THE REPULSION PROPERTY IN NONLINEAR ELASTICITY AND A NUMERICAL SCHEME TO CIRCUMVENT IT

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Abstract. For problems in the Calculus of Variations that exhibit the Lavrentiev phenomenon, it is known that a *repulsion property* may hold, that is, if one approximates the global minimizer in these problems by smooth functions, then the approximate energies will blow up. Thus, standard numerical schemes, like the finite element method, may fail when applied directly to these types of problems. In this paper we prove that a repulsion property holds for variational problems in three dimensional elasticity that exhibit cavitation. In addition, we propose a numerical scheme that circumvents the repulsion property, which is an adaptation of the Modica and Mortola functional for phase transitions in liquids, in which the phase function is coupled, via the determinant of the deformation gradient, to the stored energy functional. We show that the corresponding approximations by this method satisfy the lower bound Γ–convergence property in the multi-dimensional, non–radial, case. The convergence to the actual cavitating minimizer is established for a spherical body, in the case of radial deformations.

Key words. nonlinear elasticity, Lavrentiev phenomenon, gamma convergence, cavitation

AMS subject classifications. 74B20, 35J50, 49K20, 74G65.

1. Introduction.

One-dimensional problems in the Calculus of Variations that exhibit the Lavrentiev phenomenon [18] have been well studied (see, e.g., [5], [6]). A typical result in such problems, is that the infimum of a given integral functional

$$I(u) = \int_a^b L(x, u(x), u'(x))\, \mathrm{d}x,$$

on the admissible set of Sobolev functions

$$\mathcal{A}_p = \{u \in W^{1,p}((a,b)) \mid u(a) = \alpha,\ u(b) = \beta\},\ p > 1,$$

is strictly greater than its infimum on the corresponding set of absolutely continuous functions

$$\mathcal{A}_1 = \{u \in W^{1,1}((a,b)) \mid u(a) = \alpha,\ u(b) = \beta\},$$

i.e., for $p > 1$,

$$\inf_{u\in\mathcal{A}_1} I(u) < \inf_{u\in\mathcal{A}_p} I(u).$$

Moreover, it has been shown (see [5, Theorem 5.5]) in a number of cases that if the Lavrentiev phenomenon occurs, then a"repulsion property" holds when trying to approximate a minimiser by more regular functions: that is, if $u_0 \in \mathcal{A}_1$ is a minimiser of I on $\mathcal{A}_1$ and $(u_n) \subset \mathcal{A}_p$, $p > 1$, satisfies $u_n \to u_0$ almost everywhere, then $I(u_n) \to \infty$ as $n \to \infty$. We refer to the interesting paper [10] for results on the weak repulsion property for multi–dimensional problems of the Calculus of Variations that exhibit the Lavrentiev phenomenon. In particular, it is shown in [10] that for any minimizer of a problem that exhibits the Lavrentiev phenomenon, there exists a sequence of "smooth" functions converging (strongly) in $W^{1,p}$ to the minimizer (for some p), for which the values of the functional on the sequence tend to infinity. The

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results apply to a general class of functionals but do not take into account the local invertibility condition (2) which is a central assumption in models of hyperelasticity. The Lavrentiev phenomenon is also known to arise in problems of hyperelasticity in which condition (2) is used (cf. [3], [11]).

In the first part of this paper, we prove in Theorem 3 a repulsion property for variational problems in elasticity in $\mathbb{R}^m$ ($m = 2$ or 3) that exhibit cavitation. Our result is presented for the class of functionals given by (3), (9) and identifies the structure of the stored energy function which gives rise to the repulsion property. We show that, when approximating *any* finite-energy cavitating deformation $\boldsymbol{u} \in W^{1,p}$, $p \in (m-1, m)$ (not necessarily a minimiser) by a sequence of non-cavitating deformations $(\boldsymbol{u}_n)$ converging weakly to $\mathbf{u}$ in $W^{1,p}$, the energy of the sequence $(\boldsymbol{u}_n)$ necessarily diverges to infinity. This result does not appear to have been noted previously and has implications for the design of numerical methods to detect cavitation instabilities in nonlinear elasticity. In particular, from the proof of Theorem 3, it becomes evident that the critical term in the stored energy function, in relation to the repulsion property, is the compressibility term (the $h(\cdot)$ term in (9)). We also note that our version of the repulsion property extends previous versions in that the approximating sequence of more regular deformations is allowed to lie in the same Sobolev space as the limit cavitating deformation and we only assume weak convergence of the sequence to the limit deformation.

The numerical aspects of computing cavitated solutions are challenging due to the singular nature of such deformations. The work of Negrón–Marrero [29] generalized to the multidimensional case of elasticity a method introduced by Ball and Knowles [4] for one dimensional problems, which is based on a decoupling technique that detects singular minimizers and avoids the Lavrentiev phenomenon. The convergence result in [29] involved a very strong condition on the adjoints of the finite element approximations which among other things excluded cavitated solutions. The element removal method introduced by Li ([19], [20]) improves upon this by penalizing or excluding the elements of the finite element grid where the deformation gradient becomes very large. We refer also to the works of Henao and Xu [15] and Lian and Li ([21], [22]).

Motivated by the result in Theorem 3, we propose in Section 4 a numerical scheme for computing cavitating deformations that avoids or works around the repulsion property by using nonsingular or smooth approximations. The idea is to introduce a decoupling or phase function on the determinant of the competing deformations, together with an extra term in the energy functional that forces the phase function to assume either small or very large values, and penalizes for the corresponding transition regions. More specifically, if $W(\mathbf{F}) = \tilde{W}(\mathbf{F}) + h(\det \mathbf{F})$ represents the stored energy function of the material of the body occupying the region Ω, where $\tilde{W}$ and h satisfy certain growth conditions (cf. (10), (11)), then our proposed functional is given by

$$\begin{aligned} &\int_\Omega \left[\tilde{W}(\nabla \mathbf{u}(\mathbf{x})) + h(\det \nabla \mathbf{u}(\mathbf{x}) - v(\mathbf{x}))\right] \mathrm{d}\mathbf{x} \\ &\qquad + \int_\Omega \left[\frac{\varepsilon^\alpha}{\alpha}\|\nabla v(\mathbf{x})\|^\alpha + \frac{1}{q\varepsilon^q}\phi_\tau(v(\mathbf{x}))\right] \mathrm{d}\mathbf{x}, \end{aligned} \tag{1}$$

where $\tau > 0$ and $\varepsilon > 0$ are approximation parameters, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{q} = 1$, and $\phi_\tau : \mathbb{R} \to [0, \infty)$ is a C^1 function such that the support of ϕ_τ is $[0, \tau]$ and $\phi_\tau > 0$ on $(0, \tau)$.

The interpretation of the phase function in this model is that, in regions in which the phase function is large, the material can undergo large volume changes without a significant increase in its stored energy (one could interpret this as energetically allowing a ‘change of phase,’ analogous to the formation of vapour-filled cavities in a fluid undergoing cavitation under a large negative pressure).

The term in this functional involving the function ϕ_τ, is a variant of the corresponding term in the Modica and Mortola functional considered in [25] for phase transitions in liquids, and it penalizes for regions where the phase function v is positive but less than τ, but does not penalize for values of v greater than τ. This phase function, which in addition is required to satisfy the constraints $0 \le v < \det \nabla \mathbf{u}$, is now coupled to the mechanical energy through the compressibility term h. One major advantage of the proposed numerical scheme based on this functional is that in the limit, as $\tau \to \infty$ and $\varepsilon \to 0^+$, the phase function v marks or detects automatically those regions where fractures or cavitation may take place. For small ϵ, the second integral term in (1) approximates the surface area of the boundary between the regions in which the phase function v is zero or larger than τ, and hence models a ”surface energy”.

In Theorems 7 and 13 we show that our proposed scheme has the lower bound Γ–convergence property. Moreover, if $(\mathbf{u}_{\varepsilon\tau}, v_{\varepsilon\tau})$ denotes a minimizer of (1), then for a subsequence with $\tau \to \infty$ and $\varepsilon \to 0^+$, $(\mathbf{u}_{\varepsilon\tau})$ converges weakly in $W^{1,p}$ to a function $\mathbf{u}^*$ whose distributional determinant is a positive Radon measure. The $(v_{\varepsilon\tau})$ converge in $\mathcal{M}(\Omega)$ (the space of signed Radon measures on Ω) to the singular part of this measure and $(\det \nabla \mathbf{u}_{\varepsilon\tau} - v_{\varepsilon\tau})$ converges in $L^1(\Omega)$ to $\det \nabla \mathbf{u}^*$. The Radon measure mentioned above characterizes the points or regions in the reference configuration where discontinuities of cavitation or fracture type can occur.

Further refinements of these results, which includes a result along the lines of an upper bound Γ– convergence property (Theorem 15), are discussed in Section 5 for radial deformations of a spherical body. In Theorem 15 we show that for large boundary displacements, given a sequence (τ_j) with $\tau_j \to \infty$, one can construct a sequence (ε_j) with $\varepsilon_j \to 0$ and a corresponding sequence of admissible function pairs of the specialization of (1) to radial functions, such that the corresponding decoupled energies converge to the energy of the cavitating radial minimizer. Using this together with our previous lower bound Γ–convergence result, we then prove in Theorem 16 that the approximations of the proposed decoupled–penalized method converge to the radial cavitating solution. We also show that the minimizers of the penalized functionals (cf. (31)) satisfy the corresponding versions of the Euler–Lagrange equations and present some numerical simulations.

Our approach contrasts with that of Henao, Mora–Corral, and Xu [14] who employ two phase functions v and w, with the v coupled to the mechanical energy as a factor multiplying the original stored energy function, and w defined on the deformed configuration. The extra terms are of the Ambrosio–Tortorelli [1] type for v and of the Modica–Mortola type for w. As the approximation parameter ε in their functional goes to zero, these extra terms in the energy functional allow for the approximation of deformations that can exhibit cavitation or fracture. Our approach in this paper clearly identifies and highlights the role of the compressibility term h in the energy functional (3) as the source of the repulsion property in problems exhibiting cavitation.

2. Background. Let $\Omega \subset \mathbb{R}^m$ ($m = 2$ or $m = 3$) denote the region occupied by a nonlinearly elastic body in its reference configuration. A deformation of the body corresponds to a map $\mathbf{u} : \Omega \to \mathbb{R}^m$, $\mathbf{u} \in W^{1,1}(\Omega)$, that is one-to-one almost everywhere and satisfies the condition

$$\det \nabla \mathbf{u}(\mathbf{x}) > 0 \quad \text{for a.e. } \mathbf{x} \in \Omega. \tag{2}$$

In hyperelasticity, the energy stored under such a deformation is given by

$$E(\mathbf{u}) = \int_{\Omega} W(\nabla\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}, \tag{3}$$

where $W : \mathrm{M}_+^{m\times m} \to [0,\infty)$ is the stored energy function of the material and $\mathrm{M}_+^{m\times m}$ denotes the set of real $m \times m$ matrices with positive determinant. We consider the displacement problem in which we require

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^h(\mathbf{x}) \text{ for } \mathbf{x} \in \partial\Omega, \quad \mathbf{u}^h(\mathbf{x}) \equiv \mathbf{A}\mathbf{x}\ , \tag{4}$$

where $\mathbf{A} \in \mathrm{M}_+^{m\times m}$ is fixed. Let $\Omega \subset\subset \Omega^e$, where Ω^e is a bounded, open, connected set with smooth boundary.

2.1. The distributional determinant. If $\mathbf{u} \in W^{1,p}(\Omega)$ satisfies (4), then we define its homogeneous extension $\mathbf{u}_e : \Omega^e \to \mathbb{R}^m$ by

$$\mathbf{u}_e(\mathbf{x}) = \begin{cases} \mathbf{u}(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega, \\ \mathbf{A}\mathbf{x} & \text{if } \mathbf{x} \in \Omega^e\backslash\Omega, \end{cases} \tag{5}$$

and note that $\mathbf{u}_e \in W^{1,p}(\Omega^e;\mathbb{R}^m)$. For $p > m^2/(m+1)$,

$$\mathrm{Det}\nabla\mathbf{u}(\phi) := -\int_{\Omega} \frac{1}{m}\,([\mathrm{adj}\nabla\mathbf{u}]\mathbf{u}) \cdot \nabla\phi\,\mathrm{d}\mathbf{x}, \quad \forall\phi \in C_0^\infty(\Omega), \tag{6}$$

is a well-defined distribution. (Here adj$\nabla\mathbf{u}$ denotes the adjugate matrix of $\nabla\mathbf{u}$, that is, the transposed matrix of cofactors of $\nabla\mathbf{u}$.) The definition follows from the well-known formula for expressing $\det\nabla\mathbf{u}$ as a divergence. (See, e.g., [26] for further details and references.)

Next suppose that $\mathbf{u} \in W^{1,p}(\Omega)$, $p > m-1$, and that $\mathbf{u}_e$ satisfies condition (INV) (introduced by Müller and Spector in [28]) on Ω^e. Then $\mathbf{u}_e \in L^\infty_{loc}(\Omega^e)$ and hence $\mathrm{Det}(\nabla\mathbf{u})$ is again a well-defined distribution. Moreover, it follows from [28, Lemma 8.1] that if $\mathbf{u}$ further satisfies $\det\nabla\mathbf{u} > 0$ a.e. then $\mathrm{Det}\nabla\mathbf{u}$ is a Radon measure and

$$\mathrm{Det}\nabla\mathbf{u} = (\det\nabla\mathbf{u})\mathcal{L}^m + \mu_s, \tag{7}$$

where μ_s is singular with respect to Lebesgue measure $\mathcal{L}^m$. We first consider the case when μ_s is a Dirac measure[1] of the form $\alpha\delta_{\mathbf{x}_0}$ (where $\alpha > 0$ and $\mathbf{x}_0 \in \Omega$) which corresponds to $\mathbf{u}$ creating a cavity of volume α at the point $\mathbf{x}_0$. Note that such a cavity need not be spherical. Following [33], we fix $\mathbf{x}_0 \in \Omega$ and define the set of admissible deformations by

$$\begin{aligned} \mathcal{A}_{\mathbf{x}_0} = \{&\mathbf{u} \in W^{1,p}(\Omega) : \mathbf{u}|_{\partial\Omega} = \mathbf{u}^h,\ \mathbf{u}_e \text{ satisfies (INV) on } \Omega, \\ &\det\nabla\mathbf{u} > 0 \text{ a.e.}, \quad \mathrm{Det}\nabla\mathbf{u} = (\det\nabla\mathbf{u})\mathcal{L}^m + \alpha_{\mathbf{u}}\delta_{\mathbf{x}_0}\}, \end{aligned} \tag{8}$$

where $\alpha_{\mathbf{u}} \geq 0$ is a scalar depending on the map $\mathbf{u}$, and $\delta_{\mathbf{x}_0}$ denotes the Dirac measure with support at $\mathbf{x}_0$. Thus, $\mathcal{A}_{\mathbf{x}_0}$ contains maps $\mathbf{u}$ that produce a cavity of volume $\alpha_{\mathbf{u}}$ located at $\mathbf{x}_0 \in \Omega$. We will say that the deformation $\mathbf{u} \in \mathcal{A}_{\mathbf{x}_0}$ is *singular* if $\alpha_{\mathbf{u}} > 0$.

[1]Other assumptions on the support of the singular measure μ^s may be relevant for modelling different forms of fracture. See also [27] for further results on the singular support of the distributional Jacobian.

3. Singular Minimisers, Deformations and the Repulsion Property. In this note, for simplicity of exposition, we consider stored energy functions of the form

$$W(\mathbf{F}) = \tilde{W}(\mathbf{F}) + h(\det \mathbf{F}) \quad \text{for } \mathbf{F} \in M_+^{m\times m}, \tag{9}$$

where $\tilde{W} \geq 0$ is $W^{1,p}$–quasiconvex and satisfies that

$$k_1\|\mathbf{F}\|^p \leq \tilde{W}(\mathbf{F}) \leq k_2[\|\mathbf{F}\|^p + 1] \quad \text{for } \mathbf{F} \in M_+^{m\times m},\ p \in (m-1, m), \tag{10}$$

for some positive constants k_1, k_2, and $h(\cdot)$ is a $C^2(0,\infty)$ convex function such that

$$h(\delta) \to \infty \text{ as } \delta \to 0^+, \quad \frac{h(\delta)}{\delta} \to \infty \text{ as } \delta \to \infty. \tag{11}$$

These hypotheses are typically satisfied by many stored energy functions which exhibit cavitating minimisers, for example[2],

$$W(\mathbf{F}) = \mu\|\mathbf{F}\|^p + h(\det \mathbf{F}), \quad \mu > 0, \tag{12}$$

where h satisfies (11).

Remark 1. It is well known that, under a variety of hypotheses (see, e.g, [34]) on the stored energy function, there exists a minimiser of the energy (given by (3)) on the admissible set $\mathcal{A}_{\mathbf{x}_0}$. Moreover, it is also known that if $\mathbf{A}$ is sufficiently large, e.g., $\mathbf{A} = t\mathbf{B}$ for some $\mathbf{B} \in M_+^{m\times m}$ with $t > 0$ sufficiently large, then any minimiser $\mathbf{u}_0 \in \mathcal{A}_{\mathbf{x}_0}$ must satisfy $\alpha_{\mathbf{u}_0} > 0$ (see[35]).

Remark 2. The superlinear growth on the function h in (11), is a standard assumption in the analysis of cavitation (cf. [3]). It guarantees the existence of cavitating minimizers. The function $\tilde{W}$ by itself, because of the $W^{1,p}$ quasiconvexity, would rule out cavitation and thus the Lavrentiev phenomenon. (See also Remark 4.)

We next prove that if we attempt to approximate, even in a weak sense, a singular deformation $\mathbf{u}_0 \in \mathcal{A}_{\mathbf{x}_0}$ with finite elastic energy E (given by (3)) by a sequence of non-cavitating deformations in $\mathcal{A}_{\mathbf{x}_0}$, then the energy of the approximating sequence must necessarily diverge to infinity. In particular, this must also hold in the case of approximating a singular energy minimiser. This phenomenon of the energy diverging to infinity is essentially due to the presence of the compressibility term h which appears in the stored energy function (9).

THEOREM 3. *Let $p \in (m-1, m)$. Suppose, for some $\mathbf{A} \in M_+^{m\times m}$, that $\mathbf{u}_0 \in \mathcal{A}_{\mathbf{x}_0}$ is a deformation with finite energy and with $\alpha_{\mathbf{u}_0} > 0$. Suppose further that $(\mathbf{u}_n) \subset \mathcal{A}_{\mathbf{x}_0}$ satisfies $\alpha_{\mathbf{u}_n} = 0$, $\forall n$ and that $\mathbf{u}_n \rightharpoonup \mathbf{u}_0$ as $n \to \infty$ in $W^{1,p}(\Omega)$. Then $E(\mathbf{u}_n) \to \infty$ as $n \to \infty$.*

Proof. We first note that, since $||\mathbf{u}_n|| <$ const. uniformly in n, it follows by (10) that

$$\text{constant} \geq \int_\Omega \tilde{W}(\nabla \mathbf{u}_n)\,\mathrm{d}\mathbf{x} \quad \text{uniformly in } n.$$

We next claim that for any $R > 0$ such that $B_R(\mathbf{x}_0) \subset \Omega$ we have

$$\int_{B_R(\mathbf{x}_0)} \det(\nabla \mathbf{u}_n)\,\mathrm{d}\mathbf{x} \to \int_{B_R(\mathbf{x}_0)} \det(\nabla \mathbf{u}_0)\,\mathrm{d}\mathbf{x} + \alpha_{\mathbf{u}_0} > 0, \quad \text{as} \quad n \to \infty.$$

[2]This stored energy function is a special case of a class proposed by Ogden [30, 31] and is used to model rubber. The Ogden materials include as special cases the Mooney–Rivlin and neo–Hookean materials.

This follows from the facts (see [28, Lemma 8.1]) that

$$\begin{aligned}(\mathrm{Det}(\nabla \mathbf{u}_0))(B_R(\mathbf{x}_0)) &= \int_{B_R(\mathbf{x}_0)} \det(\nabla \mathbf{u}_0)\,\mathrm{d}\mathbf{x} + \alpha_{\mathbf{u}_0},\\ (\mathrm{Det}(\nabla \mathbf{u}_n))(B_R(\mathbf{x}_0)) &= \int_{B_R(\mathbf{x}_0)} \det(\nabla \mathbf{u}_n)\,\mathrm{d}\mathbf{x}, \quad \text{for all } n,\end{aligned}$$

and that

$$(\mathrm{Det}(\nabla \mathbf{u}_n))(B_R(\mathbf{x}_0)) \to (\mathrm{Det}(\nabla \mathbf{u}_0))(B_R(\mathbf{x}_0)) \ \text{ as } \ n \to \infty.$$

This last limit follows from classical results on the sequential weak continuity of the mapping $\mathbf{u} \to \mathrm{adj}(\nabla \mathbf{u})$ from $W^{1,p}$ into $L^{\frac{p}{m-1}}$ (see [2, Corollary 3.5]) and the compact embedding of $W^{1,p}$ into L^q_{loc} for every $q \in [1, \infty)$ for functions satisfying the (INV) condition (see [33, Lemma 3.3]).

Hence, by Jensen's Inequality, for all n we have

$$E(\mathbf{u}_n) \geq \int_{B_R(\mathbf{x}_0)} W(\nabla \mathbf{u}_n)\,\mathrm{d}\mathbf{x} \geq |B_R(\mathbf{x}_0)| h \left(\frac{\int_{B_R(\mathbf{x}_0)} \det(\nabla \mathbf{u}_n\,\mathrm{d}\mathbf{x})}{|B_R(\mathbf{x}_0)|} \right).$$

Hence

$$\begin{aligned}\mathrm{liminf}_{n\to\infty} E(\mathbf{u}_n)\,\mathrm{d}\mathbf{x} &\geq \lim_{n\to\infty} |B_R(\mathbf{x}_0)| h \left(\frac{\int_{B_R(\mathbf{x}_0)} \det(\nabla \mathbf{u}_n)\,\mathrm{d}\mathbf{x}}{|B_R(\mathbf{x}_0)|} \right)\\ &= |B_R(\mathbf{x}_0)| h \left(\frac{\int_{B_R(\mathbf{x}_0)} \det(\nabla \mathbf{u}_0)\,\mathrm{d}\mathbf{x} + \alpha_{\mathbf{u}_0}}{|B_R(\mathbf{x}_0)|} \right).\end{aligned}$$

Since this holds for all $R > 0$ sufficiently small, and since $\alpha_{\mathbf{u}_0} > 0$ by assumption, it follows by (11) that

$$\mathrm{liminf}_{n\to\infty} E(\mathbf{u}_n) = \infty.$$

∎

Remark 4. If we replace the mode of convergence in the hypotheses of the above Theorem from weak convergence in $W^{1,p}$ to strong convergence, then it follows by the dominated convergence theorem that

$$\int_\Omega \tilde{W}(\nabla \mathbf{u}_n)\,\mathrm{d}\mathbf{x} \to \int_\Omega \tilde{W}(\nabla \mathbf{u}_0)\,\mathrm{d}\mathbf{x} \ \text{ as } \ n \to \infty. \tag{13}$$

Hence, this part of the total energy can be well approximated by nonsingular deformations but the compressibility term involving h cannot.[3]

4. A decoupled method to circumvent the repulsion property. We now consider an approximation scheme that avoids or works around the repulsion property. The idea is to introduce a decoupling or phase function v in such a way that the difference between the determinant of the approximation and the phase function remains well behaved. The modified functional includes as well a penalization term

[3] We note if $\tilde{W}$ is uniformly quasiconvex, then the arguments of Evans and Gariepy [8] show that the converse is also true, i.e., that weak convergence of the sequence $(\mathbf{u}_n)$ to $\mathbf{u}$ together with convergence of the energies (13) implies that sequence $(\mathbf{u}_n)$ converges strongly to $\mathbf{u}$.

on v reminiscent of the one used in the theory of phase transitions, that penalizes if the function v is not too large or not too small.

Let the stored energy function be as in (9). For any $\tau > 0$, let $\phi_\tau : \mathbb{R} \to [0, \infty)$ be a continuous function, strictly positive in $(0, \tau)$, and vanishing in $\mathbb{R} \setminus (0, \tau)$. For $\varepsilon > 0$, we define now the modified functional:

$$\begin{aligned} I_\varepsilon^\tau(\mathbf{u}, v) &= \int_\Omega \left[\tilde{W}(\nabla \mathbf{u}(\mathbf{x})) + h(\det \nabla \mathbf{u}(\mathbf{x}) - v(\mathbf{x})) \right] \mathrm{d}\mathbf{x} \\ &\quad + \int_\Omega \left[\frac{\varepsilon^\alpha}{\alpha} \|\nabla v(\mathbf{x})\|^\alpha + \frac{1}{q\varepsilon^q} \phi_\tau(v(\mathbf{x})) \right] \mathrm{d}\mathbf{x}, \end{aligned} \tag{14}$$

where $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{q} = 1$, and $(\mathbf{u}, v) \in \mathcal{U}$ where

$$\begin{aligned} \mathcal{U} = \{(\mathbf{u}, v) \in W^{1,p}(\Omega) \times W^{1,\alpha}(\Omega) : \mathbf{u}|_{\partial\Omega} = \mathbf{u}^h,\ \mathbf{u}_e \text{ satisfies (INV) on } \Omega, \\ \det \nabla \mathbf{u} > v \geq 0 \text{ a.e.},\quad \mathrm{Det}\nabla \mathbf{u} = (\det \nabla \mathbf{u})\mathcal{L}^m,\ v|_{\partial\Omega} = 0\}. \end{aligned} \tag{15}$$

The coupled h term in this functional, because of (11), penalizes for large $\det \nabla \mathbf{u}$ and v small. The term depending on ∇v, for ε small, allows for large phase transitions in the function v. On the other hand, the term with the function ϕ_τ for ε small, forces the regions where v is positive but less than τ, to have small measure, i.e. to "concentrate".

We now show that for any given $\tau, \varepsilon > 0$, the functional (14) has a minimizer over $\mathcal{U}$.

Lemma 5. *Assume that $\tilde{W}(\cdot)$ and $h(\cdot)$ are nonnegative and that (10), (11) hold. For each $\tau > 0$ and $\varepsilon > 0$ there exists $(\mathbf{u}_\varepsilon^\tau, v_\varepsilon^\tau) \in \mathcal{U}$ such that*

$$I_\varepsilon^\tau(\mathbf{u}_\varepsilon^\tau, v_\varepsilon^\tau) = \inf_{\mathcal{U}} I_\varepsilon^\tau(\mathbf{u}, v).$$

Proof. Since $\tilde{W}(\cdot)$ and $h(\cdot)$ are nonnegative and the pair $(\mathbf{u}^h, 0)$ belongs to $\mathcal{U}$, it follows that $\inf_{\mathcal{U}} I_\varepsilon^\tau(\mathbf{u}, v)$ exists and (cf. (9))

$$\inf_{\mathcal{U}} I_\varepsilon^\tau(\mathbf{u}, v) \leq I_\varepsilon^\tau(\mathbf{u}^h, 0) = \int_\Omega W(\nabla \mathbf{u}^h)\, \mathrm{d}\mathbf{x} \equiv \ell. \tag{16}$$

Let $\{(\mathbf{u}_k, v_k)\}$ be an infimizing sequence. From the above inequality, we can assume that $I_\varepsilon^\tau(\mathbf{u}_k, v_k) \leq \ell$ for all k. It follows that

$$\int_\Omega \tilde{W}(\nabla \mathbf{u}_k(\mathbf{x}))\, \mathrm{d}\mathbf{x} \leq \ell, \quad \forall k,$$

which together with (10) implies that for a subsequence $\{\mathbf{u}_k\}$ (not relabeled), $\mathbf{u}_k \rightharpoonup \mathbf{u}_\varepsilon^\tau$ in $W^{1,p}(\Omega)$, with $\mathbf{u}_\varepsilon^\tau = \mathbf{u}^h$ over $\partial\Omega$ and $\mathbf{u}_\varepsilon^\tau$ satisfying the (INV) condition on Ω.

From (16) we get as well that

$$\int_\Omega h(\det \nabla \mathbf{u}_k(\mathbf{x}) - v_k(\mathbf{x}))\, \mathrm{d}\mathbf{x} \leq \ell, \quad \forall k.$$

This together with (11) and de la Vallée Poussin criteria, imply that for a subsequence (not relabeled), $\det \nabla \mathbf{u}_k - v_k \rightharpoonup w_\varepsilon^\tau$ in $L^1(\Omega)$, with $w_\varepsilon^\tau > 0$ a.e. Once again, (16) implies (since ε is fixed) that $\{v_k\}$ is bounded in $W^{1,\alpha}(\Omega)$, and thus for a subsequence (not relabeled) that $v_k \rightharpoonup v_\varepsilon^\tau$ in $W^{1,\alpha}(\Omega)$, with $v_\varepsilon^\tau \geq 0$ a.e. and $v_\varepsilon^\tau = 0$ on $\partial\Omega$. Thus we can conclude that $\det \nabla \mathbf{u}_k \rightharpoonup w_\varepsilon^\tau + v_\varepsilon^\tau$ in $L^1(\Omega)$. Since $\mathrm{Det}\nabla \mathbf{u}_k = (\det \nabla \mathbf{u}_k)\mathcal{L}^m$, we have that (see [36, proof of Lemma (4.5)])

$$\det \nabla \mathbf{u}_k \overset{*}{\rightharpoonup} \mathrm{Det}\nabla \mathbf{u}_\varepsilon^\tau \quad \text{in } \Omega,$$

from which it follows that $\mathrm{Det}\nabla \mathbf{u}_\varepsilon^\tau = (w_\varepsilon^\tau + v_\varepsilon^\tau)\mathcal{L}^m$. Since $w_\varepsilon^\tau + v_\varepsilon^\tau \in L^1(\Omega)$, we have from [26, Theorem 1] that

$$\mathrm{Det}\nabla \mathbf{u}_\varepsilon^\tau = (\det \nabla \mathbf{u}_\varepsilon^\tau)\mathcal{L}^m, \quad \det \nabla \mathbf{u}_\varepsilon^\tau = w_\varepsilon^\tau + v_\varepsilon^\tau.$$

Thus $(\mathbf{u}_\varepsilon^\tau, v_\varepsilon^\tau) \in \mathcal{U}$. Finally, since

$$\begin{gathered} \mathbf{u}_k \rightharpoonup \mathbf{u}_\varepsilon^\tau \text{ in } W^{1,p}(\Omega), \quad v_k \rightharpoonup v_\varepsilon^\tau \text{ in } W^{1,\alpha}(\Omega), \\ \det \nabla \mathbf{u}_k - v_k \rightharpoonup w_\varepsilon^\tau = \det \nabla \mathbf{u}_\varepsilon^\tau - v_\varepsilon^\tau \text{ in } L^1(\Omega), \end{gathered}$$

we have by the sequential weak lower semi–continuity of I_ε^τ, that

$$I_\varepsilon^\tau(\mathbf{u}_\varepsilon^\tau, v_\varepsilon^\tau) \le \varliminf_{k\to\infty} I_\varepsilon^\tau(\mathbf{u}_k, v_k) = \inf_{\mathcal{U}} I_\varepsilon^\tau(\mathbf{u}, v),$$

and thus

$$I_\varepsilon^\tau(\mathbf{u}_\varepsilon^\tau, v_\varepsilon^\tau) = \inf_{\mathcal{U}} I_\varepsilon^\tau(\mathbf{u}, v).$$

□

Our next result shows that if $\mathbf{A}$ in (4) is not too "large", then the minimizer $(\mathbf{u}_\varepsilon^\tau, v_\varepsilon^\tau)$ of Lemma 5 must be $(\mathbf{u}^h, 0)$.

PROPOSITION 6. *Assume that the function $\tilde{W}$ is quasiconvex. If $\mathbf{A}$ in (4) is such that $h'(\det \mathbf{A}) \le 0$, then the global minimizer $(\mathbf{u}_\varepsilon^\tau, v_\varepsilon^\tau)$ of $I_\varepsilon^\tau(\cdot,\cdot)$ over $\mathcal{U}$ is given by $\mathbf{u} = \mathbf{u}^h$ and $v = 0$ in Ω.*

Proof. Note that for any $(\mathbf{u}, v) \in \mathcal{U}$, we have

$$I_\varepsilon^\tau(\mathbf{u}, v) \ge \int_\Omega \left[\tilde{W}(\nabla \mathbf{u}(\mathbf{x})) + h(\det \nabla \mathbf{u}(\mathbf{x}) - v(\mathbf{x}))\right] \mathrm{d}\mathbf{x}.$$

Since $\mathrm{Det}\nabla \mathbf{u} = (\det \nabla \mathbf{u})\mathcal{L}^m$ and $\tilde{W}$ is quasiconvex, we have that

$$\int_\Omega \tilde{W}(\nabla \mathbf{u}(\mathbf{x}))\mathrm{d}\mathbf{x} \ge \int_\Omega \tilde{W}(\nabla \mathbf{u}^h(\mathbf{x}))\mathrm{d}\mathbf{x}.$$

In addition, by the convexity of $h(\cdot)$ we get:

$$h\left(\det \nabla \mathbf{u}(\mathbf{x}) - v(\mathbf{x})\right) \ge h(\det \mathbf{A}) + h'(\det \mathbf{A})\left(\det \nabla \mathbf{u}(\mathbf{x}) - v(\mathbf{x}) - \det \mathbf{A}\right).$$

Hence

$$\begin{aligned} \int_\Omega h\left(\det \nabla \mathbf{u}(\mathbf{x}) - v(\mathbf{x})\right) \mathrm{d}\mathbf{x} \quad &\ge \quad \int_\Omega h(\det \mathbf{A})\mathrm{d}\mathbf{x} - h'(\det \mathbf{A}) \int_\Omega v(\mathbf{x})\mathrm{d}\mathbf{x} \\ &\quad + h'(\det \mathbf{A}) \int_\Omega \left(\det \nabla \mathbf{u}(\mathbf{x}) - \det \mathbf{A}\right) \mathrm{d}\mathbf{x} \end{aligned}$$

Again, since $\mathrm{Det}\nabla \mathbf{u} = (\det \nabla \mathbf{u})\mathcal{L}^m$, we have that

$$\int_\Omega \left(\det \nabla \mathbf{u}(\mathbf{x}) - \det \mathbf{A}\right) \mathrm{d}\mathbf{x} = 0.$$

Using now that $h'(\det \mathbf{A}) \le 0$ and that $v \ge 0$, we get

$$\int_\Omega h\left(\det \nabla \mathbf{u}(\mathbf{x}) - v(\mathbf{x})\right) \mathrm{d}\mathbf{x} \ge \int_\Omega h(\det \mathbf{A}) \mathrm{d}\mathbf{x}.$$

Combining this with the two inequalities at the beginning of this proof, we get that

$$I_\varepsilon^\tau(\mathbf{u}, v) \quad \ge \quad \int_\Omega \left[\tilde{W}(\nabla \mathbf{u}^h(\mathbf{x})) + h(\det \nabla \mathbf{u}^h(\mathbf{x}))\right] \mathrm{d}\mathbf{x} = I_\varepsilon^\tau(\mathbf{u}^h, 0).$$

Since $(\mathbf{u}, v)$ is arbitrary in $\mathcal{U}$ and $(\mathbf{u}^h, 0) \in \mathcal{U}$, we have that $(\mathbf{u}^h, 0)$ is the global minimizer in this case. ∎

Let $\mathcal{M}(\Omega)$ be the space of signed Radon measures on Ω. If $\mu \in \mathcal{M}(\Omega)$, then

$$\langle \mu, v\rangle = \int_\Omega v \, \mathrm{d}\mu, \quad \forall\, v \in C_0(\Omega),$$

where $C_0(\Omega)$ denotes the set of continuous functions with compact support in Ω. Moreover

$$\|\mu\|_{\mathcal{M}(\Omega)} = \sup\left\{|\langle \mu, v\rangle| \,:\, v \in C_0(\Omega),\, \|v\|_{L^\infty(\Omega)} \le 1\right\}.$$

A sequence $\{\mu_n\}$ in $\mathcal{M}(\Omega)$ converges weakly $*$ to $\mu \in \mathcal{M}(\Omega)$, denoted $\mu_n \overset{*}{\rightharpoonup} \mu$, if

$$\lim_{n\to\infty} \langle \mu_n, v\rangle = \langle \mu, v\rangle, \quad \forall\, v \in C_0(\Omega).$$

Note that any function in $L^1(\Omega)$ can be regarded as belonging to $\mathcal{M}(\Omega)$ with the same norm. It follows from this observation and the weak compactness of $\mathcal{M}(\Omega)$, that if $\{v_n\}$ is a bounded sequence in $L^1(\Omega)$, then it has a subsequence $\{v_{n_k}\}$ such that $v_{n_k} \overset{*}{\rightharpoonup} \mu$ where $\mu \in \mathcal{M}(\Omega)$.

For any subset E of Ω, we define its *(Caccioppoli) perimeter in* Ω by

$$P(E, \Omega) = \sup\left\{\int_\Omega \chi_E(\mathbf{x}) \operatorname{div} \boldsymbol{\phi}(\mathbf{x}) \, \mathrm{d}\mathbf{x} \,:\, \boldsymbol{\phi} \in C_0^1(\Omega; \mathbb{R}^m),\, \|\boldsymbol{\phi}\|_{L^\infty(\Omega)} \le 1\right\}.$$

E is said to have *finite perimeter* in Ω if $P(E, \Omega) < \infty$. For a set of finite perimeter, it follows from the Gauss–Green Theorem (cf. [9, Thm. 5.16]) that

$$P(E, \Omega) = \mathcal{H}^{m-1}(\partial_* E),$$

where $\partial_* E$ is the so called *measure theoretic boundary* of E.

We now study the convergence of the minimizers in Lemma 5 as $\varepsilon \to 0$. We employ the following notation:

$$H_\tau(s) = \int_0^s \phi_\tau^{1/q}(t) \, \mathrm{d}t.$$

Using this we can now prove the following:

Theorem 7. *Assume a stored energy of the form (9)–(11) and that $p \in (m-1, m)$. Let $(\mathbf{u}_\varepsilon^\tau, v_\varepsilon^\tau) \in \mathcal{U}$ be a minimizer of I_ε^τ over $\mathcal{U}$. Then for any sequence $\varepsilon_j \to 0$, the sequences $\{\mathbf{u}_j^\tau\}$ and $\{v_j^\tau\}$, where $\mathbf{u}_j^\tau = \mathbf{u}_{\varepsilon_j}^\tau$ and $v_j^\tau = v_{\varepsilon_j}^\tau$, have subsequences $\{\mathbf{u}_{j_k}^\tau\}$ and $\{v_{j_k}^\tau\}$ with $\mathbf{u}_{j_k}^\tau \rightharpoonup \mathbf{u}^\tau$ in $W^{1,p}(\Omega)$ and $v_{j_k}^\tau \overset{*}{\rightharpoonup} \nu^\tau$ in $\mathcal{M}(\Omega)$, where ν^τ is a nonnegative Radon measure. Moreover $\mathbf{u}^\tau|_{\partial\Omega} = \mathbf{u}^h$, $\mathbf{u}_e^\tau$ satisfies (INV) on Ω, and*

$$\mathrm{Det}\nabla \mathbf{u}^\tau = (\det \nabla \mathbf{u}^\tau)\mathcal{L}^m + \nu_s^\tau,$$

where $\det\nabla\mathbf{u}^\tau\in L^1(\Omega)$ *with* $\det\nabla\mathbf{u}^\tau>0$ *a.e. in* Ω *and* ν_s^τ *is the singular part of* ν^τ *with respect to Lebesgue measure. If we let* $\hat{v}_{j_k}^\tau(\mathbf{x})=\min\left\{v_{j_k}^\tau(\mathbf{x}),\tau\right\}$*, then* $\left\{\hat{v}_{j_k}^\tau\right\}$ *has a subsequence converging in* $L^1(\Omega)$ *to a function* g^τ *that assumes only the values* 0 *and* τ *a.e., and*

$$\varliminf_{k\to\infty} I_{\varepsilon_{j_k}}^\tau(\mathbf{u}_{j_k}^\tau,v_{j_k}^\tau)\geq\int_\Omega\left[\tilde{W}(\nabla\mathbf{u}^\tau(\mathbf{x}))+h(\det\nabla\mathbf{u}^\tau(\mathbf{x})-\omega^\tau(\mathbf{x}))\right]\mathrm{d}\mathbf{x} + H_\tau(\tau)P(B_\tau,\Omega), \tag{17}$$

where $\omega^\tau\in L^1(\Omega)$ *is the derivative of* ν^τ *with respect to Lebesgue measure and satisfies that* $\det\nabla\mathbf{u}^\tau>\omega^\tau\geq0$ *a.e., and* $B_\tau=\{\mathbf{x}\in\Omega\,:\,g^\tau(\mathbf{x})=0\}$.

Proof. The inequality

$$I_{\varepsilon_j}^\tau(\mathbf{u}_j^\tau,v_j^\tau)\leq\int_\Omega W(\nabla\mathbf{u}^h)\,\mathrm{d}\mathbf{x}, \tag{18}$$

together with (10) and Poincaré's inequality, imply the existence of a subsequence $\left\{\mathbf{u}_{j_k}^\tau\right\}$ converging weakly to a function $\mathbf{u}^\tau$ in $W^{1,p}(\Omega)$. Clearly $\mathbf{u}^\tau|_{\partial\Omega}=\mathbf{u}^h$, and that $\mathbf{u}_e^\tau$ satisfies (INV) on Ω follows from [28, Lemma 3.3]. From (11) and de la Vallée Poussin criteria, it follows that there is a subsequence (with indexes written as for the previous one) $\left\{\det\nabla\mathbf{u}_{j_k}^\tau-v_{j_k}^\tau\right\}$ such that

$$\det\nabla\mathbf{u}_{j_k}^\tau-v_{j_k}^\tau\rightharpoonup w^\tau,\quad\text{in } L^1(\Omega). \tag{19}$$

Since $\det\nabla\mathbf{u}_{j_k}^\tau-v_{j_k}^\tau>0$ a.e. on Ω, the first condition in (11) implies that we must have that $w^\tau>0$ a.e. on Ω. Now from $\det\nabla\mathbf{u}_{j_k}^\tau>v_{j_k}^\tau\geq0$, it follows that

$$\int_\Omega v_{j_k}^\tau\,\mathrm{d}\mathbf{x}\leq\int_\Omega\det\nabla\mathbf{u}_{j_k}^\tau\,\mathrm{d}\mathbf{x}=|\mathbf{u}^h(\Omega)|.$$

Thus $\left\{v_{j_k}^\tau\right\}$ is bounded in $L^1(\Omega)$. Hence there exists $\nu^\tau\in\mathcal{M}(\Omega)$ such that (for a subsequence denoted the same) $v_{j_k}^\tau\overset{*}{\rightharpoonup}\nu^\tau$ in $\mathcal{M}(\Omega)$. Since $v_{j_k}^\tau\geq0$ for all k, the measure ν^τ must be non–negative. Combining this with (19) we get that

$$(\det\nabla\mathbf{u}_{j_k}^\tau)\mathcal{L}^m\overset{*}{\rightharpoonup}w^\tau\mathcal{L}^m+\nu^\tau\quad\text{in }\Omega.$$

Since $\mathrm{Det}\nabla\mathbf{u}_{j_k}^\tau=(\det\nabla\mathbf{u}_{j_k}^\tau)\mathcal{L}^m$, we have that (see [36, proof of Lemma 4.5])

$$(\det\nabla\mathbf{u}_{j_k}^\tau)\mathcal{L}^m\overset{*}{\rightharpoonup}\mathrm{Det}\nabla\mathbf{u}^\tau\quad\text{in }\Omega,$$

from which it follows that $\mathrm{Det}\nabla\mathbf{u}^\tau=w^\tau\mathcal{L}^m+\nu^\tau$. By the Lebesgue decomposition theorem, $\nu^\tau=\nu_{ac}^\tau+\nu_s^\tau$ where ν_{ac}^τ is absolutely continuous with respect to $\mathcal{L}^m$ and ν_s^τ is singular with respect to $\mathcal{L}^m$. Thus $\mathrm{Det}\nabla\mathbf{u}^\tau=w^\tau\mathcal{L}^m+\nu_{ac}^\tau+\nu_s^\tau$. Since $w^\tau\mathcal{L}^m+\nu_{ac}^\tau$ is absolutely continuous with respect to $\mathcal{L}^m$, it follows by the uniqueness in the Lebesgue decomposition theorem, that $w^\tau\mathcal{L}^m+\nu_{ac}^\tau$ is the absolutely continuous part of $\mathrm{Det}\nabla\mathbf{u}^\tau$ with respect to $\mathcal{L}^m$. Since $p>m-1$ and $\mathbf{u}_e^\tau$ satisfies (INV) on Ω, the conclusions of Theorem 1 in [26] hold. In particular, from Remark 2 of that theorem, we get that the absolutely continuous part of $\mathrm{Det}\nabla\mathbf{u}^\tau$ is $(\det\nabla\mathbf{u}^\tau)\mathcal{L}^m$. Thus, by the uniqueness in the Lebesgue decomposition theorem, we must have that $(\det\nabla\mathbf{u}^\tau)\mathcal{L}^m=w^\tau\mathcal{L}^m+\nu_{ac}^\tau$. Hence $\mathrm{Det}\nabla\mathbf{u}^\tau=(\det\nabla\mathbf{u}^\tau)\mathcal{L}^m+\nu_s^\tau$ and $\det\nabla\mathbf{u}^\tau=w^\tau+\omega^\tau$ where ω^τ is the derivative of ν^τ with respect to $\mathcal{L}^m$. Since $w^\tau > 0$ and $\omega^\tau \geq 0$ a.e., it follows that $\det \nabla \mathbf{u}^\tau > \omega^\tau \geq 0$ a.e.

Since ϕ_τ is nonnegative and $\text{supp}(\phi_\tau) \subset (0,\tau)$, it follows that $\left\{H_\tau(v^\tau_{j_k})\right\}$ is bounded in $L^1(\Omega)$. Moreover

$$\int_\Omega \left[\frac{\varepsilon^\alpha_{j_k}}{\alpha}\|\nabla v^\tau_{j_k}(\mathbf{x})\|^\alpha + \frac{1}{q\varepsilon^q_{j_k}}\phi_\tau(v^\tau_{j_k}(\mathbf{x}))\right]\mathrm{d}\mathbf{x} \geq \int_\Omega \|\nabla[H_\tau(v^\tau_{j_k}(\mathbf{x}))]\|\,\mathrm{d}\mathbf{x}.$$

If we let $\hat{v}^\tau_{j_k}(\mathbf{x}) = \min\left\{v^\tau_{j_k}(\mathbf{x}),\tau\right\}$, then

$$\int_\Omega \|\nabla[H_\tau(v^\tau_{j_k}(\mathbf{x}))]\|\,\mathrm{d}\mathbf{x} = \int_\Omega \|\nabla[H_\tau(\hat{v}^\tau_{j_k}(\mathbf{x}))]\|\,\mathrm{d}\mathbf{x}$$

It follows $\left\{H_\tau(\hat{v}^\tau_{j_k})\right\}$ is bounded in $BV(\Omega)$ (cf. [25]) and thus it has a subsequence converging in $L^1(\Omega)$. Since $\hat{v}^\tau_{j_k} : \Omega \to [0,\tau]$, we get that $\hat{v}^\tau_{j_k} \to g^\tau$ in $L^1(\Omega)$. In addition

$$\begin{aligned}\int_\Omega \phi_\tau(v^\tau_{j_k}(\mathbf{x}))\,\mathrm{d}\mathbf{x} &= \int_\Omega \phi_\tau(\hat{v}^\tau_{j_k}(\mathbf{x}))\,\mathrm{d}\mathbf{x},\\ \lim_{k\to\infty}\int_\Omega \phi_\tau(v^\tau_{j_k}(\mathbf{x}))\,\mathrm{d}\mathbf{x} &= 0, \quad \text{(cf. (18))},\end{aligned}$$

from which we get that $\int_\Omega \phi_\tau(g^\tau(\mathbf{x}))\,\mathrm{d}\mathbf{x} = 0$, i.e., that g^τ assumes only the values 0 or τ a.e. Also

$$\begin{aligned}&\varliminf_{k\to\infty}\int_\Omega \left[\frac{\varepsilon^\alpha_{j_k}}{\alpha}\|\nabla v^\tau_{j_k}(\mathbf{x})\|^\alpha + \frac{1}{q\varepsilon^q_{j_k}}\phi_\tau(v^\tau_{j_k}(\mathbf{x}))\right]\mathrm{d}\mathbf{x} \geq\\ &\qquad \varliminf_{k\to\infty}\int_\Omega \|\nabla[H_\tau(\hat{v}^\tau_{j_k}(\mathbf{x}))]\|\,\mathrm{d}\mathbf{x} \geq \int_\Omega \|\nabla[H_\tau(g^\tau(\mathbf{x}))]\|\,\mathrm{d}\mathbf{x} = H_\tau(\tau)P(B_\tau,\Omega),\end{aligned}$$

where for the second inequality we used the lower semicontinuity property of the variation measure (cf. [9, Thm. 5.2]), and the last equality follows from the Fleming–Rishel formula (cf. [25]). Finally combining this result with those from the first part of this proof and the weak lower semicontinuity property of the mechanical part of the functional (14), we get that (17) follows. ∎

Note that Theorem 7 in a sense falls short of fully characterizing any possible singular behaviour in a minimizer $\mathbf{u}^*$ of the energy functional (3). Since the parameter τ is fixed, the phase functions are not "forced" to follow or mimic the singular behaviour in $\mathbf{u}^*$ once they have crossed the barrier τ. Moreover, the actual location of the set of possible singularities in $\mathbf{u}^*$ has not been fully resolved due to the presence of the function ω^τ in the h–term of the energy functional. Thus we need to study the behaviour of the functions $\mathbf{u}^\tau$, ω^τ, g^τ, and the measures ν^τ as $\tau \to \infty$.

In the sequel we employ some of the notation within the proof of Theorem 7 as well as the following: given $\tau_1 > 0$ and a sequence $\{\varepsilon_j\}$ converging to zero, we apply Theorem 7 to get a subsequence $\{\varepsilon_{1,r}\}$ of $\{\varepsilon_j\}$ with the corresponding sequences of functions $\{\mathbf{u}_{1,r}\}$, $\{v_{1,r}\}$, etc. We keep denoting the limiting functions and measures by $\mathbf{u}^{\tau_1}$, ν^{τ_1}, etc. Now given any τ_k with $k > 1$, we apply Theorem 7 to the subsequence $\{\varepsilon_{k-1,r}\}$ obtained from τ_{k-1}, to get a new subsequence $\{\varepsilon_{k,r}\}$ of $\{\varepsilon_{k-1,r}\}$, and so on. After relabeling, we denote by $\{\mathbf{u}_{k,r}\}$, $\{v_{k,r}\}$, etc., the sequences obtained from Theorem 7 by this process for any given τ_k.

LEMMA 8. *The sequences $\{g^{\tau_k}\}$ and $\{\nu^{\tau_k}\}$ have subsequences (not relabelled) such for some $\nu, \nu^* \in \mathcal{M}(\Omega)$, we have $g^{\tau_k} \overset{*}{\rightharpoonup} \nu$ and $\nu^{\tau_k} \overset{*}{\rightharpoonup} \nu^*$ in $\mathcal{M}(\Omega)$.*

Proof. Note that

$$\int_\Omega \hat{v}_{k,r}(\mathbf{x})\,\mathrm{d}\mathbf{x} \leq \int_\Omega v_{k,r}(\mathbf{x})\,\mathrm{d}\mathbf{x} \leq |\mathbf{u}^h(\Omega)|,$$

and since $\hat{v}^{\tau}_{k,r} \to g^{\tau_k}$ in $L^1(\Omega)$ as $r \to \infty$, it follows that

$$\int_\Omega g^{\tau_k}(\mathbf{x})\,\mathrm{d}\mathbf{x} \leq |\mathbf{u}^h(\Omega)|, \quad \forall k. \tag{20}$$

Thus for some subsequence of $\{\tau_k\}$ (not relabelled), we have that $g^{\tau_k} \overset{*}{\rightharpoonup} \nu$ in $\mathcal{M}(\Omega)$, for some $\nu \in \mathcal{M}(\Omega)$.

Also, since $v_{k,r} \overset{*}{\rightharpoonup} \nu^{\tau_k}$ as $r \to \infty$, we get that for any $\phi \in C_0(\Omega)$, $\|\phi\|_{L^\infty(\Omega)} \leq 1$, we have that

$$\lim_{r\to\infty} \int_\Omega v_{k,r}(\mathbf{x})\phi(\mathbf{x})\,\mathrm{d}\mathbf{x} = \langle \nu^{\tau_k}, \phi\rangle.$$

But

$$\left|\int_\Omega v_{k,r}(\mathbf{x})\phi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| \leq \int_\Omega v_{k,r}(\mathbf{x})\,\mathrm{d}\mathbf{x} \leq |\mathbf{u}^h(\Omega)|.$$

Letting $r \to \infty$ we get that $|\langle \nu^{\tau_k}, \phi\rangle| \leq |\mathbf{u}^h(\Omega)|$, and hence that $\|\nu^{\tau_k}\|_{\mathcal{M}(\Omega)} \leq |\mathbf{u}^h(\Omega)|$. Thus by taking a subsequence of $\{\tau_k\}$ (relabeled the same), we have $\nu^{\tau_k} \overset{*}{\rightharpoonup} \nu^*$ in $\mathcal{M}(\Omega)$, for some $\nu^* \in \mathcal{M}(\Omega)$. ∎

From these results and [7, Thm. 5.1], we get the following:

LEMMA 9. *The sequences $\{\hat{v}_{k,r}\}$ and $\{v_{k,r}\}$ have subsequences $\{\hat{v}_k\}$ and $\{v_k\}$ respectively, where $\hat{v}_k = \hat{v}_{k,r_k}$ and $v_k = v_{k,r_k}$ with $r_k \to \infty$, such that $\hat{v}_k \overset{*}{\rightharpoonup} \nu$ and $v_k \overset{*}{\rightharpoonup} \nu^*$ in $\mathcal{M}(\Omega)$, as $\tau_k \to \infty$.*

The two measures ν and ν^* in general are not equal. However, we will show that both are singular with respect to $\mathcal{L}^m$ and both are concentrated over the same set. To show this we need the following assumption on the functions $\{\phi_\tau\}$: given $0 < a < b$, there exists $\varrho > 0$ and $\tau_0 > b$ such that

$$\phi_\tau(v) \geq \varrho, \quad \forall\, a \leq v \leq b, \tag{21}$$

and $\tau \geq \tau_0$. This condition rules out the possibility that $\int_\Omega \phi_{\tau_k}(v_k)\,\mathrm{d}\mathbf{x} \to 0$ as $k \to \infty$, without the functions $\{v_k\}$ concentrating as $k \to \infty$.

PROPOSITION 10. *Let condition (21) hold. Then there exist sets B and D disjoint such that $\Omega = B \cup D$, where $|D| = 0$ and $\nu^*(B) = \nu(B) = 0$, i.e., both ν and ν^* are singular with respect to Lebesgue measure $\mathcal{L}^m$.*

Proof. For each integer $k \geq 1$, let

$$E_k = \{\mathbf{x} \in \Omega \,:\, v_k(\mathbf{x}) > \tau_k\}.$$

Provided $\tau_k \geq k^2$, we have that $|E_k| \leq \frac{C}{k^2}$ for some positive constant C independent of k. Hence $\sum_k |E_k| < \infty$ and by the Borel–Cantelli lemma we get that $|D| = 0$, where

$$D = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.$$

The set D, if nonempty, is precisely where the sequence $\{v_k\}$ becomes unbounded. If we let $B = D^c$, where $D^c = \Omega \setminus D$, then B has full measure $|\Omega|$. Note that we can also write D as

$$D = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{\mathbf{x} \in \Omega \,:\, v_k(\mathbf{x}) > n\}\,.$$

Thus

$$B = \bigcup_{n=1}^{\infty} C_n, \quad C_n = \bigcap_{k=n}^{\infty} \{\mathbf{x} \in \Omega \,:\, v_k(\mathbf{x}) \leq n\}\,.$$

It follows from (14) and (16), that

$$\lim_{k\to\infty} \int_\Omega \phi_{\tau_k}(v_k(\mathbf{x}))\, \mathrm{d}\mathbf{x} = 0.$$

Since on C_n, we have $v_k \leq n$ for all $k \geq n$, it follows from the above limit and condition (21) that $v_k \to 0$ a.e. on C_n. Thus by the Bounded Convergence Theorem,

$$\nu^*(C_n) = \lim_{k\to\infty} \int_{C_n} v_k(\mathbf{x})\, \mathrm{d}\mathbf{x} = 0.$$

Hence

$$\nu^*(B) \leq \sum_{n=1}^{\infty} \nu^*(C_n) = 0.$$

Moreover, as $\hat{v}_k \leq v_k$, we get that $\nu(B) \leq \nu^*(B)$, and thus that $\nu^*(B) = \nu(B) = 0$. ☐

Our next result establishes a connection between the limit (as $\tau \to \infty$) of the sets $\{B_\tau\}$ in Theorem 7 with the set B in Proposition 10.

PROPOSITION 11. *Let* $B_k = \{\mathbf{x} \in \Omega \,:\, g^{\tau_k}(\mathbf{x}) = 0\}$ *and*

$$\hat{B} = \varliminf_k B_k = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} B_k = \lim_n \bigcap_{k=n}^{\infty} B_k.$$

Then $|\hat{B}| = |B| = |\Omega|$ *and* $B \subset \hat{B} \cup U$ *with* $|U| = 0$. *Moreover*

$$P(B, \Omega) \leq \varliminf_k P(B_k, \Omega). \tag{22}$$

Proof. Since g^{τ_k} assumes only the values 0 or τ_k, we have from (20), and provided $\tau_k \geq k^2$, that $|\hat{B}^c| = 0$ and thus that $\hat{B}$ has full measure $|\Omega|$.

From [24, Prop. 1] we have that the sequence $\{g^{\tau_k}\}$ converges to zero a.e. on Ω. In particular, $\{g^{\tau_k}\}$ converges to zero a.e. on each C_n, where C_n is as in the proof of Proposition 10. Recall that on C_n we have that $v_k \leq n$ for all $k \geq n$. If we let k_n be such that $\tau_k > n$ for all $k \geq k_n$, then we have that $g^{\tau_k} = 0$ a.e. on C_n for all $k \geq k_n$, that is $C_n \setminus U_n \subset \hat{C}_{k_n}$ where $|U_n| = 0$ and

$$\hat{C}_{k_n} = \bigcap_{k=k_n}^{\infty} B_k.$$

From this we get that $C_n \subset \hat{C}_{k_n} \cup U_n$, from which it follows that $B \subset \hat{B} \cup U$ with $|U| = 0$.

For the last part of the proposition, since $\hat{B} = \underline{\lim}_k B_k$ it follows that $\chi_{\hat{B}} = \underline{\lim}_k \chi_{B_k}$. Thus for any $\boldsymbol{\phi} \in C_0^1(\Omega;\mathbb{R}^n)$, with $\|\boldsymbol{\phi}\|_{L^\infty(\Omega)} \leq 1$, we have that (cf. [32, Ex. 12, Pag. 90])

$$\begin{aligned}\int_\Omega \chi_B(\mathbf{x}) \operatorname{div} \boldsymbol{\phi}(\mathbf{x})\, \mathrm{d}\mathbf{x} &= \int_\Omega \chi_{\hat{B}}(\mathbf{x}) \operatorname{div} \boldsymbol{\phi}(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &\leq \underline{\lim}_k \int_\Omega \chi_{B_k}(\mathbf{x}) \operatorname{div} \boldsymbol{\phi}(\mathbf{x})\, \mathrm{d}\mathbf{x} \leq \underline{\lim}_k P(B_k, \Omega),\end{aligned}$$

from which we get (22). ◻

We now give the corresponding convergence results for the sequences $\{\mathbf{u}^{\tau_k}\}$ and $\{\omega^{\tau_k}\}$.

Proposition 12. *Let $\{\tau_k\}$ be a sequence such that $\tau_k \to \infty$. Then the sequences $\{\mathbf{u}^{\tau_k}\}$ and $\{\omega^{\tau_k}\}$ have subsequences relabeled the same, such that $\mathbf{u}^{\tau_k} \rightharpoonup \mathbf{u}^*$ in $W^{1,p}(\Omega)$, $\det \nabla \mathbf{u}^{\tau_k} \rightharpoonup \det \nabla \mathbf{u}^*$ in $L^1(\Omega)$, and $\omega^{\tau_k} \to 0$ in $L^1(\Omega)$. Moreover, the function $\mathbf{u}^*$ is such that $\mathbf{u}^*|_{\partial\Omega} = \mathbf{u}^h$, $\mathbf{u}_e^*$ satisfies (INV) on Ω, and*

$$\operatorname{Det}\nabla \mathbf{u}^* = (\det \nabla \mathbf{u}^*)\mathcal{L}^m + \nu^*, \tag{23}$$

with $\det \nabla \mathbf{u}^* \in L^1(\Omega)$ and $\det \nabla \mathbf{u}^* > 0$ a.e. in Ω.

Proof. Since, by Lemma 8, $\nu^{\tau_k} \overset{*}{\rightharpoonup} \nu^*$, we have that $\nu^{\tau_k}(B) \to \nu^*(B) = 0$, where B is as in Proposition 10. As ω^{τ_k} is the derivative of ν^{τ_k}, we get that

$$\int_B \omega^{\tau_k}\, \mathrm{d}\mathbf{x} \leq \nu^{\tau_k}(B).$$

As $\omega^{\tau_k} \geq 0$ a.e., the above implies that

$$\lim_{k\to\infty} \int_B \omega^{\tau_k}\, \mathrm{d}\mathbf{x} = 0,$$

which implies that $\omega^{\tau_k} \to 0$ in $L^1(\Omega)$, where we used that $|B| = |\Omega|$.

From (10), (17), (18), and Poincaré's inequality, we get that for a subsequence of $\{\mathbf{u}^{\tau_k}\}$ (not relabeled), we have $\mathbf{u}^{\tau_k} \rightharpoonup \mathbf{u}^*$ in $W^{1,p}(\Omega)$ for some function $\mathbf{u}^* \in W^{1,p}(\Omega)$. Clearly $\mathbf{u}^*|_{\partial\Omega} = \mathbf{u}^h$, and that $\mathbf{u}_e^*$ satisfies (INV) on Ω follows from [28, Lemma 3.3] and the fact that each $\mathbf{u}_k$ satisfies (INV).

From (11) and de la Vallée Poussin criteria, it follows that there is a subsequence (with indexes written as for the previous one) $\{\det \nabla \mathbf{u}^{\tau_k} - \omega^{\tau_k}\}$ such that

$$\det \nabla \mathbf{u}^{\tau_k} - \omega^{\tau_k} \rightharpoonup w^*, \quad \text{in } L^1(\Omega).$$

Since $\det \nabla \mathbf{u}^{\tau_k} - \omega^{\tau_k} > 0$ a.e. on Ω, the first condition in (11) implies that we must have that $w^* > 0$ a.e. on Ω. Now $\det \nabla \mathbf{u}^{\tau_k} > \omega^{\tau_k} \geq 0$ a.e. on Ω, and since $\omega^{\tau_k} \to 0$ in $L^1(\Omega)$, we get from the previous convergence that

$$\det \nabla \mathbf{u}^{\tau_k} \rightharpoonup w^*, \quad \text{in } L^1(\Omega).$$

It follows now from [28, Theorem 4.2], that $\det \nabla \mathbf{u}^* = w^*$. From the proof of Theorem 7, we have that $\det \nabla \mathbf{u}^{\tau_k} = w^{\tau_k} + \omega^{\tau_k}$ from which it follows that $w^{\tau_k} \rightharpoonup w^*$, in $L^1(\Omega)$. Also $\operatorname{Det}\nabla \mathbf{u}^{\tau_k} = w^{\tau_k}\mathcal{L}^m + \nu^{\tau_k}$ and since $\operatorname{Det}\nabla \mathbf{u}^{\tau_k} \overset{*}{\rightharpoonup} \operatorname{Det}\nabla \mathbf{u}^*$, we get that (23) holds. ◻

We now have one of the main results of this paper.

THEOREM 13. *Let $\{\tau_k\}$ and $\{\varepsilon_r\}$ be sequences such that $\tau_k \to \infty$ and $\varepsilon_r \to 0^+$, and let $(\mathbf{u}_{k,r}, v_{k,r})$ be a minimizer of $I^{\tau_k}_{\varepsilon_r}$ over $\mathcal{U}$. Then there exist a subsequence of $\{\tau_k\}$ relabelled the same, and a subsequence $\{\varepsilon_{r_k}\}$, such that if $(\mathbf{u}_k, v_k) = (\mathbf{u}_{k,r_k}, v_{k,r_k})$, then $\mathbf{u}_k \rightharpoonup \mathbf{u}^*$ in $W^{1,p}(\Omega)$ and $v_k \overset{*}{\rightharpoonup} \nu^*$ in $\mathcal{M}(\Omega)$ as $k \to \infty$. Moreover, with $B_k = \{\mathbf{x} \in \Omega : g^{\tau_k}(\mathbf{x}) = 0\}$, we have that*

$$\varliminf_{k\to\infty} I^{\tau_k}_{\varepsilon_{r_k}}(\mathbf{u}_k, v_k) \geq \int_\Omega W(\nabla \mathbf{u}^*(\mathbf{x}))\,\mathrm{d}\mathbf{x} + c, \tag{24}$$

where

$$c = \varliminf_{k\to\infty} H_{\tau_k}(\tau_k) P(B_k, \Omega).$$

Proof. The existence and the convergence of the subsequence $\{v_k\}$ with $v_k = v_{k,r_k}$, follows from the boundedness of $\{v_{k,r}\}$ in $L^1(\Omega)$, Theorem 7, Lemma 8, and [7, Thm. 5.1]. For the existence and the convergence of the subsequence $\{\mathbf{u}_k\}$ with $\mathbf{u}_k = \mathbf{u}_{k,r_k}$, it follows from the boundedness of this sequence in $W^{1,p}(\Omega)$ (cf. (18)), Theorem 7, Proposition 12, and [7, Thm. 5.1].

Without loss of generality, we can assume that for each k, the r_k is chosen so that

$$I^{\tau_k}_{\varepsilon_{k,r_k}}(\mathbf{u}_{k,r_k}, v_{k,r_k}) > \varliminf_{r\to\infty} I^{\tau_k}_{\varepsilon_{k,r}}(\mathbf{u}_{k,r}, v_{k,r}) - \frac{1}{k}.$$

We get now using (17) that

$$\begin{aligned} I^{\tau_k}_{\varepsilon_{k,r_k}}(\mathbf{u}_k, v_k) \quad &\geq \quad \int_\Omega \left[\tilde{W}(\nabla \mathbf{u}^{\tau_k}(\mathbf{x})) + h(\det \nabla \mathbf{u}^{\tau_k}(\mathbf{x}) - \omega^{\tau_k}(\mathbf{x}))\right] \mathrm{d}\mathbf{x} \\ &\quad + H_{\tau_k}(\tau_k) P(B_k, \Omega) - \frac{1}{k}. \end{aligned} \tag{25}$$

As the energies $\left\{I^{\tau_k}_{\varepsilon_{k,r_k}}(\mathbf{u}_k, v_k)\right\}$ are bounded, the constant c in the statement of the theorem must be finite. The result (24) now follows from this, (25), and the convergence results in Proposition 12 for the sequences $\{\mathbf{u}^{\tau_k}\}$, $\{\det \nabla \mathbf{u}^{\tau_k}\}$, and $\{\omega^{\tau_k}\}$. □

The measure ν^* in this theorem, according to Proposition 10, is concentrated on the set D which is the complement of B. In addition, by the extended Lebesgue Decomposition Theorem (cf [12], [16]), ν^* is the sum of a discrete measure and a continuous one, both singular with respect to Lebesgue measure. The discrete part of ν^* corresponds to points in the reference configuration where singularities of cavitation type may occur, while the continuous part corresponds to lower dimensional surfaces in the reference configuration where fractures or other type of nonzero dimensional singularities might take place. We should mention that by [28, Thm. 8.4], if the perimeter $P(\mathrm{im}(\mathbf{u}^*(\Omega)))$ is finite, then ν^* must be discrete.

5. The radial problem. For ease of exposition we limit ourselves in this section to the case where $m = 3$. We recall that if $\tilde{W}$ is frame indifferent and isotropic then there is a symmetric function $\tilde{\Phi}$ such that

$$\tilde{W}(\mathbf{F}) = \tilde{\Phi}(v_1, v_2, v_3), \tag{26}$$

where v_1, v_2, v_3 are the singular values of the matrix $\mathbf{F}$. For the function $h(\cdot)$ in (11) we assume that it is strictly convex so that it has a unique minimum at d_0, and that

$$h(d) \sim C d^\gamma, \quad d \to \infty, \tag{27}$$

where $\gamma > 1$ and C is some positive constant.

For Ω equal to the unit ball with center at the origin, the radial deformation

$$\mathbf{u}(\mathbf{x}) = \frac{r(R)}{R}\,\mathbf{x}, \quad R = \|\mathbf{x}\|, \tag{28}$$

has energy (up to a constant) given by:

$$E_{\mathrm{rad}}(r) = \int_0^1 R^2 \left[\tilde{\Phi}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) + h\left(r'(R)\left(\frac{r(R)}{R}\right)^2\right)\right] \mathrm{d}R. \tag{29}$$

It is well known (cf. [3], [37]) that for $p \in (1,3)$ in (10), there exists $\lambda_c > d_0^{\frac{1}{3}}$ such that for $\lambda > \lambda_c$, the minimizer r_c of $E_{\mathrm{rad}}(\cdot)$ over the set

$$\mathcal{A}_{\mathrm{rad}} = \left\{r \in W^{1,1}(0,1) \,:\, r'(R) > 0 \text{ a.e.}, \, r(0) \geq 0, \, r(1) = \lambda\right\}, \tag{30}$$

exists and has $r_c(0) > 0$.

With v a radial function now, the modified functional (14) reduces up to a constant to:

$$\begin{aligned} I_\varepsilon^\tau(r,v) \;&=\; \int_0^1 R^2 \left[\tilde{\Phi}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) + h\left(r'(R)\left(\frac{r(R)}{R}\right)^2 - v(R)\right)\right] \mathrm{d}R \\ &\quad + \int_0^1 R^2 \left[\frac{\varepsilon^\alpha}{\alpha}|v'(R)|^\alpha + \frac{1}{q\varepsilon^q}\,\phi_\tau(v(R))\right] \mathrm{d}R, \end{aligned} \tag{31}$$

and the set $\mathcal{U}$ becomes

$$\begin{aligned} \mathcal{U}_{\mathrm{rad}} = \{(r,v) \in W^{1,1}(0,1) \times W^{1,\alpha}(0,1) : r(0) = 0, \ r(1) = \lambda, \\ r'(R)(r(R)/R)^2 > v(R) \geq 0 \text{ a.e.}, \ v(1) = 0\}. \end{aligned} \tag{32}$$

As a special case of Proposition 6, we now have the following result:

PROPOSITION 14. *Assume that the stored enery function (26) is quasiconvex. Then for $\lambda \leq d_0^{\frac{1}{3}}$, the global minimizer of $I_\varepsilon^\tau(\cdot,\cdot)$ over $\mathcal{U}_{\mathrm{rad}}$ is given by $r(R) = \lambda R$ and $v(R) = 0$ for all R.*

Note that if $(r,0) \in \mathcal{U}_{\mathrm{rad}}$, then $r(0) = 0$, and quasiconvexity implies that

$$I_\varepsilon^\tau(r,0) \geq I_\varepsilon^\tau(\lambda R, 0). \tag{33}$$

Moreover, since $I_\varepsilon^\tau(r,0) = E_{\mathrm{rad}}(r)$, we have that

$$I_\varepsilon^\tau(\lambda R, 0) > E_{\mathrm{rad}}(r_c), \quad \lambda > \lambda_c. \tag{34}$$

In our next result we show that for large boundary displacements λ, given a sequence (τ_j) with $\tau_j \to \infty$, one can construct a sequence (ε_j) with $\varepsilon_j \to 0$ and a corresponding sequence of admissible function pairs for (29) over $\mathcal{A}_{\mathrm{rad}}$, such that the corresponding decoupled energies converge to the energy of the cavitating radial minimizer. Using this together with the lower bound Γ–convergence result of Section 4, we then prove in Theorem 16 that the approximations of the proposed decoupled–penalized method, converge to the radial cavitating solution.

THEOREM 15. *Let $\lambda > \lambda_c$ and $\gamma > 1$ be as in (27). Assume that*

$$\int_0^\tau \phi_\tau(u)\,\mathrm{d}u = O(\tau^a) \quad \text{as } \tau \to \infty, \tag{35}$$

for some $a > 1$. Then for any τ sufficiently large, there exists $\varepsilon(\tau) > 0$ with $\varepsilon(\tau) \to 0^+$ as $\tau \to \infty$, and an admissible pair $(\tilde{r}_\tau, \tilde{v}_\tau) \in \mathcal{U}_{\mathrm{rad}}$ with $\tilde{v}_\tau$ non–constant, such that

$$\lim_{\tau\to\infty} I^\tau_{\varepsilon(\tau)}(\tilde{r}_\tau, \tilde{v}_\tau) = E_{\mathrm{rad}}(r_c).$$

In particular, any minimizer (r_τ, v_τ) of $I^\tau_{\varepsilon(\tau)}$ must have v_τ non–constant, and

$$\varliminf_{\tau\to\infty} I^\tau_{\varepsilon(\tau)}(r_\tau, v_\tau) \leq E_{\mathrm{rad}}(r_c). \tag{36}$$

Proof. We now construct $(\tilde{r}, \tilde{v})$, $\tilde{v}$ non constant such that

$$I^\tau_\varepsilon(\tilde{r}, \tilde{v}) < I^{\tau(\varepsilon)}_\varepsilon(\lambda R, 0),$$

for τ sufficiently large and ε sufficiently small. For any $\delta > 0$ we let

$$\tau = \left(\frac{r_c(\delta)}{\delta}\right)^3 - d_0. \tag{37}$$

Since $r_c(0) > 0$, we have that $\tau \to \infty$ as $\delta \to 0^+$. For δ sufficiently small, we let $\eta \in (0, \delta)$ and define:

$$\tilde{r}(R) = \begin{cases} \left[\frac{r_c(\delta)}{\delta}\right] R & , \quad 0 \leq R \leq \delta, \\ r_c(R) & , \quad \delta \leq R \leq 1, \end{cases}$$

$$\tilde{v}(R) = \begin{cases} \tau & , \quad 0 \leq R \leq \delta - \eta, \\ \frac{\tau}{\eta}(\delta - R) & , \quad \delta - \eta \leq R \leq \delta, \\ 0 & , \quad \delta \leq R \leq 1. \end{cases}$$

For this test pair we have that

$$\begin{aligned} I^\tau_\varepsilon(\tilde{r}, \tilde{v}) &= \int_0^{\delta-\eta} R^2 \left[\tilde{\Phi}\left(\tilde{r}'(R), \frac{\tilde{r}(R)}{R}, \frac{\tilde{r}(R)}{R}\right) + h\left(\tilde{r}'(R)\left[\frac{\tilde{r}(R)}{R}\right]^2 - \tilde{v}(R)\right)\right] \mathrm{d}R \\ &+ \int_{\delta-\eta}^{\delta} R^2 \left[\tilde{\Phi}\left(\tilde{r}'(R), \frac{\tilde{r}(R)}{R}, \frac{\tilde{r}(R)}{R}\right) + h\left(\tilde{r}'(R)\left[\frac{\tilde{r}(R)}{R}\right]^2 - \tilde{v}(R)\right)\right] \mathrm{d}R \\ &+ \int_{\delta}^{1} R^2 \left[\tilde{\Phi}\left(\tilde{r}'(R), \frac{\tilde{r}(R)}{R}, \frac{\tilde{r}(R)}{R}\right) + h\left(\tilde{r}'(R)\left[\frac{\tilde{r}(R)}{R}\right]^2 - \tilde{v}(R)\right)\right] \mathrm{d}R \\ &+ \int_{\delta-\eta}^{\delta} R^2 \left[\frac{\varepsilon^\alpha}{\alpha}\,|\tilde{v}'(R)|^\alpha + \frac{1}{q\varepsilon^q}\,\phi_\tau(\tilde{v}(R))\right] \mathrm{d}R \equiv I_1 + I_2 + I_3 + I_4. \end{aligned}$$

From the definition of $(\tilde{r}, \tilde{v})$, it follows that:

1.

$$I_1 = \int_0^{\delta-\eta} R^2 \left[\tilde{\Phi}\left(\frac{r_c(\delta)}{\delta}, \frac{r_c(\delta)}{\delta}, \frac{r_c(\delta)}{\delta}\right) + h(d_0)\right] \mathrm{d}R$$

$$= \frac{(\delta-\eta)^3}{3}\left[\tilde\Phi\left(\frac{r_c(\delta)}{\delta},\frac{r_c(\delta)}{\delta},\frac{r_c(\delta)}{\delta}\right)+h(d_0)\right].$$

By taking

$$\eta=\delta^{\beta_1},\quad \beta_1>1, \tag{38}$$

we get from that I_1 can be made arbitrarily small with δ.

2. For the term I_2, first note that since

$$\tilde v(R)\le\tau=\left(\frac{r_c(\delta)}{\delta}\right)^3-d_0.$$

we have that

$$d_0\le\left(\frac{r_c(\delta)}{\delta}\right)^3-\tilde v(R)\le\left(\frac{r_c(\delta)}{\delta}\right)^3.$$

Since $h(\cdot)$ is increasing on (d_0,∞), it follows that

$$h\left(\left(\frac{r_c(\delta)}{\delta}\right)^3-\tilde v(R)\right)\le h\left(\left(\frac{r_c(\delta)}{\delta}\right)^3\right).$$

Thus

$$\begin{aligned} I_2 &= \int_{\delta-\eta}^{\delta} R^2\left[\tilde\Phi\left(\frac{r_c(\delta)}{\delta},\frac{r_c(\delta)}{\delta},\frac{r_c(\delta)}{\delta}\right)+h\left(\left(\frac{r_c(\delta)}{\delta}\right)^3-\tilde v(R)\right)\right]\mathrm{d}R\\ &\le \int_{\delta-\eta}^{\delta} R^2\left[\tilde\Phi\left(\frac{r_c(\delta)}{\delta},\frac{r_c(\delta)}{\delta},\frac{r_c(\delta)}{\delta}\right)+h\left(\left(\frac{r_c(\delta)}{\delta}\right)^3\right)\right]\mathrm{d}R. \end{aligned}$$

Now

$$\int_{\delta-\eta}^{\delta} R^2\tilde\Phi\left(\frac{r_c(\delta)}{\delta},\frac{r_c(\delta)}{\delta},\frac{r_c(\delta)}{\delta}\right)\mathrm{d}R\le\eta\delta^2\tilde\Phi\left(\frac{r_c(\delta)}{\delta},\frac{r_c(\delta)}{\delta},\frac{r_c(\delta)}{\delta}\right).$$

It follows from (10) and (38) that the right hand side of the above inequality goes to zero with δ. For the other term in I_2 we have:

$$\int_{\delta-\eta}^{\delta} R^2 h\left(\left(\frac{r_c(\delta)}{\delta}\right)^3\right)\mathrm{d}R\le C\frac{\eta\delta^2}{\delta^{3\gamma}},$$

for some constant $C>0$ and where $\gamma>1$ is the growth rate of $h(d)$ as $d\to\infty$ (cf. (27)). If we further assume that $\beta_1>3\gamma-2$, then I_2 goes to zero with δ.

3. Since $\tilde r(R)=r_c(R)$ and $\tilde v(R)=0$ for $\delta\le R\le 1$, we have that

$$\begin{aligned} I_3 &= \int_{\delta}^{1} R^2\left[\tilde\Phi\left(r_c'(R),\frac{r_c(R)}{R},\frac{r_c(R)}{R}\right)+h\left(r_c'(R)\left[\frac{r_c(R)}{R}\right]^2\right)\right]\mathrm{d}R\\ &= E_{\mathrm{rad}}(r_c)-\int_{0}^{\delta} R^2\left[\tilde\Phi\left(r_c'(R),\frac{r_c(R)}{R},\frac{r_c(R)}{R}\right)+h\left(r_c'(R)\left[\frac{r_c(R)}{R}\right]^2\right)\right]\mathrm{d}R. \end{aligned}$$

But $R^2\left[\tilde\Phi\left(r_c'(R),\frac{r_c(R)}{R},\frac{r_c(R)}{R}\right)+h\left(r_c'(R)\left[\frac{r_c(R)}{R}\right]^2\right)\right]\in L^1(0,1)$. Hence

$$\int_{0}^{\delta} R^2\left[\tilde\Phi\left(r_c'(R),\frac{r_c(R)}{R},\frac{r_c(R)}{R}\right)+h\left(r_c'(R)\left[\frac{r_c(R)}{R}\right]^2\right)\right]\mathrm{d}R$$

can be made arbitrarily small with δ.

4. For the last term in $I^\tau_\varepsilon(\tilde r,\tilde v)$:

$$\begin{aligned} I_4 &= \int_{\delta-\eta}^{\delta} R^2\left[\frac{\varepsilon^\alpha}{\alpha}\,|\tilde v'(R)|^\alpha + \frac{1}{q\varepsilon^q}\,\phi_\tau(\tilde v(R))\right]\mathrm{d}R \\ &\le \eta\delta^2\left[C_1\frac{\varepsilon^\alpha}{\delta^{3\alpha}\eta^\alpha} + \frac{C_2\eta}{\varepsilon^q\delta^{3(a-1)}}\right]. \end{aligned}$$

Here we used (37), condition (35), and that

$$\int_{\delta-\eta}^{\delta}\phi_\tau(\tilde v(R))\,\mathrm{d}R = \frac{\eta}{\tau}\int_0^\tau \phi_\tau(u)\,\mathrm{d}u.$$

We set

$$\frac{\varepsilon^\alpha}{\delta^{3\alpha}\eta^\alpha} = \frac{\eta}{\varepsilon^q\delta^{3(a-1)}},$$

so that both terms on the right hand side of the inequality for I_4 above are of the same order, which upon recalling (38), leads to

$$\varepsilon^{\alpha+q} = \delta^{(\beta_1+3)\alpha+\beta_1-3(a-1)}. \tag{39}$$

Thus provided $\beta_1 > 3a$, we have that given $\delta > 0$, if ε is chosen according to (39), then $\varepsilon \to 0^+$ and $\tau \to \infty$ (cf. (37)) as $\delta \to 0^+$. Thus

$$\eta\delta^2\frac{\eta}{\varepsilon^q} = \delta^{2\beta_1+2-\frac{q}{\alpha+q}((\beta_1+3)\alpha+\beta_1)} = \delta^{\frac{\alpha}{\alpha+q}(\beta_1-q)+\frac{3q}{\alpha+q}(a-1)},$$

and both terms in I_4 go to zero with δ provided $\beta_1 > \max\{q, 3a\}$.

Thus we can conclude that

$$I^\tau_{\varepsilon(\tau)}(\tilde r,\tilde v) \to E_{\mathrm{rad}}(r_c), \quad \text{as } \tau\to\infty.$$

If (r_τ, v_τ) is a minimizer of $I^\tau_{\varepsilon(\tau)}$, then $I^\tau_{\varepsilon(\tau)}(r_\tau, v_\tau) \le I^\tau_{\varepsilon(\tau)}(\tilde r,\tilde v)$, and (36) follows upon taking $\liminf$ on both sides of this inequality. If the minimizing pair (r_τ, v_τ) would have $v_\tau \equiv 0$ for τ sufficiently large, then

$$E_{\mathrm{rad}}(r_c) < E_{\mathrm{rad}}(r_H) \le E_{\mathrm{rad}}(r_\tau) = I^\tau_{\varepsilon(\tau)}(r_\tau, 0) \le I^\tau_{\varepsilon(\tau)}(\tilde r,\tilde v),$$

where the inequality $E_{\mathrm{rad}}(r_H) \le E_{\mathrm{rad}}(r_\tau)$, follows from the fact that $r_\tau(0) = 0$ and that $r_H(R) = \lambda R$ is the global minimizer among such functions. Letting $\tau\to\infty$ in the inequality above leads to a contradiction. Hence v_τ must be non–constant for τ sufficiently large. ∎

Now, in the radial case, the limiting function $\mathbf{u}^*$ of Theorem 13 must be radial, and the limiting measure ν^* must be a non–negative multiple of the Dirac delta distribution centered at the origin. Since $\mathbf{u}^*$ is radial we must have, with Ω the unit ball, that

$$\int_\Omega W(\nabla\mathbf{u}^*)\,\mathrm{d}\mathbf{x} \ge E_{\mathrm{rad}}(r_c).$$

Thus, combining this with (24) and (36), we get that the constant c in Theorem 13 must be zero, and that

$$E_{\mathrm{rad}}(r_c) = \int_\Omega W(\nabla\mathbf{u}^*)\,\mathrm{d}\mathbf{x} = \varliminf_{\tau\to\infty}\inf_{\mathcal{U}_{\mathrm{rad}}} I^\tau_{\varepsilon(\tau)}(r,v),$$

where $\mathbf{u}^*$ is given by (28) using r_c. Thus we have proved the following:

THEOREM 16. *Assume that* (35) *holds. Fix* $\lambda > \lambda_c$ *and let* $(r_\varepsilon^\tau, v_\varepsilon^\tau)$ *be a minimizer of* I_ε^τ *over* $\mathcal{U}_{\text{rad}}$ *and* $\mathbf{u}_\varepsilon^\tau$ *be the radial map* (28) *corresponding to* r_ε^τ*. Let* $\{\tau_j\}$ *be a sequence such that* $\tau_j \to \infty$*. Then for a subsequence of* $\{\tau_j\}$*, there exists a sequence* $\{\varepsilon_j\}$ *with* $\varepsilon_j \to 0^+$*, such that the sequences* $\{\mathbf{u}_j\}$ *and* $\{v_j\}$*, where* $\mathbf{u}_j = \mathbf{u}_{\varepsilon_j}^{\tau_j}$ *and* $v_j = v_{\varepsilon_j}^{\tau_j}$*, have subsequences (relabeled the same)* $\{\mathbf{u}_j\}$ *and* $\{v_j\}$ *with* $\mathbf{u}_j \rightharpoonup \mathbf{u}^*$ *in* $W^{1,p}(\Omega)$ *and* $v_j \overset{*}{\rightharpoonup} \nu$ *in* $\mathcal{M}(\Omega)$*, where* $\mathbf{u}^*$ *is given by* (28) *using* r_c *(the minimizer of* $E_{\text{rad}}(\cdot)$ *over the set* (30)*) and* $\nu = \kappa\delta_{\mathbf{0}}$ *with* $\kappa > 0$*. Moreover*

$$E_{\text{rad}}(r_c) = \varliminf_{j\to\infty} I_{\varepsilon_j}^{\tau_j}(r_j, v_j).$$

5.1. The Euler–Lagrange equations. In this section we show that the minimizers of (31) over (32), satisfy the Euler–Lagrange equations for this functional. The analysis is not straightforward, basically due to the singular behaviour of the function $h(\cdot)$ (cf. (11)), and the inequality constraints involving the phase function v, that is, its non–negativity and the inequality involving the determinant of the deformation r. The proof is a variation of that in [3].

For the following discussion we use the notation:

$$\hat{\Phi}(v_1, v_2, v_3, v_4) = \tilde{\Phi}(v_1, v_2, v_3) + h(v_1 v_2 v_3 - v_4). \tag{40}$$

Also we shall write

$$\hat{\Phi}(r(R), v(R)) = \hat{\Phi}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}, v(R)\right), \quad \text{etc.}$$

The functional (31) can now be written as:

$$\begin{aligned} I_\varepsilon^\tau(r,v) \quad &= \quad \int_0^1 R^2 \hat{\Phi}\left(r(R), v(R)\right) \mathrm{d}R \\ &\quad + \int_0^1 R^2 \left[\frac{\varepsilon^\alpha}{\alpha}|v'(R)|^\alpha + \frac{1}{q\varepsilon^q}\phi_\tau(v(R))\right] \mathrm{d}R, \end{aligned} \tag{41}$$

where $(r, v) \in \mathcal{U}_{\text{rad}}$ (cf. (32)).

For the analysis in this section we take $\tilde{\Phi}$ in (40) as

$$\tilde{\Phi}(v_1, v_2, v_3) = \sum_{i=1}^{3} \psi(v_i), \tag{42}$$

where ψ is a non-negative convex C^3 function over $(0, \infty)$, and for some positive constants $K > 0$ and $0 < \gamma_0 < 1$:

$$|v\,\psi'(cv)| \leq K\psi(v), \tag{43}$$

for all $v > 0$ and $c \in [1-\gamma_0, 1+\gamma_0]$. However, our results hold as well for more general stored energy functions under suitable assumptions. We now have:

THEOREM 17. *Let* (r, v) *be any minimizer of* I_ε^τ *over* (32)*. Assume that the functions* $h(\cdot)$ *and* $\psi(\cdot)$ *in* (40) *together with* (42)*, satisfy* (11) *and* (43) *respectively. Then* $(r, v) \in C^1(0,1] \times C^1(0,1]$*,* $r'(R) > 0$ *for all* $R \in (0,1]$*,* $R^2\hat{\Phi}_1(r(R), v(R))$ *is* $C^1(0,1]$*, and*

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[R^2\hat{\Phi}_{,1}(r(R), v(R))\right] = 2R\hat{\Phi}_{,2}(r(R), v(R)), \quad 0 < R < 1, \tag{44a}$$

$$v^{\frac{1}{2}}(R)\Bigg(\varepsilon^\alpha \frac{\mathrm{d}}{\mathrm{d}R}[R^2|v'(R)|^{\alpha-1}\operatorname{sgn}(v'(R))]$$

$$-R^2\left[\hat{\Phi}_{,4}(r(R),v(R)) + \tfrac{1}{q\varepsilon^q}\phi'_\tau(v(R))\right]\Bigg) = 0, \quad 0 < R < 1, \tag{44b}$$

with boundary conditions:

$$r(0)=0, \quad r(1)=\lambda, \quad \lim_{R\to 0^+} R^2|v'(R)|^{\alpha-1}\operatorname{sgn}(v'(R))v^{\frac{1}{2}}(R)=0, \quad v(1)=0. \tag{45}$$

Proof. If we let $v = u^2$, then our problem is equivalent to that of minimizing

$$\begin{aligned} \hat{I}^\tau_\varepsilon(r,u) &= \int_0^1 R^2\hat{\Phi}\left(r(R),u^2(R)\right)\mathrm{d}R \\ &\quad + \int_0^1 R^2\left[\frac{\varepsilon^\alpha}{\alpha}|2u(R)u'(R)|^\alpha + \frac{1}{q\varepsilon^q}\phi_\tau(u^2(R))\right]\mathrm{d}R, \end{aligned} \tag{46}$$

over

$$\begin{aligned} \hat{\mathcal{U}}_{\mathrm{rad}} = \{&(r,u)\in W^{1,1}(0,1)\times W^{1,\alpha}(0,1) : r(0)=0,\ r(1)=\lambda, \\ &r'(R)(r(R)/R)^2 > u^2(R) \text{ a.e.},\ u(1)=0\}. \end{aligned} \tag{47}$$

Note that since $u \in W^{1,\alpha}(0,1)$, then u is continuous in $[0,1]$. Hence both u^2 and uu' belong to $L^\alpha(0,1)$.

Let (r,u) be any minimizer of $\hat{I}^\tau_\varepsilon$ over (47). We first consider variations only in r, keeping u fixed. We make the change of variables $w = r^3(R)$ and $\rho = R^3$. It follows now that

$$\dot{w}(\rho) = \frac{\mathrm{d}w}{\mathrm{d}\rho}(\rho) = r'(R)\left(\frac{r(R)}{R}\right)^2.$$

The first part of the functional (46) can now be written as

$$\int_0^1 f(\rho,w,\dot{w},u^2)\,\mathrm{d}\rho,$$

where

$$3f(\rho,w,\dot{w},u^2) = \tilde{\Phi}((\rho/w)^{\frac{2}{3}}\dot{w},(w/\rho)^{\frac{1}{3}},(w/\rho)^{\frac{1}{3}}) + h(\dot{w}-u^2).$$

For $k \geq 1$ we define

$$S_k = \left\{\rho \in \left(\frac{1}{k},1\right) : \frac{1}{k} \leq \dot{w}(\rho) - u^2(\rho) \leq k\right\},$$

and let χ_k be its characteristic function. Let $\omega \in L^\infty(0,1)$ be such that

$$\int_{S_k}\omega(s)\,\mathrm{d}s = 0,$$

and for any $\gamma > 0$, define the variations

$$w_\gamma(\rho) = w(\rho) + \gamma\int_0^\rho \chi_k(s)\omega(s)\,\mathrm{d}s.$$

Note that $w_\gamma(0) = 0$ and $w_\gamma(1) = \lambda^3$. The rest of the proof, using (43), is as in [3], from which it follows (after changing back to R and r) that $r \in C^1(0,1]$, $r'(R) > 0$

for all $R \in (0,1]$, $R^2\hat{\Phi}_1(r(R),u(R))$ is $C^1(0,1]$, and that equations (44a) and the first two boundary conditions in (45) hold.

We now consider variations in u keeping r fixed. For any $k \geq 1$, let $z \in W^{1,\infty}(0,1)$ have support in $(\frac{1}{k},1)$. and let

$$u_\gamma = u + \gamma z.$$

Note that $u_\gamma(1) = 0$. Moreover, since $r \in C^1(0,1]$ and $u \in C[0,1]$, it follows that

$$r'(R)\left(\frac{r(R)}{R}\right)^2 > u_\gamma^2(R), \quad R \in \left[\frac{1}{k},1\right],$$

for γ sufficiently small. It follows now, upon setting $\delta(R) = r'(R)(r(R)/R)^2$, that

$$\begin{aligned}\frac{\hat{I}_\varepsilon^\tau(r,u_\gamma) - \hat{I}_\varepsilon^\tau(r,u)}{\gamma} &= \frac{1}{\gamma}\int_0^1 R^2\left[h(\delta - u_\gamma^2) - h(\delta - u^2)\right]\mathrm{d}R \\ &+\frac{1}{\gamma}\int_0^1 \frac{\varepsilon^\alpha}{\alpha}R^2\left[|2u_\gamma u_\gamma'|^\alpha - |2uu'|^\alpha\right]\mathrm{d}R \\ &+\frac{1}{\gamma}\int_0^1 \frac{1}{q\varepsilon^q}R^2\left[\phi_\tau(u_\gamma^2) - \phi_\tau(u^2)\right]\mathrm{d}R.\end{aligned}$$

Now

$$\begin{aligned}&\frac{1}{\gamma}\int_0^1 R^2\left[h(\delta - u_\gamma^2) - h(\delta - u^2)\right]\mathrm{d}R = \\ &\qquad \frac{1}{\gamma}\int_0^1 R^2\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}[h(\delta - (tu_\gamma^2 + (1-t)u^2))]\,\mathrm{d}t\mathrm{d}R = \\ &\qquad -\int_{\frac{1}{k}}^1 R^2 z(2u+\gamma z)\int_0^1 h'(\delta - (tu_\gamma^2 + (1-t)u^2))\,\mathrm{d}t\mathrm{d}R \\ &\qquad\qquad \to -\int_{\frac{1}{k}}^1 2h'(\delta - u^2)uzR^2\mathrm{d}R,\end{aligned}$$

as $\gamma \to 0$. Similarly

$$\begin{aligned}\frac{1}{\gamma}\int_0^1 \frac{\varepsilon^\alpha}{\alpha}R^2\left[|2u_\gamma u_\gamma'|^\alpha - |2uu'|^\alpha\right]\mathrm{d}R &\to \int_{\frac{1}{k}}^1 \varepsilon^\alpha|2uu'|^{\alpha-1}\,\mathrm{sgn}(2uu')2(uz)'R^2\,\mathrm{d}R, \\ \frac{1}{\gamma}\int_0^1 \frac{1}{q\varepsilon^q}R^2\left[\phi_\tau(u_\gamma^2) - \phi_\tau(u^2)\right]\mathrm{d}R &\to \int_{\frac{1}{k}}^1 \frac{1}{q\varepsilon^q}\phi_\tau'(u^2)2uzR^2\,\mathrm{d}R,\end{aligned}$$

as $\gamma \to 0$. Since

$$\lim_{\gamma\to 0}\frac{\hat{I}_\varepsilon^\tau(r,u_\gamma) - \hat{I}_\varepsilon^\tau(r,u)}{\gamma} = 0,$$

we get, combining our previous results that

$$\int_{\frac{1}{k}}^1 [-h'(\delta - u^2)uz + \varepsilon^\alpha|2uu'|^{\alpha-1}\,\mathrm{sgn}(2uu')(uz)' + \frac{1}{q\varepsilon^q}\phi_\tau'(u^2)uz]R^2\mathrm{d}R = 0,$$

or after collecting terms,

$$\int_{\frac{1}{k}}^{1}[\varepsilon^\alpha|2uu'|^{\alpha-1}\operatorname{sgn}(2uu')uz'+(\varepsilon^\alpha|2uu'|^{\alpha-1}\operatorname{sgn}(2uu')u'+\frac{1}{q\varepsilon^q}\phi_\tau'(u^2)u-h'(\delta-u^2)u)z]R^2\mathrm{d}R=0,$$

for all $z\in W^{1,\infty}(0,1)$ with support in $(\frac{1}{k},1)$. The coefficient of z in this expression is in $L^1(\frac{1}{k},1)$. Hence the above equation is equivalent to

$$\int_{\frac{1}{k}}^{1}\left[\varepsilon^\alpha|2uu'|^{\alpha-1}\operatorname{sgn}(2uu')uR^2+\int_R^1(\varepsilon^\alpha|2uu'|^{\alpha-1}\operatorname{sgn}(2uu')u'+\frac{1}{q\varepsilon^q}\phi_\tau'(u^2)u-h'(\delta-u^2)u)\xi^2\,\mathrm{d}\xi\right]z'\mathrm{d}R=0.$$

The arbitrariness of z implies now that for some constant C independent of k, we have

$$\varepsilon^\alpha|2uu'|^{\alpha-1}\operatorname{sgn}(2uu')uR^2+\int_R^1(\varepsilon^\alpha|2uu'|^{\alpha-1}\operatorname{sgn}(2uu')u'+\frac{1}{q\varepsilon^q}\phi_\tau'(u^2)u-h'(\delta-u^2)u)\xi^2\,\mathrm{d}\xi=C,$$

over $(0,1)$. It follows from this equation that over the intervals where $u\neq 0$, the function $|2uu'|^{\alpha-1}\operatorname{sgn}(2uu')R^2$ is absolutely continuous. Hence after differentiating and simplifying, the equation above yields that

$$\left(\varepsilon^\alpha\frac{\mathrm{d}}{\mathrm{d}R}\left[|2uu'|^{\alpha-1}\operatorname{sgn}(2uu')R^2\right]-\left(\frac{1}{q\varepsilon^q}\phi_\tau'(u^2)-h'(\delta-u^2)\right)R^2\right)u=0,$$

i.e., that (44b) holds after reverting the substitution $v=u^2$. A standard argument now using variations z not vanishing at $R=0$, yields the third boundary condition in (45). ∎

Remark 18. Note that the pair $r(R)=\lambda R$ and $v(R)=0$ is a solution of (44)-(45) for all λ. By Proposition 14, this pair is a global minimizer for $\lambda<d_0^{\frac{1}{3}}$. However for $\lambda>\lambda_c$, ε sufficiently small, and τ sufficiently large, we get from Theorems 15 and 17, that the minimizer must have v non–constant, with segments in which v vanishes, and (non–trivial) segments in which the differential equation

$$\varepsilon^\alpha\frac{\mathrm{d}}{\mathrm{d}R}[R^2|v'(R)|^{\alpha-1}\operatorname{sgn}(v'(R))]=R^2\left[\hat{\Phi}_{,4}(r(R),v(R))+\frac{1}{q\varepsilon^q}\phi_\tau'(v(R))\right],$$

holds.

5.2. Numerical results. To approximate the minimum of (31) over (32), let $\Delta R=1/n$ and $R_i=ih$, $0\le i\le n$, where $n\ge 1$. We write (r_i,v_i) for any approximation of $(r(R_i,v(R_i)))$, $0\le i\le n$, and

$$R_{i-\frac{1}{2}}=\frac{R_i+R_{i-1}}{2},\quad \delta r_{i-\frac{1}{2}}=\frac{r_i-r_{i-1}}{\Delta R},\quad \left(\frac{r}{R}\right)_{i-\frac{1}{2}}=\frac{r_i+r_{i-1}}{R_i+R_{i-1}},\quad i=1,\ldots,n.$$

Now we discretize I_ε^τ as follows:

ε^2	$I^\tau_{\varepsilon,h}$	$\delta r_{\frac{1}{2}}$	$v$max
10^{-5}	6.101645	5.412	169.2
10^{-6}	6.105267	1.560	0.0020
10^{-7}	6.105291	1.590	8.008×10^{-4}
10^{-8}	5.634048	32.85	3.606×10^{4}
10^{-9}	4.771748	50.47	1.499×10^{5}
10^{-10}	4.535530	49.91	1.455×10^{5}

Table 1

Convergence of the decoupled penalized scheme in the radial case using (50) *and* (51) *with data* (52).

$$I^\tau_{\varepsilon,h} = \Delta R \sum_{i=1}^{n} R^2_{i-\frac{1}{2}} \left[\tilde{\Phi}\left(\delta r_{i-\frac{1}{2}}, \left(\frac{r}{R}\right)_{i-\frac{1}{2}}, \left(\frac{r}{R}\right)_{i-\frac{1}{2}} \right) + h\left(\delta r_{i-\frac{1}{2}} \left(\frac{r}{R}\right)^2_{i-\frac{1}{2}} - v_{i-\frac{1}{2}} \right) \right] + \Delta R \sum_{i=1}^{m} R^2_{i-\frac{1}{2}} \left[\frac{\varepsilon^\alpha}{\alpha} |\delta v_{i-\frac{1}{2}}|^\alpha + \frac{1}{q\varepsilon^q}\, \phi_\tau(v_{i-\frac{1}{2}}) \right], \tag{48}$$

subject to $r_0 = 0$, $r_n = \lambda$, $v_n = 0$ and

$$v_i \geq 0, \quad 0 \leq i \leq n, \quad \delta r_{i-\frac{1}{2}} \left(\frac{r}{R}\right)^2_{i-\frac{1}{2}} - v_{i-\frac{1}{2}} > 0, \quad 1 \leq i \leq n. \tag{49}$$

We compute (relative) minimizers of (48) over (49) using the function `fmincon` of MATLAB with the option for an interior point algorithm. With this routine the first set of conditions in (49) can be directly specified as lower bounds on the v_i's, while the second set of constraints is specified with the option for inequality constraints. The strict sign in the second set of conditions in (49) is indirectly handled by the interior point algorithm with the h playing the role of an interior penalty function (since $h(d) \to \infty$ as $d \searrow 0$). For the various functions in the functional above we used the following:

$$\tilde{\Phi}(v_1, v_2, v_3) = \mu\left(v_1^p + v_2^p + v_3^p\right), \quad h(d) = c_1 d^\gamma + c_2 d^{-\delta}, \tag{50}$$

$$\phi_\tau(v) = \begin{cases} K v^2 (v-\tau)^2 & , \quad v \in [0,\tau], \\ 0 & , \quad \text{elsewhere}, \end{cases} \tag{51}$$

where $p \in [1,3)$, $\mu, c_1, c_2 \geq 0$, $\gamma, \delta \geq 1$, and $K > 0$. One can easily check now that conditions (21) and (35) hold for ϕ_τ. In the calculations below we use $n = 100$ and the following values for the various constants:

$$\mu = 1.0,\ c_1 = 1.0,\ p = 2.0,\ \alpha = 2.0,\ \gamma = 2.0,\ \delta = 2.0,\ \tau = 3.0,\ \lambda = 1.5, \tag{52}$$

with $c_2 = (p\mu + \gamma c_1)/\delta$ so as to make the reference configuration stress free. In this case the minimizer r_c of (29) over (30) has $E_{\text{rad}}(r_c) \approx 4.5396$ with $r_c(0) \approx 1.222$, while the affine deformation $r^h(R) = \lambda R$ has energy $E_{\text{rad}}(r^h) \approx 6.1053$.

In Table 1 we show the computed minimum energies for different values of ε^2. In each case the iterations were started from the discretized versions of the affine deformation r^h and $v = 0$. From the values in the table we see that the approximations of r for $\varepsilon^2 = 10^{-5}, 10^{-6}, 10^{-7}$ stay "close" to the affine deformation r^h but developing a steep slope close to $R = 0$. This process picks up after $\varepsilon^2 = 10^{-8}$, where the energies get very close to the energy $E_{\text{rad}}(r_c) \approx 4.5396$ of the cavitated solution, and with very

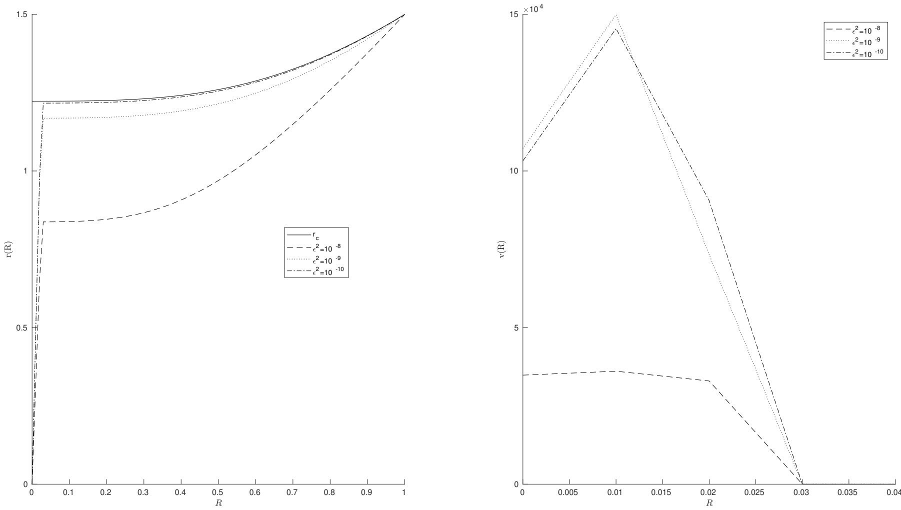

Fig. 1. *Numerical results for the data in* (52).

large slopes close to $R = 0$. The last column in Table 1 shows the maximum value of computed phase functions v for the different values of ε^2. In Figure 1 (left) we show the computed r approximations for $\varepsilon^2 = 10^{-8}, 10^{-9}, 10^{-10}$ which are clearly converging to the cavitated solution r_c. On the other hand, Figure 1 (right) shows the corresponding approximations of v restricted to the interval $[0, 0.04]$, which are clearly developing a singularity close to $R = 0$ to match the corresponding singular behaviour of the determinants corresponding to the r approximations.

6. Concluding Remarks. From the proof of Theorem 3 it becomes clear that the critical term in the stored energy function, in relation to the repulsion property, is the compressibility term, i.e., the function $h(\cdot)$ in (9). This result is the main idea behind the method proposed in Section 4 and might explain why previous numerical schemes, such as the element removal method developed by Li and coworkers (see, e.g., [20]) or the use of "punctured domains" (see, e.g., [36]), have been successful.

As a practical matter, we mention that the numerical routine that one employs to solve the discrete versions of the minimization of (14) over (15), must be "aggressive" enough, specially during the early stages of the minimization, to allow for actual increases in the intermediate approximate energies, which rules out the use of strictly descent methods. The reason for this is that, when needed, the scheme has to increase the phase function v in regions where the determinant of the deformation gradient might become large. To do so, it might be necessary to increase v past τ in the penalty function ϕ_τ (cf. (14)), resulting in an increase in the computed energy. One could try to avoid this by taking initial candidates for v large, but this requires identifying regions where this is to be done, which in turn presumes knowledge of the location of the singularities. Although in general one can not assume such knowledge, it might be the case if the locations of possible flaws in the material are known before hand.

The results in the paper for non–radial problems can be extended to more general displacement type boundary conditions and for mixed type boundary conditions. We refer to [28] or [33] for the corresponding technical details.

Finally we did not address the question of the convergence of the minimizers of the discretized versions of (14) over (15). Also we need to test the method on more general problems, like the one for non radially symmetric deformations, and in problems in which the Lavrentiev phenomenon takes place for boundary value problems in two dimensional elasticity among admissible continuous deformations. (See [11].) These questions shall be pursued elsewhere.

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