:

Lemma 2.1. *Assume that C_l satisfies assumptions* (4) *and that $C_l'(0) > a$. Then* (7) *has at least one solution.*

Proof: Let $g(\theta) = C_l'(\theta) - a\cos\theta$. Then $g(0) = C_l'(0) - a > 0$. Moreover, since $C_l'(\theta_u) < 0$, we have that $g(\theta_u) < 0$. Thus g has a least one root in $[0, \theta_u]$. □

Note that, If the number of solutions is finite and each root is simple, then the number of solutions of (7) must be odd.

Under the hypothesis of the lemma, $\beta'(0) > 0$. We further assume that θ_u in (4) is such that

$$\beta(\theta) > 0, \quad \theta \in (0, \theta_u), \quad \beta(\theta_u) = 0. \tag{8}$$

3 Stability Analysis

The dynamics of the kite is given (see e.g. [3], [4]) by the following system of equations:

$$\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \eta(t), \tag{9a}$$

$$\begin{aligned}\frac{\mathrm{d}\eta(t)}{\mathrm{d}t} = & -\frac{g}{r}\cos\theta(t) + \frac{1}{2mr}\rho A\lambda_e^2(\theta(t),\eta(t))\cos\theta_e(\theta(t),\eta(t)) \times \\ & \left(C_l(\theta_e(\theta(t),\eta(t))) - a\sin\theta_e(\theta(t),\eta(t))\right),\end{aligned} \tag{9b}$$

where the *effective* quantities λ_e, θ_e are given by:

$$\lambda_e^2(\theta,\eta) = \lambda^2 + r^2\eta^2 + 2r\lambda\eta\sin\theta, \quad \lambda_e(\theta,\eta)\cos\theta_e(\theta,\eta) = \lambda\cos\theta. \tag{10}$$

Note that if $\eta \equiv 0$ so that $\theta(t) = \text{constant}$, then $\lambda_e \equiv \lambda$ and $\theta_e = \theta$. Hence the steady states or equilibrium states of the system (9) are precisely the solutions of (5).

Let θ_s be any solution of the equilibrium equation (5). We study the linearization of the system (9) about the steady state $(\theta,\eta) = (\theta_s, 0)$. For this purpose we use the following partial derivatives which can be computed from (10):

$$\frac{\partial\lambda_e}{\partial\theta}(\theta_s,0) = 0, \quad \frac{\partial\lambda_e}{\partial\eta}(\theta_s,0) = r\sin\theta_s, \tag{11a}$$

$$\frac{\partial\theta_e}{\partial\theta}(\theta_s,0) = 1, \quad \frac{\partial\theta_e}{\partial\eta}(\theta_s,0) = \frac{r}{\lambda}\cos\theta_s. \tag{11b}$$

If we denote by $H(\theta, \eta)$ the right hand side of (9c), then using (11) we get that:

$$\frac{\partial H}{\partial \theta}(\theta_s, 0) = \frac{\rho A \lambda^2}{2mr} \cos\theta_s \left(C_l'(\theta_s) - a\cos\theta_s\right) \equiv \Phi(\theta_s), \tag{12a}$$
$$\frac{\partial H}{\partial \eta}(\theta_s, 0) = \frac{\rho A \lambda}{2m} \cos\theta_s \left(C_l(\theta_s)\sin\theta_s + C_l'(\theta_s)\cos(\theta_s) - a\right) \equiv \Delta(\theta_s). \tag{12b}$$

Using these expressions we get that the linearization of (9) about the equilibrium state $(\theta_s, 0)$ is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \theta(t) \\ \eta(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Phi(\theta_s) & \Delta(\theta_s) \end{pmatrix} \begin{pmatrix} \theta(t) \\ \eta(t) \end{pmatrix}. \tag{13}$$

Using (5) we can get the following alternate formulas for Φ, Δ:

$$\Phi(\theta_s) = \frac{g}{r}\frac{\beta'(\theta_s)}{\beta(\theta_s)} \cos\theta_s, \tag{14}$$
$$\Delta(\theta_s) = \sqrt{\frac{a\rho A g}{2m}} \left[\frac{\beta(\theta_s)\sin\theta_s + \beta'(\theta_s)\cos\theta_s}{\sqrt{\beta(\theta_s)}}\right] \cos\theta_s. \tag{15}$$

Note that the zeros of Φ are exactly those of β' which in turn are those predicted by Lemma (2.1). If we let θ_b be the smallest such root, then since $\beta'(0) > 0$, we have that

$$\beta'(\theta) > 0, \quad \theta \in [0, \theta_b).$$

Thus from (15) it follows as well that

$$\Delta(\theta) > 0, \quad \theta \in [0, \theta_b].$$

Since $\Delta(\theta_u) < 0$, it follows that there exists a θ_h, with $\theta_h > \theta_b$, such that $\Delta(\theta_h) = 0$. We can now prove the following theorem.

Theorem 3.1. *Let θ_h be the smallest zero of Δ, $\theta_h > \theta_b$. Assume that θ_h is a root of odd algebraic multiplicity of Δ. Then the equilibrium states of the system* (9) *are unstable for $\theta_s < \theta_h$, they are stable for $\theta_s > \theta_h$ up to the next root of Δ if any, and there is a Hopf bifurcation at $\theta_s = \theta_h$.*

Proof: From (15) we get that $\Delta(\theta) \leq 0$ only if $\beta'(\theta) \leq 0$, or equivalently form (14) that $\Delta(\theta) \leq 0$ only if $\Phi(\theta) \leq 0$. Thus if θ_h is a root of odd algebraic multiplicity of Δ, we have the following pattern of signs for Φ and Δ:

$$\Delta(\theta_s) > 0, \quad 0 < \theta_s < \theta_h, \tag{16}$$
$$\Phi(\theta_s) < 0, \quad \Delta(\theta_s) < 0, \quad \theta_s > \theta_h, \text{ up to next root of } \Delta. \tag{17}$$

The eigenvalues of the coefficient matrix of the linearized system (13) are given by:

$$\lambda_\pm = \frac{1}{2}\left[\Delta(\theta_s) \pm \sqrt{\Delta^2(\theta_s) + 4\Phi(\theta_s)}\right].$$

Thus for the pattern (16), the eigenvalues are either real with at least one of them positive, or complex with positive real part. Hence the corresponding equilibrium states are unstable. For the pattern (17) the eigenvalues are either complex with negative real part or real with both negative and hence, the corresponding equilibrium states are stable. (See e.g. [2].) Furthermore, at $\theta_s = \theta_h$ we have that:

$$\lambda_{\pm}(\theta_h) = \pm 2i\sqrt{-\Phi(\theta_h)}, \quad \Phi(\theta_h) \neq 0,$$

and the odd algebraic multiplicity condition on θ_h implies that we have a strict crossing of the two eigenvalues on the imaginary axis. Hence there is a Hopf bifurcation at θ_h. (See e.g. [2].) □

In general there can be multiple points of Hopf bifurcations and branches of stable solutions, the stability of which changes every time a Hopf bifurcation is crossed. In the next section we give a numerical example of this situation.

4 Numerical Examples

In this section we describe some numerical examples illustrating the results of the previous section in particular those in Theorem (3.1). The examples are constructed in such a way that the function $\beta(\theta)$ has an M–shaped profile. We first construct a fourth degree polynomial that interpolates $\cos\theta$ at three point in $(0, \pi/2)$, which would be the solutions of (7) in this case, and with a value at zero greater than one and negative at $\pi/2$. This polynomial will be set equal to $C_l'(\theta)/a$, which upon integration using the condition $C_l(0) = 0$, give us $C_l(\theta)/a$. The three interpolation points and the values at zero and $\pi/2$ are chosen in such a way that conditions (4) are satisfied.

An example in which only one Hopf bifurcation occurs happens for the following lift coefficient function:

$$\frac{1}{a} C_l(\theta) = -0.6425\theta^5 + 0.87370\theta^4 + 0.6498\theta^3 - 1.8459\theta^2 + 2.0000\theta. \tag{18}$$

Conditions (4) can be checked numerically. For this case $\theta_d = 1.156$ and $\theta_u = 1.3381$ approximately. In Figure (1) we show $C_l(\theta)/a$ and the corresponding function $\beta(\theta)$ with θ on the vertical axis. In Figure (2) we show the corresponding plots for $\Phi(\theta)$ and $\Delta(\theta)$. It follows from the last figure that the dotted part of the curve for $\beta(\theta)$ represents unstable equilibria while those on the solid curve are stable. Also we get that there is a Hopf bifurcation, indicated by a circle in the figure, at approximately $\theta_h = 1.1502$.

An example in which three Hopf bifurcations occur happens for the following lift coefficient function:

$$\frac{1}{a} C_l(\theta) = -0.7246\theta^5 + 1.0754\theta^4 + 0.6479\theta^3 - 2.0412\theta^2 + 2.0000\theta. \tag{19}$$

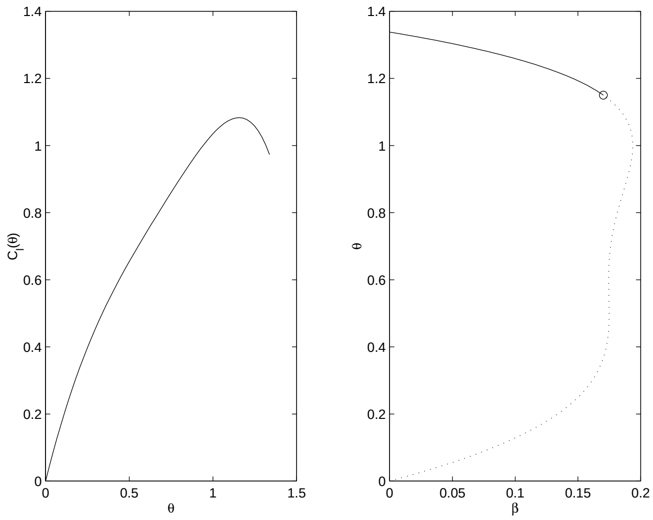

Figure 1: The lift coefficient function (18) (left) and the corresponding $\beta(\theta)$. The dotted curve represents unstable equilibria while those on the solid curve are stable. There is one point of Hopf bifurcation (circle).

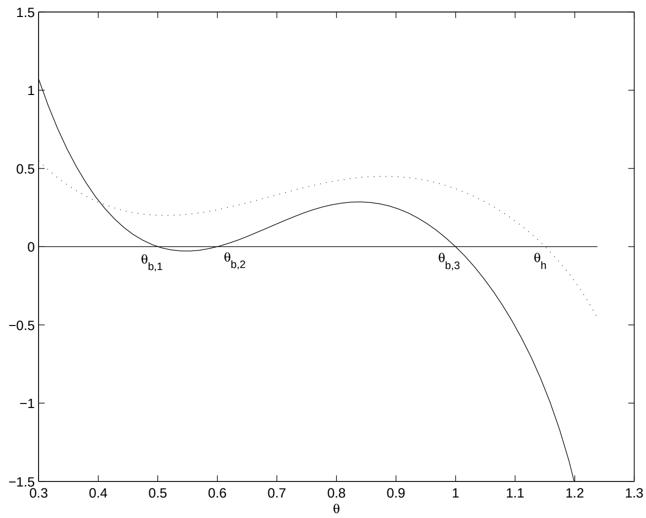

Figure 2: The function $\Phi(\theta)$ (solid), $\Delta(\theta)$ (dotted) corresponding to the lift coefficient function (18).

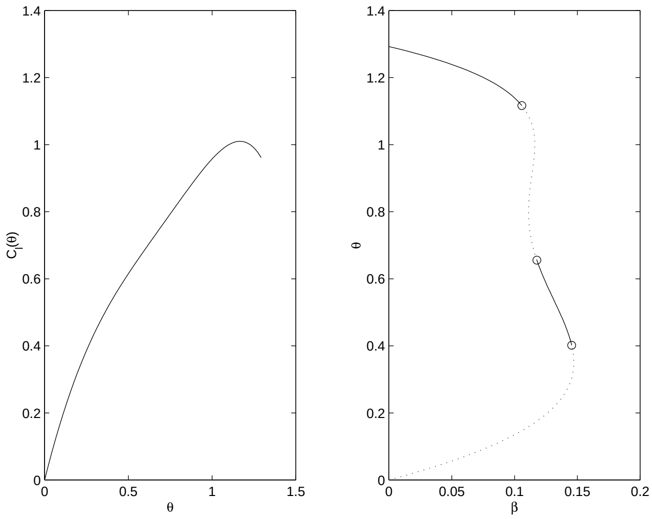

Figure 3: The lift coefficient function (19) (left) and the corresponding $\beta(\theta)$. The dotted curves represent unstable equilibria while those on the solid curves are stable. There are three points of Hopf bifurcation (circles).

For this case $\theta_d = 1.166$ and $\theta_u = 1.2928$ approximately. In Figure (3) we show $C_l(\theta)/a$ and the corresponding function $\beta(\theta)$ with θ on the vertical axis. In Figure (4) we show the corresponding plots for $\Phi(\theta)$ and $\Delta(\theta)$. It follows from the last figure that the dotted parts of the curve for $\beta(\theta)$ represents unstable equilibria while those on the solid curves are stable. Also we get that there are Hopf bifurcations, indicated by the three circles in the figure, at approximately $\theta_h = 0.4016, 0.6557, 1.1166$.

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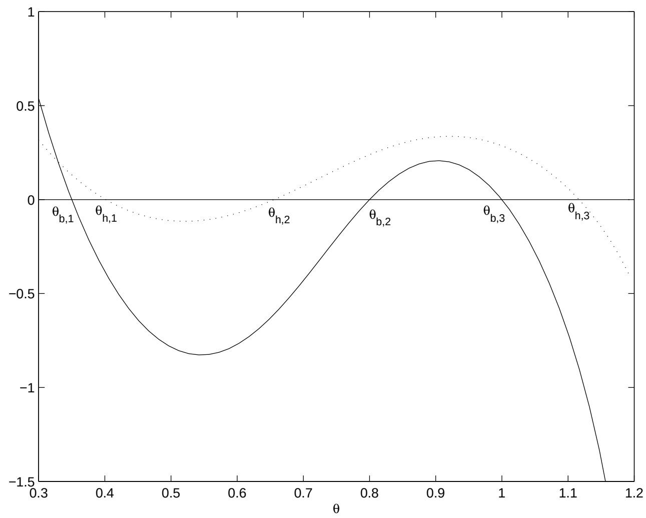

Figure 4: The function $\Phi(\theta)$ (solid), $\Delta(\theta)$ (dotted) corresponding to the lift coefficient function (19).