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The Numerical Computation of the Critical Boundary Displacement for Radial Cavitation for Composite Materials

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Abstract

The phenomenon of void formation in bodies under tension has been observed in laboratory experiments. Ball (1982) showed, in the context of nonlinear elasticity, that void formation or "cavitation" can decrease the (potential) energy of a body in tension when the tension is sufficiently large. An important related problem is that of characterizing or computing the critical boundary displacement at which cavitation occurs. As cavitation can point to the initiation of fracture or rupture in a body, the computation of such critical boundary displacement is important from the point of view of design. In Negrón-Marrero and Sivaloganathan [4] a numerical scheme for computing the critical boundary displacement for cavitation is proposed that applies to a very general class of compressible homogeneous materials. In this paper we study the generalization or extension of this method to composite (non-homogeneous) materials.
Keywords: Cavitation, Critical Boundary Displacement, Numerical Scheme.

1 Introduction

Void formation, also known as "cavitation" can point to the initial fracture of a body in tension (Figure 1). Over the years, this phenomenon has been studied and it was shown by Ball (1982) that void formation decreases the potential energy of the body. This happens when the boundary displacement is sufficiently large. The particular boundary displacement at which cavitation appears is called the *critical boundary displacement for cavitation* and is denoted by λ_{crit}. The computation of such critical boundary displacement is important from the point of view of design. Most of the attempts of computing λ_{crit} have been based on finding exact solutions of the equations describing such deformations. For a nice review on these and other related results on cavitation we refer to Horgan and Polignone [3]. In Negrón-Marrero and Sivaloganathan [4] a numerical scheme for computing the critical boundary displacement for cavitation is proposed that applies to a very general class of compressible homogeneous materials. This method is based on the solution of a sequence of problems with punctured domains. That is, a small hole is put in the center of the body, and the problem is solved for such a domain. Then we proceed to make the hole smaller and repeat the process. It is known, Sivaloganathan [5], that this process converges to a solution of the corresponding problem for the solid body. In the method of Negrón-Marrero and Sivaloganathan [4], which is called the *inverse method*, in addition to the punctured domain, the inner cavity size of the deformed body is specified as well, and a sequence of problems with both the hole in the reference configuration and that of the deformed configuration approaching zero, is solved. In Negrón-Marrero and Sivaloganathan [4] it is shown that this process converges to λ_{crit}. With the specification of the inner cavity, one then is confronted with solving a sequence of initial value problems (c.f. (10), (11)) instead of a sequence of nonlinear boundary value problems (c.f. (6), (7), (8), (9)).

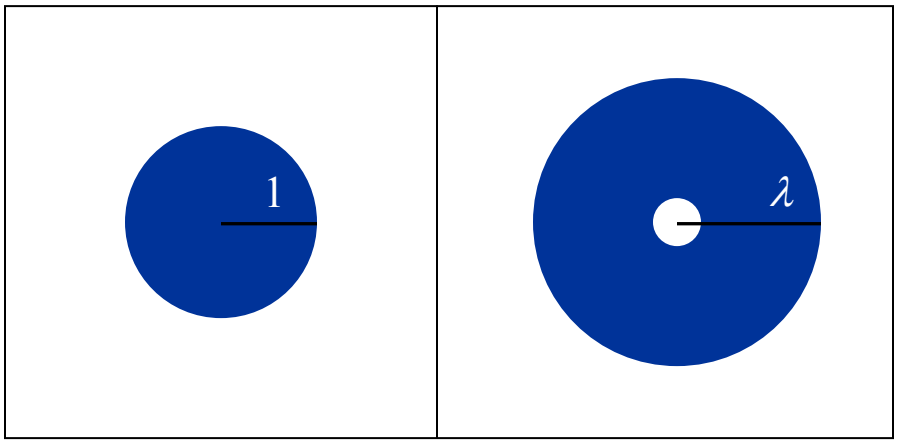

Figure 1. Void formation or cavitation on a spherically symmetric body.

The inverse method proposed by Negrón-Marrero and Sivaloganathan [4] and its convergence properties are for homogeneous materials. In this paper we consider the generalization of this method to bodies composed of two different homogeneous materials. We consider a sphere or ball with a center core and an outer core of different materials each (Figure 2). We study how the critical boundary displacement depends on the properties of the two material and the relative sizes of the cores. For the purpose of this work we consider cores composed of materials that can be described by the stored energy function (1):

$$\Phi(v_1, v_2, v_3) = v_1^{-2} + v_2^{-2} + v_3^{-2} + cv_1v_2v_3. \tag{1}$$

Each core will be described by a function of this type. For this particular example we study how λ_{crit} varies as the material parameter c changes for each core, and how it depends on the relative sizes of the cores.

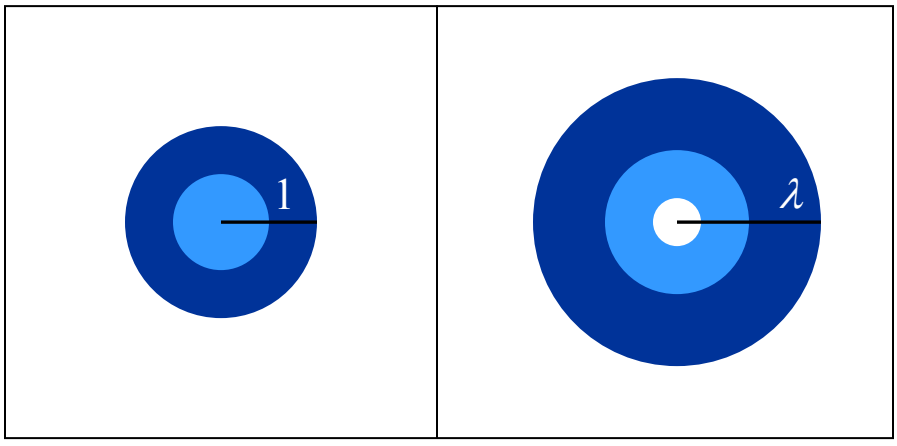

Figure 2. Non-homogeneous material (left). Non-homogeneous material with cavitation (right).

2 Formulation of the Problem

We considered a unit sphere B as the reference configuration of the body. A deformation of *B* is a function $u : B \to \Re^3$. The derivative $\nabla u(x)$ is called the *deformation gradient*. The requirement that the deformation *u preserves orientation* is equivalent to:

$$\det \nabla u(x) > 0, \quad x \in B. \tag{2}$$

If $W : B \times M_+ \to \Re$, where $M_+ = \{F \in \Re^{3\times 3} : \det F > 0\}$, represents the stored energy function for the material of the body, then the total stored energy associated with the deformation *u* is given by:

$$E(u) = \int_B W(x, \nabla u(x))\, dx. \tag{3}$$

We look for deformations *u* that minimize this total stored energy functional among an appropriate class of functions and satisfying the boundary condition:

$$u(x) = \lambda x, \quad x \in \partial B.$$

In this paper we look for solutions of this problem that are *radially symmetric*, that is, solutions *u* of the form:

$$u(x) = r(R)\frac{x}{R}, \quad x \in B,$$

where $R = \|x\|$, and $r : [0,1] \to \Re$. It follows that the condition (2) is satisfied provided:

$$r'(R), \frac{r(R)}{R} > 0, \quad R \in [0,1].$$

The stored energy functions *W* that we consider will be of the following form:

$$W(x,F) = \begin{cases} \Phi_1(v_1, v_2, v_3), & 0 < R < a, \\ \Phi_2(v_1, v_2, v_3), & a < R < 1, \end{cases} \tag{4}$$

where $v_1, v_2, v_3,$ are the proper values of $(F'F)^{1/2}$, called the principal stretches, and $a > 0$ is the radial size of the inner core. The total stored energy in the body due to deformation $r(R)$ is now given by:

$$I(r) = \int_0^a R^2 \Phi_1\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) dR + \int_a^1 R^2 \Phi_2\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) dR. \tag{5}$$

In the following we use the notation $\Phi_{i,j}$ for the partial derivative of Φ_i with respect to the *j*-th variable, and

$$\Phi_{i,j}(r(R)) = \Phi_{i,j}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right), \quad \text{etc..}$$

By considering smooth variations v such that $v(1) = 0$, one can show that the Euler-Lagrange equations for the functional (4) are given by:

$$\frac{d}{dR}\left[R^2 \Phi_{1,1}(r(R))\right] = 2R\Phi_{1,2}(r(R)), \quad 0 < R < a, \tag{6}$$

$$\frac{d}{dR}\left[R^2 \Phi_{2,1}(r(R))\right] = 2R\Phi_{2,2}(r(R)), \quad a < R < 1, \tag{7}$$

with boundary conditions:

$$r(0) \geq 0, \quad r(1) = \lambda, \quad \Phi_{1,1}\left(r'(a^-), \frac{r(a)}{a}, \frac{r(a)}{a}\right) = \Phi_{2,1}\left(r'(a^+), \frac{r(a)}{a}, \frac{r(a)}{a}\right). \tag{8}$$

Moreover, if $r(0) > 0$, then we must have that[1]:

$$\lim_{R \to 0^+} \left(\frac{R}{r(R)}\right)^2 \Phi_{1,1}(r(R)) = 0. \tag{9}$$

Note that in general $r(R)$ will be a continuous function, differentiable everywhere except at $R = a$. However the radial stress is in general continuous for all R. In fact, the last condition in (8) is just a statement that this radial stress is continuous across $R = a$.

3 Numerical Scheme

In this section we describe the numerical scheme that was used to approximate the critical boundary displacement for cavitation. As mentioned in the introduction, the method is based on the solution of a sequence of problems with punctured domains. The punctured domains are given by:

$$B_\varepsilon = \{x : \varepsilon < \|x\| < 1\},$$

where $\varepsilon > 0$. We denote by $r_\varepsilon(R)$ the solution of (6), (7), (8), and (9) in this new domain. In this case the condition (9) reduces to:

$$\Phi_{1,1}\left(r'_\varepsilon(\varepsilon), \frac{r_\varepsilon(\varepsilon)}{\varepsilon}, \frac{r_\varepsilon(\varepsilon)}{\varepsilon}\right) = 0.$$

Under some physically reasonable assumptions on the stored energy functions Φ_1, Φ_2, we can get that the equations:

$$\Phi_{i,1}(\nu, \tau, \tau) = P, \quad i = 1,2,$$

are equivalent to

$$\nu = \phi_i(\tau, P), \quad i = 1,2,$$

where $\phi_i : (0,\infty) \times \Re \to (0,\infty), \quad i = 1,2$, are smooth functions.

If instead of λ in (8), we prescribe $r_\varepsilon(\varepsilon) = c$, then the problem of finding $r_\varepsilon(R)$ can be stated now as the following initial value problems:

$$\frac{d}{dR}\left[R^2 \Phi_{1,1}\left(r_\varepsilon(R)\right)\right] = 2R\Phi_{1,2}\left(r_\varepsilon(R)\right), \quad \varepsilon < R < a, \tag{10}$$

$$r_\varepsilon(\varepsilon) = c, \quad r'_\varepsilon(\varepsilon) = \phi_1\left(\frac{c}{\varepsilon}, 0\right),$$

$$\frac{d}{dR}\left[R^2 \Phi_{2,1}\left(r_\varepsilon(R)\right)\right] = 2R\Phi_{2,2}\left(r_\varepsilon(R)\right), \quad a < R < 1, \tag{11}$$

$$r_\varepsilon(a^+) = r_\varepsilon(a^-), \quad r'_\varepsilon(a^+) = \phi_2\left(\frac{r_\varepsilon(a)}{a}, P^-\right), \quad P^- = \Phi_{1,1}\left(r'_\varepsilon(a^-), \frac{r_\varepsilon(a)}{a}, \frac{r_\varepsilon(a)}{a}\right).$$

The idea now is to solve these problems for a sequence of ε, c converging to zero. For the case of the single core it is shown in Negrón-Marrero and Sivaloganathan [4] that the sequence of boundary displacements generated

according to $\lambda = r_\varepsilon(1)$, converges to the critical boundary displacement for cavitation λ_{crit}. This procedure can be described by the following pseudo-algorithm:

3.1 Procedure

Let $\{(\varepsilon_k, c_k)\}$ be a sequence converging to (0,0).

1. For k = 0, 1, 2,…,

   a) Compute an approximate solution $v_{k,1}$ of the equation: $\Phi_{1,1}\left(v_{k,1}, \frac{c_k}{\varepsilon_k}, \frac{c_k}{\varepsilon_k}\right) = 0.$

   b) Compute an approximate solution $r_{k,1}(R)$ of the IVP given by the equation (3) on (ε_k, a) subject to

   $$r(\varepsilon_k) = c_k,\ r'(\varepsilon_k) = v_{k,1}.$$

   c) Compute an approximate solution $v_{k,2}$ of the equation:

   $$\Phi_{1,1}\left(r'_{k,1}(a), \frac{r_{k,1}(a)}{a}, \frac{r_{k,1}(a)}{a}\right) = \Phi_{2,1}\left(v_{k,2}, \frac{r_{k,2}(a)}{a}, \frac{r_{k,2}(a)}{a}\right).$$

   d) Compute an approximate solution $r_{k,2}(R)$ of the IVP given by the equation (5) on $(a,1)$ subject to

   $$r(a) = r_{k,1}(a),\ r'(a) = v_{k,2}.$$

   e) Set $\lambda_k = r_{k,2}(1)$.

2. Repeat steps (a) to (e) until $\{\lambda_k\}$ satisfies a certain stopping criteria.

4 Numerical Results

Procedure 3.1 was implemented in MATLAB. This computational environment provides for very efficient routines for solving initial value problems. For the numerical simulations we used the following stored energy functions for the cores:

$$\Phi_i(v_1, v_2, v_3) = v_1^{-2} + v_2^{-2} + v_3^{-2} + c_i v_1 v_2 v_3, \quad i = 1,2,$$

where $i = 1,2$ denote the inner an outer cores respectively.

We ran several simulations in which we used Procedure 3.1 to compute λ_{crit} for different values of a, c_1, c_2. In the first case $a = 0.2, c_1 = 1.5$ and we vary c_2. As c_2 increases from 1.5 to 5, the outer core becomes "harder". We can see (Figure 3) that it becomes "easier" (smaller λ_{crit}) to open a hole in the center as expected due to the "harder" outer core.

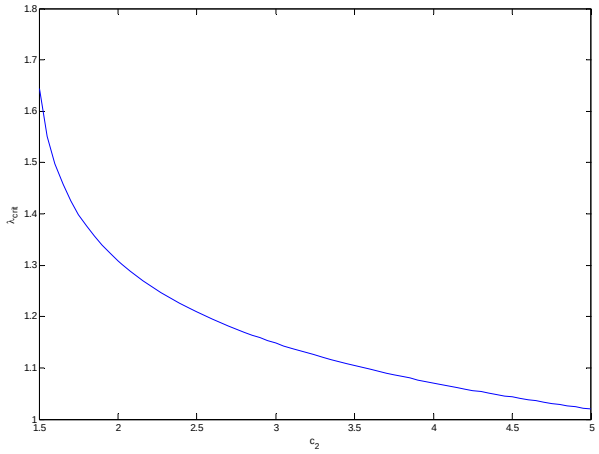

Figure 3. λ_{crit} as a function of c_2 for $a = 0.2, c_1 = 1.5$.

In the next simulation we have $c_1 = 2, c_2 = 5$. Thus the outer core is harder than the inner one. We vary the inner core radius. We can (Figure 4) that as the inner core radius increases, it becomes "harder" (larger λ_{crit}) to open a hole at the center. This is the effect of the increasing inner core. As a gets close to 1, one can see that λ_{crit} approaches the value of 1.3087 which is that corresponding to a single core of $c_1 = 2$.

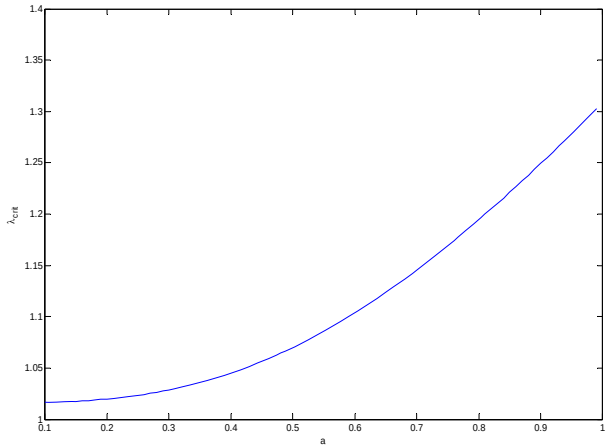

Figure 4. λ_{crit} as a function of a for $c_1 = 2, c_2 = 5$.

In the last simulation we consider variations both in c_2, a while c_1 is fixed at 2. c_2 varies between 1.5 and 5, while a changes between 0.1 up to 0.5. We show (Figure 5) the corresponding surface of λ_{crit} as a function of c_2, a. It is interesting to observe that for values of c_2 between 1.5 and 2, the value of λ_{crit} is a decreasing function of a. For a small, the softer material given by c_2 occupies most of the body. One has to pull "harder" to open a

hole in the "harder" center ($c_1 = 2$) because the outer material yields more easily. As a increases, this effect becomes less marked. The opposite behavior holds for values of c_2 between 2 and 5.

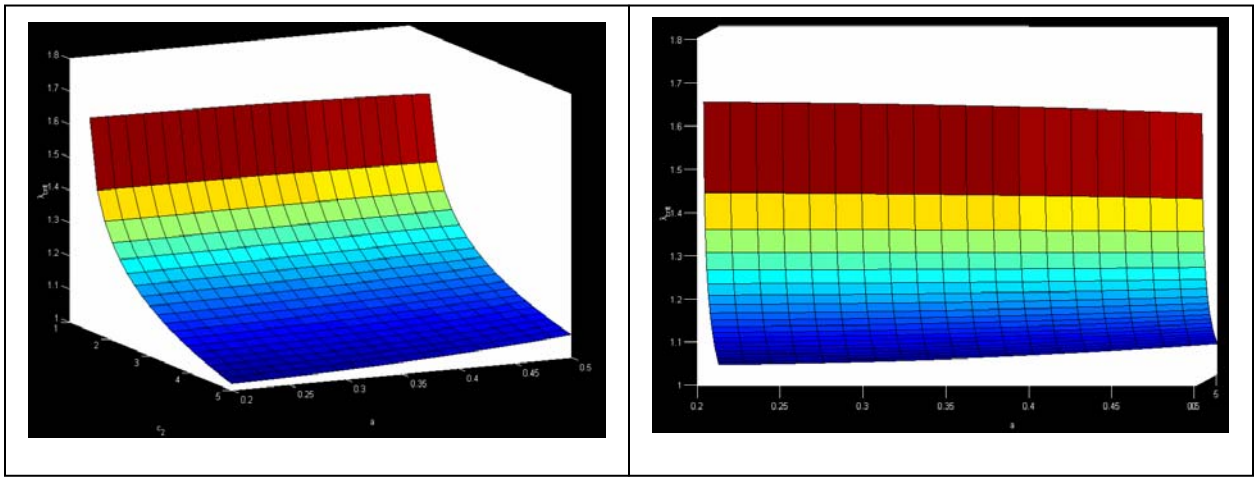

Figure 5. λ_{crit} as a function of both c_2 and a.

5 Conclusions

In the more general case of a composite material, the inverse method proved to be a useful scheme for computing the critical displacement for cavitation. We studied the behavior of λ_{crit} as a function of some of the constitutive parameters of the materials of the cores and as the size of the cores changed. The stored energy function used for the simulations is good only for small deformations because the term corresponding to the determinant, which is $v_1 v_2 v_3$, is finite under extreme compressions. In a future paper we consider more realistic stored energy functions (3) as well as other types of non-homogeneities. Also one needs to study the theoretical convergence properties of the scheme under these more general conditions.

6 Acknowledgements

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