# Global Continuation Methods in Three Dimensional Elasticity

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# Preface

Historically the problems posed by continuum mechanics have induced major developments in important areas of mathematics, such as the theory of partial differential equations both linear and nonlinear, the calculus of variations, and bifurcation theory (cf. [44], [5]). On the other hand, there has been an increasing interest among the mathematical community in nonlinear problems and techniques to tackle such problems. An example of this has been the use of degree theoretic techniques, e.g. the Leray–Schauder degree in nonlinear differential equations. These mutual influences between continuum mechanics and several areas of mathematics research are still going on. In this lecture series we will study some important recent developments, explained in more details below, in degree theoretic techniques and their applications to problems in nonlinear elastostatics. These recent developments in this field have an applicability that extends beyond its original motivation from elasticity theory and consequently are worth knowing by other mathematicians interested in nonlinear problems and theories. Moreover, this area of research has many interesting open problems and would benefit from new researchers, specially young talented people.

For many years one of the most difficult open problems of non–linear elasticity theory has been the use of global continuation methods (via degree theory) to study the governing system of partial differential equations of three–dimensional models, c.f. [7], and [32]. The use of Leray–Schauder degree techniques in elasticity has a long and successful story that we will not review here but we refer to [6] for examples and its extensive literature review. However, for the most part, those applications have been limited to one-dimensional problems. Not until recently, in [21], such a major enterprize was carried out for the three dimensional displacement problem of nonlinear elasticity. On the other hand, the full nonlinearity of traction boundary conditions renders more general boundary value problems out of reach to the traditional Leray–Schauder degree.

A more general degree based on proper Fredholm maps of index zero ([14], [28], [26]) avoids the transformation of the original problem into one in terms of a compact perturbation of the identity but requires properness of the nonlinear operator and some a priori estimates on solutions of the linear problem and its spectrum. This generalized degree has the same important properties of the classical Leray–Schauder degree. In particular, the homotopy invariance property (cf. Proposition 4.12 in [26]) of the new degree supports its applicability to study global bifurcation in the sense of Rabinowitz (c.f. [37]). For the three dimensional mixed problem of nonlinear elasticity the required spectral estimates were obtained in [26] and together with the estimates of [4] for elliptic systems, Healey and Simpson were able to apply the generalized degree to get the existence of a global branch of solutions of this problem. Therefore, the new methods of Healey and Simpson make it possible, for the first time, to tackle global bifurcation problems in non–linear three–dimensional elasticity.

A first step in the analysis of local continuation or bifurcation is to study each linearized problem around the corresponding trivial solution. For global continuation or bifurcation one further needs to study the linearized problem about an arbitrary deformation. These linear problems correspond to elliptic systems of partial differential equations on a domain determined by the geometry of the physical problem. In order to apply the more general degree mentioned above, the linear operators must be Fredholm of index zero and certain spectral estimates are needed. When the domain is smooth, one can use Schauder estimates ([4]) to get the required properties. On other types of regions one might use hidden symmetries in the problem to get the required properties ([22], [23], [24]).

The behavior of the global solution branches predicted in [26] is characterized, in addition to the two Rabinowitz alternatives, cf. [37], by the possibility that they terminate due to loss of local injectivity and/or ellipticity and/or the failure of the complementing condition. An open problem then is to find physically meaningful restrictions on the constitutive laws which rule out some of these alternatives. In [25] the failure of local injectivity on bounded solution branches is obviated for a general class of stored energy functions subject to mild growth conditions. Thus, the existence of unbounded solution branches is obtained. With slightly stronger, but nonetheless, physically realistic, growth conditions, a similar result is obtained in [24] for a class of boundary value problems involving traction-free boundary conditions.

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# Chapter 1

# Boundary Value Problems of Nonlinear Elastostatics

## 1.1 Notation

We denote vectors in $\mathbb{R}^n$ by bold–face, lower–case symbols, e. g., $\mathbf{a}$, $\mathbf{b}$, $\mathbf{x}$, etc., the components of which with respect to a particular basis will be denoted $(a_i)$, $(b_i)$, $(x_i)$, etc.. The vector cross product will be denoted by $\mathbf{a} \times \mathbf{b}$, the Euclidean inner ("dot") product by $\mathbf{a} \cdot \mathbf{b}$, and the corresponding Euclidean norm by $|\mathbf{a}|$.

The second order tensors consists of the set of all linear transformations on $\mathbb{R}^n$ and will be denoted $L(\mathbb{R}^n)$. Elements of $L(\mathbb{R}^n)$ will be denoted by bold–face, upper–case symbols, e. g., $\mathbf{A}$, $\mathbf{B}, \mathbf{C}$, etc., the components of which with respect to a particular basis will be denoted $(A_{ij})$, $(B_{ij})$, $(C_{ij})$, etc.. We will also employ the following notations:

$\mathbf{a} \otimes \mathbf{b} \in L(\mathbb{R}^n)$ is the tensor product of $\mathbf{a}$ and $\mathbf{b}$. In components, $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$.

$\mathbf{A} \cdot \mathbf{B} = \operatorname{trace}(\mathbf{A}^t \mathbf{B})$ is the Euclidean inner product on $L(\mathbb{R}^n)$. In components $\mathbf{A} \cdot \mathbf{B} = A_{ij} B_{ij}$.

$|\mathbf{A}| = (\mathbf{A} \cdot \mathbf{A})^{1/2}$ is the corresponding Euclidean norm.

$\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$.

$\mathbf{A}^{-1}$ denotes the inverse of $\mathbf{A}$ provided $\mathbf{A}$ is invertible.

$\det \mathbf{A}$ denotes the determinant of $\mathbf{A}$.

$\operatorname{Cof} \mathbf{A} \in L(\mathbb{R}^n)$ is the cofactor tensor which satisfies $\mathbf{A}^T \operatorname{Cof} \mathbf{A} = (\det \mathbf{A}) \mathbf{I}$.

$\mathrm{GL}^+(\mathbb{R}^n) = \{ \mathbf{A} \in L(\mathbb{R}^n) : \det \mathbf{A} > 0 \}$.

$\mathrm{SO}(n) = \{ \mathbf{Q} \in L(\mathbb{R}^n) : \mathbf{Q}\mathbf{Q}^T = \mathbf{I}, \quad \det(\mathbf{Q}) = 1 \}$.

For any bounded open subset $\Omega$ of $\mathbb{R}^n$ we let $L^p(\Omega)$, $1 < p < \infty$, denote the real Banach space of (classes of) real valued measurable functions $\mathbf{u} : \Omega \to \mathbb{R}^n$ such that $|\mathbf{u}|^p$ is integrable. The norm over $L^p(\Omega)$ is given by:

$$\|\mathbf{u}\|_p = \left[ \int_\Omega |\mathbf{u}(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} \right]^{1/p}. \tag{1.1}$$

For $k \geq 1$ an integer, and $1 < p < \infty$, we let $W^{k,p}(\Omega)$ be the *Sobolev* space of classes of functions $\mathbf{u} \in L^p(\Omega)$ such that $\mathbf{u}$ has generalized derivatives of order less than or equal to $k$ that belong to $L^p(\Omega)$. The norm in $W^{k,p}(\Omega)$ is given by

$$\|\mathbf{u}\|_{k,p} = \left[ \sum_{|\boldsymbol{\beta}| \leq k} \left\| \mathrm{D}^{\boldsymbol{\beta}} \mathbf{u} \right\|_p^p \right]^{1/p}. \tag{1.2}$$

We let $C^k(\overline{\Omega})$ be the Banach space of continuous functions with derivatives up to order $k$ continuous also over $\overline{\Omega}$. The norm in $C^k(\overline{\Omega})$ is given by

$$\|\mathbf{u}\|_{C^k(\overline{\Omega})} = \sum_{|\boldsymbol{\beta}| \leq k} \max_{\mathbf{x} \in \overline{\Omega}} \left| \mathrm{D}^{\boldsymbol{\beta}} \mathbf{u}(\mathbf{x}) \right|. \tag{1.3}$$

The Schauder space $C^{k,\alpha}(\overline{\Omega})$, $0 < \alpha \leq 1$, denotes the Banach space of functions $\mathbf{u} \in C^k(\overline{\Omega})$ such that

$$\sup_{\substack{\mathbf{x},\mathbf{y} \in \overline{\Omega} \\ \mathbf{x} \neq \mathbf{y}}} \frac{\left| \mathrm{D}^{\boldsymbol{\beta}} \mathbf{u}(\mathbf{x}) - \mathrm{D}^{\boldsymbol{\beta}} \mathbf{u}(\mathbf{y}) \right|}{|\mathbf{x} - \mathbf{y}|^\alpha} < \infty,$$

for any multi–index $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| = k$. The norm in $C^{k,\alpha}(\overline{\Omega})$ is given by

$$\|\mathbf{u}\|_{k,\alpha} = \|\mathbf{u}\|_{C^k(\overline{\Omega})} + \sum_{|\boldsymbol{\beta}| = k} \sup_{\substack{\mathbf{x},\mathbf{y} \in \overline{\Omega} \\ \mathbf{x} \neq \mathbf{y}}} \frac{\left| \mathrm{D}^{\boldsymbol{\beta}} \mathbf{u}(\mathbf{x}) - \mathrm{D}^{\boldsymbol{\beta}} \mathbf{u}(\mathbf{y}) \right|}{|\mathbf{x} - \mathbf{y}|^\alpha}. \tag{1.4}$$

## 1.2 The Governing Equations of Nonlinear Elasticity

Nonlinear Elasticity is a vast subject, as demonstrated by the existence of numerous books devoted to it, e. g., [6], [11], [32], [35]. We make no attempt at a thorough introduction here, but rather employ the mathematically expedient device of obtaining the Euler–Lagrange equations from the principle of stationary potential energy:

Consider an elastic body occupying a bounded domain $\Omega \subset \mathbb{R}^n$. A *deformation* of the body is a $C^1$ mapping, $\mathbf{f} : \overline{\Omega} \longrightarrow \mathbb{R}^n$, satisfying

$$\det \mathbf{F}(\mathbf{x}) > 0 \quad \text{in } \overline{\Omega}, \tag{1.5}$$

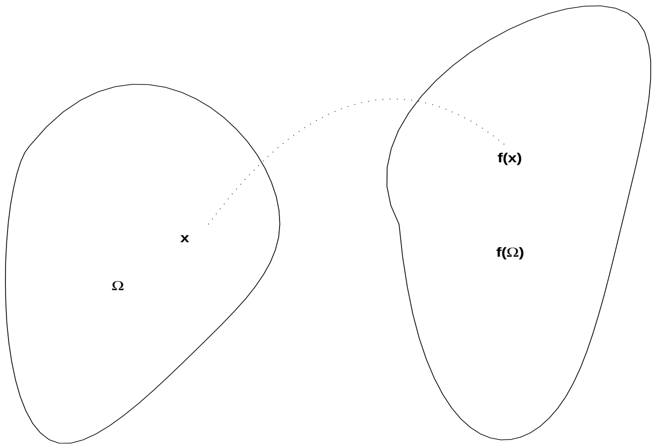


Figure 1.1: Geometry of Deformation. The body occupying the region $\Omega$ in its reference configuration (left) and its deformed configuration $\mathbf{f}(\Omega)$ (right).

where

$$\mathbf{F}(\mathbf{x}) \equiv \nabla \mathbf{f}(\mathbf{x}), \tag{1.6}$$

is the *deformation gradient*, defined by

$$\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) = \nabla \mathbf{f}(\mathbf{x})\,\mathbf{h} + o(|\mathbf{h}|)\,, \quad \text{as } |\mathbf{h}| \to 0\,.$$

The *displacement* of the body is defined by:

$$\mathbf{u}(\mathbf{x}) \equiv \mathbf{f}(\mathbf{x}) - \mathbf{x}\,. \tag{1.7}$$

The region $\Omega$ is called the *reference configuration*, and the image $\mathbf{f}(\Omega) \subset \mathbb{R}^n$ is called the *deformed configuration*. (See Figure (1.1).)

We postulate the existence of a *stored energy* density $W : \mathrm{GL}^+(\mathbb{R}^n) \longrightarrow \mathbb{R}$, such that

$$E(\mathbf{f}) \equiv \int_\Omega W(\nabla \mathbf{f}(\mathbf{x}))\,\mathrm{d}V, \tag{1.8}$$

represents the *total internal potential energy* of the body in the deformed configuration. We make the physically realistic assumption that

$$W(\mathbf{F}) \to \infty \text{ as } |\mathbf{F}| \to \infty\,, \text{ and as } \det \mathbf{F} \searrow 0\,. \tag{1.9}$$

We require that $W$ also satisfy *material objectivity*:

$$W(\mathbf{QF}) = W(\mathbf{F})\ \forall\, \mathbf{Q} \in \mathrm{SO}(3)\,. \tag{1.10}$$

We suppose that the body is acted on by various external (prescribed), sufficiently smooth fields:

1. $\boldsymbol{\tau}(\mathbf{u},\cdot)$ – prescribed traction on $\partial\Omega_1$.
2. $\mathbf{u}=\mathbf{d}(\cdot)$, on $\partial\Omega_2$, where $\partial\Omega_1, \partial\Omega_2$ are disjoint relatively open subsets of $\partial\Omega$ with $\overline{\partial\Omega_1}\cup\overline{\partial\Omega_2}=\partial\Omega$.
3. $\mathbf{b}(\mathbf{u},\nabla\mathbf{u}.\cdot)$–prescribed body force per unit volume in $\Omega$.

In particular the later two fields need not be conservative (i.e. derivable from scalar fields. Accordingly, for equilibrium we take the first variation of (1.8) and equate it to the *virtual work* of the external fields acting through an arbitrary *admissible variation*. Consider a smooth field:

$$\boldsymbol{\eta}:\overline{\Omega}\to\mathbb{R}^n\ ,\ \text{such that } \boldsymbol{\eta}|_{\partial\Omega_2}=\mathbf{0}\,. \tag{1.11}$$

The *first variation* of (1.8) in the direction of $\boldsymbol{\eta}$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\,E(\mathbf{f}+\alpha\boldsymbol{\eta})|_{\alpha=0} \;=\; \int_\Omega \frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla\mathbf{f})\cdot\nabla\boldsymbol{\eta}\,\mathrm{d}V\,. \tag{1.12}$$

For *equilibrium* we require (the principle of virtual work):

$$\int_\Omega \frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla\mathbf{f})\cdot\nabla\boldsymbol{\eta}\,dV=\int_\Omega \mathbf{b}\cdot\boldsymbol{\eta}\,\mathrm{d}V+\int_{\partial\Omega_2}\boldsymbol{\tau}\cdot\boldsymbol{\eta}\,\mathrm{d}S\,, \tag{1.13}$$

for all admissible variations $\boldsymbol{\eta}$. Assuming sufficiently smooth fields satisfying (1.11), integration by parts and the divergence theorem yield the (Euler–Lagrange) equilibrium equations:

$$\mathrm{Div}\left(\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\mathbf{I}+\nabla\mathbf{u})\right)+\mathbf{b}(\mathbf{u},\nabla\mathbf{u}) \;=\; \mathbf{0} \text{ in } \Omega\,, \tag{1.14a}$$

$$\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\mathbf{I}+\nabla\mathbf{u})\mathbf{n} \;=\; \boldsymbol{\tau}(\mathbf{u}) \text{ on } \partial\Omega_1\,, \tag{1.14b}$$

$$\mathbf{u} \;=\; \mathbf{d} \text{ on } \partial\Omega_2\,, \tag{1.14c}$$

where $\mathbf{n}(\cdot)$ is the outward unit normal field to $\partial\Omega$.

Of course, (1.13) is simply the weak form of (1.14). The tensor

$$\mathbf{S}(\mathbf{F})\equiv\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\mathbf{F}), \tag{1.15}$$

is the *first Piola–Kirchhoff stress tensor* at $\mathbf{F}$.

Expanding the first term in (1.14a) yields the quasi–linear form

$$\mathrm{Div}\left(\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\mathbf{I}+\nabla\mathbf{u})\right)=\frac{\mathrm{d}^2W}{\mathrm{d}\mathbf{F}^2}(\mathbf{I}+\nabla\mathbf{u})[\nabla^2\mathbf{u}]\,, \tag{1.16}$$

where the (vector–valued) right side of (1.16) is defined componentially (using summation convention) via

$$\mathbf{e}_i\cdot\left(\frac{\mathrm{d}^2W}{\mathrm{d}\mathbf{F}^2}(\mathbf{F})[\nabla^2\mathbf{u}]\right)\equiv\frac{\partial^2W(\mathbf{F})}{\partial F_{ij}\partial F_{kl}}\;\frac{\partial^2u_k}{\partial x_j\partial x_l}\,. \tag{1.17}$$

Here $\{\mathbf{e}_i : i = 1, 2, \ldots, n\}$ denotes the standard orthonormal basis for $\mathbb{R}^n$.

The fourth–order tensor

$$\mathsf{C}(\mathbf{F}) \equiv \frac{\mathrm{d}^2 W}{\mathrm{d}\mathbf{F}^2}(\mathbf{F}), \tag{1.18}$$

is called the *elasticity tensor* at $\mathbf{F}$, and in view of (1.17), we have the componential form

$$\mathsf{C}_{ijkl}(\mathbf{F}) = \frac{\partial^2 W(\mathbf{F})}{\partial F_{ij} \partial F_{kl}}. \tag{1.19}$$

Note that $\mathsf{C}$ is symmetric, that is $\mathsf{C}_{ijkl} = \mathsf{C}_{klij}$ for all $i$, $j$, $k$, $l$.

Clearly we may consider $\mathsf{C}(\mathbf{F})$ (for fixed $\mathbf{F}$) as either a linear transformation of third–order tensors into vectors, as in (1.16), or as a linear mapping of $L(\mathbb{R}^n)$ into itself. For the later we write

$$\mathsf{C}(\mathbf{F})[\mathbf{A}] \in L(\mathbb{R}^n) \ \ \forall \mathbf{A} \in L(\mathbb{R}^n), \tag{1.20}$$

and in components, we have

$$\mathbf{e}_i \cdot \mathsf{C}(\mathbf{F})[\mathbf{A}]\mathbf{e}_j = \mathsf{C}_{ijkl}(\mathbf{F}) A_{kl}. \tag{1.21}$$

At each $\mathbf{F} \in \mathrm{GL}^+(\mathbb{R}^n)$, we assume the *strong ellipticity* condition:

$$\mathbf{a} \otimes \mathbf{b} \cdot \mathsf{C}(\mathbf{F})[\mathbf{a} \otimes \mathbf{b}] > 0, \ \ \forall\, \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \backslash \{\mathbf{0}\}. \tag{1.22}$$

In components, inequality (1.22) reads

$$\mathsf{C}_{ijkl}(\mathbf{F}) a_i a_k b_j b_l > 0.$$

We say that the material of the body is *uniformly elliptic* at $\mathbf{F}$ if there exists a constant $\alpha > 0$ such that

$$\mathbf{a} \otimes \mathbf{b} \cdot \mathsf{C}(\mathbf{F})[\mathbf{a} \otimes \mathbf{b}] \geq \alpha\, |\mathbf{a}|\, |\mathbf{b}|. \tag{1.23}$$

**Remark 1.1.** *In contrast to the situation for scalar–valued second order elliptic partial differential equations, observe that strong ellipticity (1.22) is weaker than positive definiteness of the "coefficient matrix"* $\mathsf{C}(\mathbf{F})$. *The latter condition reads*

$$\mathbf{H} \cdot \mathsf{C}(\mathbf{F})[\mathbf{H}] > 0, \ \ \forall\, \mathbf{H} \in L(\mathbb{R}^n) \backslash \{\mathbf{0}\},$$

*which, in view of (1.16), implies the (strict) convexity of* $W(\mathbf{F})$. *The latter is untenable in nonlinear elasticity for many reasons, e. g., it insures uniqueness of solution in situations where one expects non–uniqueness (buckling), and it violates material objectivity (1.10), cf. [6], [11]. On the other hand, strong ellipticity (1.22) insures that* $W(\mathbf{F})$ *is strictly "rank–one" convex (since any* $\mathbf{H} = \mathbf{a} \otimes \mathbf{b}$, $\mathbf{a}$, $\mathbf{b} \neq \mathbf{0}$, *has rank equal to one).*

Throughout this work we assume that the external fields depend upon a single "control" parameter $\lambda \in \mathbb{R}$, the simplest example of which is "proportional loading": $\mathbf{u} = \lambda\mathbf{d}$ on $\partial\Omega_2$, $\lambda\boldsymbol{\tau}(\mathbf{u})$–prescribed traction on $\partial\Omega_1$, and $\lambda\mathbf{b}(\nabla\mathbf{u},\mathbf{u})$–prescribed body force. More generally we consider continuous parameter dependence, viz. , $\boldsymbol{\tau}(\lambda,\mathbf{u})$, $\mathbf{d}(\lambda)$, $\mathbf{b}(\lambda,\mathbf{u},\nabla\mathbf{u})$, each of which vanish at $\lambda = 0$:

$$\begin{aligned} \boldsymbol{\tau}(0,\mathbf{u},\cdot) &\equiv \mathbf{0} \text{ on } \partial\Omega_1\,, \\ \mathbf{d}(0,\cdot) &\equiv \mathbf{0} \text{ on } \partial\Omega_2\,, \\ \mathbf{b}(0,\mathbf{u},\mathbf{A},\cdot) &\equiv \mathbf{0} \text{ in } \Omega\,. \end{aligned} \tag{1.24}$$

We now record the final form of the boundary value problem, employing (1.15), (1.18), and (1.24):

$$\begin{aligned} \mathsf{C}(\mathbf{I}+\nabla\mathbf{u})[\nabla^2\mathbf{u}] + \mathbf{b}(\lambda,\mathbf{u},\nabla\mathbf{u}) &= \mathbf{0} \text{ in } \Omega\,, \\ \mathbf{S}(\mathbf{I}+\nabla\mathbf{u})\mathbf{n} &= \boldsymbol{\tau}(\lambda,\mathbf{u}) \text{ on } \partial\Omega_1\,, \\ \mathbf{u} &= \mathbf{d}(\lambda) \text{ on } \partial\Omega_2\,. \end{aligned} \tag{1.25}$$

Finally, we note that while (1.22) insures that (1.25), is an elliptic system (c.f. Chapter (2)), the growth condition (1.9) at the boundary of $\mathrm{GL}^+(\mathbb{R}^n)$ generally precludes *uniform ellipticity.* For example, consider

$$W(\mathbf{F}) = \frac{1}{2}\mathbf{F}\cdot\mathbf{F} + \Gamma(\det\mathbf{F})\,, \tag{1.26}$$

with $\Gamma : (0,\infty) \to \mathbb{R}$ convex and $\Gamma(\alpha)\,,\ \Gamma''(\alpha) \to \infty$ as $\alpha \searrow 0$. A straight forward calculation yields

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} W(\mathbf{F} + t\mathbf{a}\otimes\mathbf{b})|_{t=0} &\equiv \mathbf{a}\otimes\mathbf{b}\cdot\mathsf{C}(\mathbf{F})[\mathbf{a}\otimes\mathbf{b}] \\ &= (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) + \Gamma''(\det\mathbf{F})(\mathrm{Cof}\,\mathbf{F}\cdot\mathbf{a}\otimes\mathbf{b})^2\,, \end{aligned}$$

from which we conclude that

$$\sup_{|\mathbf{a}|=|\mathbf{b}|=1} \mathbf{a}\otimes\mathbf{b}\cdot\mathsf{C}(\mathbf{F})[\mathbf{a}\otimes\mathbf{b}] \geq 1 + \frac{|\mathrm{Cof}\,\mathbf{F}|^2}{3}\Gamma''(\det\mathbf{F})\,, \tag{1.27}$$

which "blows up" at the boundary of $\mathrm{GL}^+(\mathbb{R}^n)$, provided that $|\mathrm{Cof}\,\mathbf{F}| > 0$ as $\det\mathbf{F} \searrow 0$. If $\Gamma''(\alpha) \sim \alpha^{-(s+2)}$, $(s > 0)$, it's not hard to show that the right side of (1.27) always goes to infinity in the limit as $\det\mathbf{F} \searrow 0$ (without assuming that $|\mathrm{Cof}\,\mathbf{F}|$ is bounded away from zero).

**Remark 1.2.** *In general the stored energy $W(\cdot)$ is not known. The role of analysis here is to obtain existence results with physically meaningful conditions placed upon $W(\cdot)$, e. g., (1.9), (1.10), (1.22). Even though the beginnings of the theory date back to Cauchy, there is still no complete existence theory in nonlinear elasticity.*

# Chapter 2

# Linearization, Ellipticity and the Fredholm Property

We examine the elliptic linearized partial differential operators occurring in elasticity and determine conditions under which the Fredholm property holds for corresponding boundary value problems.

The equations of nonlinear elasticity are of the form

$$\operatorname{Div}\mathbf{S}(\nabla\mathbf{f})+\mathbf{b}=\mathbf{0} \qquad \text{on } \Omega. \tag{2.1}$$

Here $\mathbf{S}:\mathrm{GL}^{+}(\mathbb{R}^n)\to L(\mathbb{R}^n)$ is the Piola-Kirchhoff stress tensor given by (1.15). Also $\Omega$ is some region in $\mathbb{R}^n$ ($\Omega$ is open, bounded with sufficiently smooth boundary, $\partial\Omega$) and $\mathbf{f}:\overline{\Omega}\to\mathbb{R}^n$ represents the deformation of an elastic body initially occupying $\Omega$ in the reference configuration. See Antman [6], Ciarlet [11], Gurtin [19], Valent [45].

We consider two types of boundary conditions. In the displacement type, $\mathbf{f}$ is specified on $\partial\Omega$ and in the traction type, $\mathbf{S}(\nabla f)\mathbf{n}$ is specified on $\partial\Omega$; $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. When (2.1) is written in components we have

$$\sum_{j=1}^{n}\frac{\partial}{\partial x_j}S_{ij}(\nabla f(\mathbf{x}))=b_i(\mathbf{x}), \qquad \mathbf{x}\in\Omega,$$

for $i\in\{1,\ldots,n\}$. The displacement boundary condition is

$$f_i(\mathbf{x})=d_i(\mathbf{x}), \quad \mathbf{x}\in\partial\Omega, \tag{2.2}$$

$i=1,\ldots,n$, and the traction boundary condition is

$$\sum_{j=1}^{n}S_{ij}(\nabla f(\mathbf{x}))n_j(\mathbf{x})=\tau_i(\mathbf{x}), \quad \mathbf{x}\in\partial\Omega, \tag{2.3}$$

$i\in\{1,\ldots,n\}$, where $\mathbf{d}(\mathbf{x})=(d_1(\mathbf{x}),\ldots,d_n(\mathbf{x}))$, $\boldsymbol{\tau}(\mathbf{x})=(\tau_1(\mathbf{x}),\ldots,\tau_n(\mathbf{x}))$ are specified functions on $\partial\Omega$. $\boldsymbol{\tau}$ is the imposed surface traction and $\mathbf{n}(\mathbf{x})=(n_1(\mathbf{x}),\ldots,n_n(\mathbf{x}))$ is the outward unit normal to $\partial\Omega$.

To linearize the equation above, first fix a deformation $\mathbf{f}_0 : \Omega \to \mathbb{R}^n$ and write $\mathsf{C}(\mathbf{x})$ in place of $\mathsf{C}(\nabla\mathbf{f}_0(\mathbf{x}))$, $\mathbf{x} \in \Omega$. We linearize about $\mathbf{f}_0$ and denote by $\mathbf{u} : \Omega \to \mathbb{R}^n$ and infinitesimal displacement. Then (2.1) and boundary conditions (2.2) and (2.3) yield the linear boundary value problem

$$\begin{aligned} \operatorname{Div} \mathsf{C}(\mathbf{x})[\nabla\mathbf{u}(\mathbf{x})] &= \mathbf{b}(\mathbf{x}), && \mathbf{x} \in \Omega, \\ \left.\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{d}(\mathbf{x}), \\ \text{or, } \mathsf{C}(\mathbf{x})[\nabla\mathbf{u}(\mathbf{x})]\mathbf{n}(\mathbf{x}) &= \mathbf{t}(\mathbf{x}) \end{aligned}\right\} & && \mathbf{x} \in \partial\Omega. \end{aligned} \tag{2.4}$$

Here $\mathbf{b}$, $\mathbf{d}$, $\mathbf{t}$ are specified functions[1] on either $\Omega$ or $\partial\Omega$ with values in $\mathbb{R}^n$, $\mathbf{u} : \Omega \to \mathbb{R}^n$ is the solution function to be found. In (2.4) there are two types of boundary condition on $\partial\Omega$: displacement or linearized traction ($\mathsf{C}[\nabla\mathbf{u}]\mathbf{n}$). We denote the boundary operator by $B(\mathbf{u}) = \mathbf{u}$ or $\mathsf{C}[\nabla\mathbf{u}]\mathbf{n}$ in $(2.4)_2$ or $(2.4)_3$ respectively; we sometimes write $\mathsf{C}$ for $\mathsf{C}(\mathbf{x})$. The linear operator represented in (2.4) is

$$L(\mathbf{u}) = (\operatorname{Div} \mathsf{C}[\nabla\mathbf{u}], B(\mathbf{u})) \tag{2.5}$$

and the problem in (2.4) is equivalent to $L(\mathbf{u}) = (\mathbf{b}, \mathbf{d})$ or $(\mathbf{b}, \mathbf{t})$. The differential operator on the left in $(2.4)_1$ is of second order

$$\operatorname{Div} \mathsf{C}(\mathbf{x})[\nabla\mathbf{u}(\mathbf{x})]_i = \sum_{j,k,l=1}^{n} \frac{\partial}{\partial x_j}\left[\mathsf{C}(\mathbf{x})_{ijkl}\frac{\partial u_k}{\partial x_l}(\mathbf{x})\right], \qquad \mathbf{x} \in \Omega,$$

$1 \le i \le n$. The operator in $(2.4)_3$ is of first order

$$(\mathsf{C}(\mathbf{x})[\nabla\mathbf{u}(\mathbf{x})]\mathbf{n}(\mathbf{x}))_i = \sum_{j,k,l=1}^{n} \mathsf{C}(\mathbf{x})_{ijkl}\frac{\partial u_k}{\partial x_l}(\mathbf{x})n_j(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega,$$

$1 \le i \le n$, and that on the left in $(2.4)_2$ is of zeroth order (displacement or Dirichlet type).

To determine the injectivity, surjectivity and Fredholm properties of $L$ we shall work in either the Sobolev spaces $W^{k,p}(\Omega)$ or the Holder spaces $C^{k,\alpha}(\bar{\Omega})$, $k \ge 0$, $1 < p < \infty$, $0 < \alpha < 1$. Here $\mathbf{u}$, $\mathbf{b}$, $\mathbf{d}$, $\mathbf{t}$ will lie in these spaces and $\mathsf{C}(x)_{ijkl}$, $\partial\Omega$ will assume certain smoothness conditions given below. For simplicity we denote the Banach space $X = W^{k,p}(\Omega)$ or $C^{k,\alpha}(\bar{\Omega})$ with the usual norms (cf. Adams [1]). In all that follows we assume either the *pure* displacement problem on all of $\partial\Omega$ or *pure* traction problem on all of $\partial\Omega$.

We make certain assumptions for the tensor $\mathsf{C}$ and the region $\Omega$ (here $k$ is a positive integer):

(H1) $\Omega \subset \mathbb{R}^n$ is open, bounded; and either $\partial\Omega$ is locally $C^k$-smooth if $X = W^{k,p}(\Omega)$, $k \ge 2$, $1 < p < \infty$; or $\partial\Omega$ is locally $C^{k,\alpha}$-smooth if $X = C^{k,\alpha}(\bar{\Omega})$, $k \ge 2$, $0 < \alpha < 1$ (see Adams [1, p. 67]). Then $\mathbf{n} : \partial\Omega \to \mathbb{R}^n$ is $C^{k-1}$- or $C^{k-1,\alpha}$-smooth respectively.

[1]Here $\mathbf{b}$, $\mathbf{d}$ represent linearizations of the corresponding functions in (1.14) which for simplicity we denote again with the same symbols and $\mathbf{t}$ the corresponding linearization of $\boldsymbol{\tau}$.

(H2) $\mathsf{C}(\cdot)_{ijkl} : \bar{\Omega} \to \mathbb{R}$ are $C^{k-1}$-smooth functions on $\bar{\Omega}$ if $X = W^{k,p}(\Omega)$ or $C^{k-1,\alpha}$-smooth functions on $\bar{\Omega}$ if $X = C^{k-1,\alpha}(\bar{\Omega})$ for each $i, j, k, l \in \{1, \ldots, n\}$. Also $\mathsf{C}(\cdot)_{ijkl} = \mathsf{C}(\cdot)_{klij}$ on $\bar{\Omega}$ for all $i$, $j$, $k$, $l$ (symmetry).

(H3) Ellipticity: the $n \times n$ matrix function $\mathbf{M}(\mathbf{x})$ with components $M_{ik}(\mathbf{x}) = \sum_{j,l=1}^{n} \mathsf{C}(\mathbf{x})_{ijkl}\xi_j\xi_l$ has nonzero determinant for each $\mathbf{x} \in \bar{\Omega}$ and each (real) $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ (i.e. $\boldsymbol{\xi} \neq \mathbf{0}$).

(H4) We assume that the Complementing Condition (CC) (see Definition (3.1)) holds at every $\mathbf{x}_0 \in \partial\Omega$.

(H5) Either the pure displacement problem holds on all of $\partial\Omega$ or the pure traction problem holds on all of $\partial\Omega$.

**Remark 2.1.** *The conditions in (H1) are, at the level of generality considered here, almost minimal for the classical elliptic estimates (Agmon, Douglis, Nirenberg [4], Lions, Magenes [31], Wloka [46], [47]) to hold, as well as the consequent regularity theory of solutions to follow. In fact $\partial\Omega$ may be assumed locally $C^{k-1,1}$-smooth in case $X = W^{k,p}(\Omega)$ without changing any results. We shall define a constant $\omega$ which bounds the map that "straightens the boundary" in $C^k$- or $C^{k,\alpha}$-norm as well as its inverse. On the other hand if (H1) fails and $\partial\Omega$ is allowed to have, for example, corners or edges then the elliptic estimates and Fredholm properties may fail. We do not consider these cases here, see Grisvard [17], Dauge [12] for results.*

**Remark 2.2.** *The conditions (H2) are sufficient for the* general *elliptic estimates and regularity theory. We define a constant $m$ which bounds the $C^{k-1}$- or $C^{k-1,\alpha}$-norm of $\mathsf{C}$ on $\bar{\Omega}$. These smoothness requirements for the components of the tensor $\mathsf{C}$ follow from corresponding ones for $\mathbf{f}_0$ and $\mathbf{S}$ (cf. Valent [45]).*

**Remark 2.3.** *The ellipticity condition stated above is sometimes referred to as ordinary ellipticity. Two particular strengthened versions of this are called strong ellipticity and positive definiteness (cf. Antman [6], Gurtin [19], Valent [45]). Strong ellipticity is: the matrix $\mathbf{M}(\mathbf{x})$ is positive definite for each $\mathbf{x} \in \bar{\Omega}$ (note symmetry $\mathsf{C}_{ijkl} = \mathsf{C}_{klij}$ implies $\mathbf{M}$ is symmetric). Positive definiteness is: $\sum_{i,j,k,l=1}^{n} \mathsf{C}(x)_{ijkl}H_{ij}H_{kl} > 0$ for each $\mathbf{x} \in \bar{\Omega}$, $\mathbf{H} \in L(\mathbb{R}^n) \setminus \mathbf{0}$. Clearly, positive definiteness implies strong ellipticity which implies ordinary ellipticity. Ordinary ellipticity is the weakest type (most general) to guarantee the elliptic estimates (the latter implies the former, see Agmon, Douglis, Nirenberg [4] for a proof). Also strong ellipticity is equivalent to real wave speeds in the linearized dynamic problem associated to* (2.4) *(adding the term $\rho(\mathbf{x})\frac{\partial^2\mathbf{u}}{\partial t^2}$, see Gurtin [19]). Following (H3) we define an ellipticity constant $E$ which bounds below the determinant of $\mathbf{M}$ on $\bar{\Omega}$ for each unit vector $\boldsymbol{\xi} \in \mathbb{R}^n$.*

**Remark 2.4.** *The complementing condition, which is discussed in details in Chapter* (3)*, is sufficient and necessary (as well as the other requirements) for the elliptic estimates to hold and this also reflects in the Fredholm property. See Agmon, Douglis, Nirenberg [4] for a proof of this. It is related to certain high* *frequency behavior of infinitesimal solutions near the boundary in tangential directions, see Simpson, Spector [40]. It is possible to define a constant $\Delta > 0$ on $\partial\Omega$ which measures how close the complementing condition is to failing at any point $\mathbf{x}_0$ on $\partial\Omega$.*

We also have:

**Theorem 2.5.** *If $B(\mathbf{u}) = \mathbf{u}$ on all of $\partial\Omega$ (pure displacement problem), then the* CC *holds at every $\mathbf{x}_0 \in \partial\Omega$.*

This follows from Garding's inequality. In addition, if the **C**-tensor is perturbed slightly in the $C^0$-norm over $\bar{\Omega}$ then CC still holds, see Simpson, Spector [40].

Again we denote $X$ as above with corresponding norm: $X = W^{k,p}(\Omega)$ or $X = C^{k,\alpha}(\bar{\Omega})$, $k \geq 2$, $1 < p < \infty$, $0 < \alpha < 1$. Also let

$$Z = \begin{cases} W^{k-2,p}(\Omega), & \text{if } X = W^{k,p}(\Omega), \\ C^{k-2,\alpha}(\bar{\Omega}), & \text{if } X = C^{k,\alpha}(\bar{\Omega}), \end{cases} \tag{2.6}$$

and

$$V = \begin{cases} W^{k-1/p,p}(\partial\Omega), & \text{if } X = W^{k,p}(\Omega), \\ C^{k,\alpha}(\partial\Omega), & \text{if } X = C^{k,\alpha}(\bar{\Omega}), \end{cases} \tag{2.7}$$

for the displacement problem, or

$$V = \begin{cases} W^{k-1-1/p,p}(\partial\Omega), & \text{if } X = W^{k,p}(\Omega), \\ C^{k-1,\alpha}(\partial\Omega), & \text{if } X = C^{k,\alpha}(\bar{\Omega}), \end{cases} \tag{2.8}$$

for the traction problem. Put $Y = Z \times V$ with corresponding norm $\|(f,g)\|_Y = \|f\|_Z + \|g\|_V$, $f \in Z$, $g \in V$. Here $V$ consists of trace spaces and involves the boundary data. Note $X \to Z$ is a compact embedding (see Adams [1]). With this, we have that $L$ in (2.5) is a bounded linear operator $L : X \to Y$ and $B : X \to V$ is also bounded and linear. This uses (H2) and the trace theorems for Sobolev spaces.

We have some definitions: $L$ is said to be *injective* if $\ker L = \operatorname{kernel} L = \{0\}$ ($L$ is one-one). $L$ is *surjective* if $\operatorname{ran} L = \operatorname{range} L = Y$ ($L$ is onto). $L$ is *bijective* if it is injective and surjective. $L$ is a *semi-Fredholm* operator if $\ker L$ is finite dimensional ($\dim\ker L < \infty$) and $\operatorname{ran} L$ is a closed subspace of $Y$. $L$ is a *Fredholm* operator if $L$ is semi-Fredholm and $\operatorname{ran} L$ has finite codimension in $Y$ ($\operatorname{codim}\operatorname{ran} L < \infty$). If $L$ is semi-Fredholm its *index* is $\operatorname{ind} L = \operatorname{codim}\operatorname{ran} L - \dim\ker L$. ($L$ is Fredholm iff $\operatorname{ind} L < \infty$). See Kato [27].

Some classic results follow.

**Lemma 2.6 (Peetre's Lemma).** *Suppose $X$, $Y$, $Z$ are Banach spaces such that $X \to Z$ is a compact embedding. Let $L : X \to Y$ be a bounded linear map. Then $L$ is semi-Fredholm iff there exists $c > 0$ such that*

$$\|u\|_X \leq c(\|Lu\|_Y + \|u\|_Z)$$

*for all $u \in X$. If, in addition, $L$ is injective then the term $\|u\|_Z$ may be deleted.*

See Peetre [36], Lions, Magenes [31].

**Theorem 2.7 (Stability of Fredholm index).** *Suppose $L : X \to Y$ is semi-Fredholm. Then there exists $c > 0$ such that if $L' : X \to Y$ is a bounded linear map and $\|L - L'\| < c$ then $L'$ is semi-Fredholm and* $\operatorname{ind} L = \operatorname{ind} L'$. *If $K : X \to Y$ is compact, linear then $L + K$ is semi-Fredholm with* $\operatorname{ind}(L + K) = \operatorname{ind} L$.

See Kato [27, Chap. 4, Thm. 5.17].

**Theorem 2.8 (Elliptic Estimates).** *Assume (H1)–(H5), $k \geq 2$, $1 < p < \infty$, $0 < \alpha < 1$, and let $Y = Z \times V$ where $Z$ and $V$ are given by* (2.6), (2.7) *or* (2.8). *Then there exists $c > 0$ such that*

$$\|\mathbf{u}\|_X \leq c(\|L(\mathbf{u})\|_Y + \|\mathbf{u}\|_Z) \tag{2.9}$$

*for all $\mathbf{u} \in X$. The constant $c$ depends only on $\omega$, $m$, $E$, $\Delta$ (see Remarks above) as well as $k$, $p$, $\alpha$ in the definition of $X$. Furthermore, if $\mathbf{u} \in W^{2,p}(\Omega)$ and $L(\mathbf{u}) \in W^{k-2,p}(\Omega) \times W^{k-1/p,p}(\partial\Omega)$ (displacement) or $L(\mathbf{u}) \in W^{k-2,p}(\Omega) \times W^{k-1-1/p,p}(\partial\Omega)$ (traction) ($k \geq 2$, $1 < p < \infty$) then $\mathbf{u} \in W^{k,p}(\Omega)$. Similarly if $\mathbf{u} \in C^{2,\alpha}(\bar{\Omega})$ ($0 < \alpha < 1$) and $L(\mathbf{u}) \in C^{k-2,\alpha}(\bar{\Omega}) \times (C^{k,\alpha}(\partial\Omega)$ or $C^{k-1,\alpha}(\partial\Omega))$ (displ. or traction) then $\mathbf{u} \in C^{k,\alpha}(\bar{\Omega})$.*

See Agmon, Douglis, Nirenberg [4] and also Lions, Magenes [31], Triebel [43], Wloka [46], [47]. In the case of the Holder spaces, (2.9) are called the Schauder estimates. We note higher regularity results are included in this theorem when $k > 2$. Also note

$$\|L(\mathbf{u})\|_Y = \begin{cases} \|\operatorname{Div} \mathbf{C}[\nabla \mathbf{u}]\|_{k-2,p} + \|\mathbf{u}\|_{W^{k-1/p,p}(\partial\Omega)}, & \text{if } X = W^{k,p}(\Omega), \\ \|\operatorname{Div} \mathbf{C}[\nabla \mathbf{u}]\|_{k-2,\alpha} + \|\mathbf{u}\|_{C^{k,\alpha}(\partial\Omega)}, & \text{if } X = C^{k,\alpha}(\bar{\Omega}), \end{cases}$$

for the displacement problem, or

$$\|L(\mathbf{u})\|_Y = \begin{cases} \|\operatorname{Div} \mathbf{C}[\nabla \mathbf{u}]\|_{k-2,p} + \|\mathbf{C}[\nabla \mathbf{u}]\mathbf{n}\|_{W^{k-1-1/p,p}(\partial\Omega)}, & \text{if } X = W^{k,p}(\Omega), \\ \|\operatorname{Div} \mathbf{C}[\nabla \mathbf{u}]\|_{k-2,\alpha} + \|\mathbf{C}[\nabla \mathbf{u}]\mathbf{n}\|_{C^{k-1,\alpha}(\partial\Omega)}, & \text{if } X = C^{k,\alpha}(\bar{\Omega}), \end{cases}$$

for traction problem.

Putting these three theorems together yields ($\mathbb{C}$ denotes the set of complex numbers):

**Theorem 2.9.** *Assume (H1)–(H5), $k \geq 2$, $1 < p < \infty$, $0 < \alpha < 1$. Then $L$ in* (2.5) *is semi-Fredholm. Also for each $\lambda \in \mathbb{C}$, $L_\lambda$ is semi-Fredholm and* $\operatorname{ind} L = \operatorname{ind} L_\lambda$ *where* $L_\lambda(\mathbf{u}) = (\operatorname{Div} \mathbf{C}[\nabla \mathbf{u}] - \lambda \mathbf{u}, B(\mathbf{u}))$, $\mathbf{u} \in X$.

**Remark 2.10.** *Note that adding zeroth- (e.g. $-\lambda \mathbf{u}$) or first-order terms (with sufficiently smooth coefficients) to* $\operatorname{Div} \mathbf{C}[\nabla \mathbf{u}]$, *or zeroth-order terms to the traction operator $\mathbf{C}[\nabla \mathbf{u}]\mathbf{n}$ on $\partial\Omega$ is equivalent to adding a compact operator $K$ to $L$ (cf. compact embedding of $X$ in lower order spaces). The last statement in Theorem* (2.7) *above then applies.*

Next we determine when $L$ has index *zero*. We refer to a result of Schechter [38], also in Lions, Magenes [31] in the $W^{k,2}(\Omega)$ case (Hilbert space).

**Theorem 2.11.** *Suppose (H1)–(H5) hold with a strengthening of (H1) and (H2): $\partial\Omega$ is locally $C^\infty$-smooth and $\mathsf{C}(\cdot)_{ijkl}$ are $C^\infty$-smooth functions on $\bar{\Omega}$ ($k = \infty$ in (H1), (H2)). Let $p = 2$: $X = W^{k,2}(\Omega)$, $Y = W^{k-2,2}(\Omega) \times W^{k-1/2,2}(\partial\Omega)$ (displacement) or $Y = W^{k-2,2}(\Omega) \times W^{k-3/2,2}(\partial\Omega)$ (traction), $k \geq 2$. Assume that $L : X \to Y$ is an injective map. Then $L$ is surjective. The same statement applies with $L_\lambda$ replacing $L$ for any $\lambda \in \mathbb{C}$.*

**Remark 2.12.** *This theorem asserts the classic statement that "uniqueness implies solvability" since surjectivity of $L$ is equivalent to solvability of the boundary value problem in* (2.4) *for any $\mathbf{b} \in W^{k-2,p}(\Omega)$, $\mathbf{d} \in W^{k-1/p,p}(\partial\Omega)$, $\mathbf{t} \in W^{k-1-1/p,p}(\partial\Omega)$ or $\mathbf{b} \in C^{k-2,\alpha}(\bar{\Omega})$, $\mathbf{d} \in C^{k,\alpha}(\partial\Omega)$, $\mathbf{t} \in C^{k-1,\alpha}(\partial\Omega)$.*

The $C^\infty$-smoothness assumptions in this theorem are made for convenience in the proof in [31] and can be relaxed so that no restrictions are imposed beyond (H1) and (H2). This is accomplished as follows. First, suppose the coefficients of $L$ (i.e. the $\mathsf{C}$-tensor) satisfy (H2) and $L : X \to Y$ is injective. Then approximate $\mathsf{C}$ with a sequence $\mathsf{C}_n$ of $C^\infty$-smooth tensors in $C^{k-1}$-norm over $\bar{\Omega}$ still satisfying (H3) and (H4). We call the corresponding operators $L_n$. Then the elliptic estimates (2.9) hold with constant $c$ independent of $n$ (the constants $m$, $E$, $\Delta$ are independent of $n$) and with the term $\|\mathbf{u}\|_Z$ deleted (see Peetre's Theorem, e.g.,

$$\begin{aligned}\|\mathbf{u}\|_X &\leq c\|L(\mathbf{u})\|_Y \leq c(\|L_n(\mathbf{u})\|_Y + \|(L - L_n)(\mathbf{u})\|_Y) \\ &\leq c(\|L_n(\mathbf{u})\|_Y + \epsilon_n\|\mathbf{u}\|_X)\end{aligned}$$

where $\epsilon_n \to 0$). Then $L_n$ is injective and satisfies the hypotheses of the theorem above. Let $\mathbf{b} \in W^{k-2,2}(\Omega)$, $\mathbf{d} \in W^{k-1/2,2}(\partial\Omega)$. By surjectivity there exist $\mathbf{u}_n \in W^{k,2}(\Omega)$ such that $L_n(\mathbf{u}_n) = (\mathbf{b}, \mathbf{d})$ for each $n$. Then the $\mathbf{u}_n$ are Cauchy in $W^{k,2}(\Omega)$: $\|\mathbf{u}_n\|_X \leq c\|L_n(\mathbf{u}_n)\|_Y = c\|(\mathbf{b}, \mathbf{d})\|_Y$ ($\mathbf{u}_n$ are uniformly bounded in $X$), and

$$\begin{aligned}\|\mathbf{u}_n - \mathbf{u}_m\|_X &\leq c\|L_n(\mathbf{u}_n - \mathbf{u}_m)\|_Y \\ &= c\,\|L_n(\mathbf{u}_n) - L_m(\mathbf{u}_m) + L_m(\mathbf{u}_m) - L_n(\mathbf{u}_m)\| \\ &= c\|(L_m - L_n)(\mathbf{u}_m)\|_Y \leq c\epsilon_{n,m}\|\mathbf{u}_m\|_X \to 0,\end{aligned}$$

as $n, m \to \infty$. If $\mathbf{u}$ is the limit of $\mathbf{u}_n$ in $X$ then it easily follows that $L(\mathbf{u}) = (\mathbf{b}, \mathbf{d})$ so $L$ is surjective. Similarly for the traction problem. See also Agmon, Douglis, Nirenberg [4] for this argument.

Second, suppose $\partial\Omega$ satisfies (H1). Then $\Omega$ may be approximated from the outside by a sequence of domains $\Omega_n$ ($\bar{\Omega} \subset \Omega_n$) with $C^\infty$-smooth boundaries such that the distance from $\bar{\Omega}$ to $\partial\Omega_n$ tends to zero as $n \to \infty$ and the constants $\omega$ in (H1) uniformly approach that for $\partial\Omega$ all along $\partial\Omega$. Then the unit normal $\mathbf{n}$ is also uniformly approximated. To do this one may construct a function $F : \mathbb{R}^n \to \mathbb{R}$ with the same smoothness as $\Omega$ in (H1) such that $\Omega = \{\mathbf{x} : F(\mathbf{x}) >$

$0\}$, $\partial\Omega = \{\mathbf{x} : F(\mathbf{x}) = 0\}$ and $\mathbb{R}^n \setminus \Omega = \{\mathbf{x} : F(\mathbf{x}) < 0\}$. Then uniformly approximating $F$ with $C^\infty$-smooth functions $F_n$ yields sets $\Omega_n = \{F_n > 0\}$ as desired. By extending the $\mathbf{C}$-tensor smoothly to $\Omega_n$ we may assume (H3), (H4) still hold on $\bar{\Omega}_n$, $\partial\Omega_n$ respectively. The spaces $X$, $Y$ change accordingly. Then $L$ still satisfies (2.9) and is injective on each domain $\Omega_n$ (for $n$ sufficiently large; again the $\|\mathbf{u}\|_Z$ term is missing). Let $(\mathbf{b}, \mathbf{d}) \in W^{k-2,2}(\Omega) \times W^{k-1/2,2}(\partial\Omega)$ and we extend $\mathbf{b}$, $\mathbf{d}$ to functions in a neighborhood of $\bar{\Omega}$. Again by surjectivity there exist $\mathbf{u}_n \in W^{k,2}(\Omega_n)$ such that $L(\mathbf{u}_n) = (\mathbf{b}, \mathbf{d}) \in W^{k-2,2}(\Omega_n) \times W^{k-1/2,2}(\partial\Omega_n)$. By (2.9) $\mathbf{u}_n$ is bounded in $W^{k,2}(\Omega)$ so has a weakly convergent subsequence (again called $\mathbf{u}_n$); say $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in $W^{k,2}(\Omega)$. Then easily $L(\mathbf{u}) = (\mathbf{b}, \mathbf{d})$ in the original space $Y$. Thus $L$ is surjective. Similarly for the traction problem. See also Necas [34] (continuous dependence on domains).

Here is a sketch of the proof of Schechter's Theorem (Theorem (2.11)). Let

$$X_B = \left\{\mathbf{u} \in W^{k,2}(\Omega) \,|\, B(\mathbf{u}) = \mathbf{0} \text{ on } \partial\Omega\right\},$$

and consider the bilinear form over $X_B$,

$$A(\mathbf{w}, \mathbf{v}) = \int_\Omega L(\mathbf{w}) \cdot L(\mathbf{v}) \, \mathrm{d}\mathbf{x}, \quad \mathbf{w}, \mathbf{v} \in X_B.$$

By the elliptic estimates (2.9) and injectivity of $L$, we see $A$ is equivalent to the $W^{k,2}(\Omega)$-norm on $X_B$. Let $\mathbf{b} \in W^{k-2,2}(\Omega)$. By the Riesz Representation Theorem there is $\mathbf{w} \in X_B$ such that $A(\mathbf{w}, \mathbf{v}) = \int_\Omega \mathbf{b} \cdot \mathbf{v} \, dx$ for all $\mathbf{v} \in X_B$. Let $\mathbf{u} = L(\mathbf{w}) \in W^{k-2,2}(\Omega)$. Then by the difference quotient method it can be shown $\mathbf{u} \in W^{k,2}(\Omega)$ (regularity). By symmetry of $L$ and integration by parts it is easy to see $\operatorname{Div} \mathbf{C}[\nabla \mathbf{u}] = \mathbf{b}$ on $\Omega$, $B(\mathbf{u}) = \mathbf{0}$ on $\partial\Omega$. Now assume the traction problem and let $(\mathbf{b}, \mathbf{t}) \in W^{k-2,2}(\Omega) \times W^{k-3/2,2}(\partial\Omega)$. It can be shown that $B$ is surjective onto $V$ (see Adams [1], Triebel [43]), so there exists $\mathbf{u}_0 \in W^{k,2}(\Omega)$ such that $B(\mathbf{u}_0) = \mathbf{t}$. By the above there exists $\mathbf{u} \in W^{k,2}(\Omega)$ such that $L(\mathbf{u}) = (\mathbf{b} - \operatorname{Div} \mathbf{C}[\nabla \mathbf{u}_0], \mathbf{0}) \iff L(\mathbf{u} + \mathbf{u}_0) = (\mathbf{b}, \mathbf{t})$. Thus $L$ is surjective. Similarly for the displacement problem.

Thus we may assume the $C^\infty$-smoothness assumptions are dropped in the theorem above. So in the case $X = W^{k,2}(\Omega)$ $(k \geq 2)$ we have: $L$ is injective $\Rightarrow$ $\operatorname{ind} L = 0$. Similarly $L_\lambda$ is injective for some $\lambda \in \mathbb{C} \Rightarrow \operatorname{ind} L = 0$. To extend this result to the general spaces $X = W^{k,p}(\Omega)$ $(1 < p < \infty)$ or $C^{k,\alpha}(\bar{\Omega})$ $(0 < \alpha < 1)$, $k \geq 2$ we appeal to the following.

Let $X_p = W^{k,p}(\Omega)$, $Y_p = W^{k-2,p}(\Omega) \times \left(W^{k-1/p,p}(\partial\Omega) \text{ or } W^{k-1-1/p,p}(\partial\Omega)\right)$, $X_\alpha = C^{k,\alpha}(\bar{\Omega})$, $Y_\alpha = C^{k-2,\alpha}(\bar{\Omega}) \times \left(C^{k,\alpha}(\partial\Omega) \text{ or } C^{k-1,\alpha}(\partial\Omega)\right)$ and let $L^{(p)}$ denote the operator $L_\lambda$ above mapping $X_p$ into $Y_p$, and similarly $L^{(\alpha)}$ maps $X_\alpha$ into $Y_\alpha$ ($\lambda \in \mathbb{C}$ is fixed). We define a Property $P$:

$$\begin{aligned} &1 < p, q < \infty, \; 0 < \alpha < 1, \; k \geq 2, \\ &\text{if } \mathbf{u} \in W^{k,p}(\Omega) \text{ and } L^{(p)}(\mathbf{u}) \in Y_q, \text{ then} \\ &\mathbf{u} \in W^{k,q}(\Omega); \\ &\text{if } \mathbf{u} \in W^{k,p}(\Omega) \text{ and } L^{(p)}(\mathbf{u}) \in Y_\alpha, \text{ then} \\ &\mathbf{u} \in C^{k,\alpha}(\bar{\Omega}). \end{aligned} \tag{$P$}$$

**Theorem 2.13.**

*i) Assume* $1 < p, q < \infty$, $0 < \alpha < 1$ *and (H1)–(H5). Then Property* $P$ *implies*

*a)* $\dim\ker L^{(p)} = \dim\ker L^{(q)} = \dim\ker L^{(\alpha)}$;

*b)* $\operatorname{codim}\operatorname{ran} L^{(p)} = \operatorname{codim}\operatorname{ran} L^{(q)} = \operatorname{codim}\operatorname{ran} L^{(\alpha)}$;

*c)* $\operatorname{ind} L^{(p)} = \operatorname{ind} L^{(q)} = \operatorname{ind} L^{(\alpha)}$.

*If* $\operatorname{ind} L^{(p)}$, $\operatorname{ind} L^{(q)}$, $\operatorname{ind} L^{(\alpha)}$ *are finite then the implication is an equivalence.*

*ii) Property* $P$ *is true.*

**Remark 2.14.** *In Theorem 2.13, if the Holder spaces are referred to, then the corresponding portions of (H1), (H2) are assumed.*

**Remark 2.15.** *Part (ii) is the regularity result. Its proof can be found in Morrey [33, Thm. 6.3.7] using potential integrals or can be proved more directly by adapting a method of Grisvard [17], see Lemma 2.4.1.4 there.*

**Remark 2.16.** *Part (i) is a purely functional analytic result and shows the index of* $L$ *is independent of* $p$ *and* $\alpha$ *provided Property* $P$ *holds. Similarly the index of* $L$ *is independent of* $k$ *— see the elliptic estimate theorem above.*

Thus we have

**Theorem 2.17.** *Suppose* $X = W^{k,p}(\Omega)$ *or* $C^{k,\alpha}(\bar{\Omega})$, $k \geq 2$, $1 < p < \infty$, $0 < \alpha < 1$. *Suppose (H1)–(H5) hold. If there exists* $\lambda \in \mathbb{C}$ *such that* $L_\lambda : X \to Y$ *is injective then* $\operatorname{ind} L_\mu = 0$ *for all* $\mu \in \mathbb{C}$.

We remark that if spectral constraints are imposed on $L$, e.g. the spectrum lies in a sector $S$ in $\mathbb{C}$, $S = \{\lambda \in \mathbb{C} : \; c_1|\operatorname{Im}\lambda| \leq \operatorname{Re}\lambda - c_0\}$ (see Kato [27]) then the resolvent set of $L$ is nonempty and Theorem 2.17 yields the index of $L$ is zero. This is related to Agmon's condition (see Simpson, Spector [40]), also closely related to CC. It implies $\|\mathbf{u}\|_X \leq c\|(L - \lambda I)(\mathbf{u})\|_Y$ for all $\mathbf{u} \in X$, $\operatorname{Re}\lambda \leq c_0$ with $c$ independent of such $\mathbf{u}$, $\lambda$ (see Agmon [3]). Also in the special case $L_\lambda(\mathbf{u}) = \operatorname{Div} \mathbf{C}[\nabla\mathbf{u}] - \lambda\mathbf{u}$, $X = W^{k,2}(\Omega)$, $k \geq 2$, we have $L_\lambda$ is injective for nonreal $\lambda$, $\operatorname{Im}\lambda \neq 0$: if $\mathbf{u} \in \ker L_\lambda$ then

$$0 = \int_\Omega \bar{\mathbf{u}} \cdot L_\lambda(\mathbf{u})\,\mathrm{d}\mathbf{x} = -\int_\Omega \nabla\bar{\mathbf{u}} \cdot \mathbf{C}[\nabla\mathbf{u}]\,\mathrm{d}\mathbf{x} - \lambda\int_\Omega |\mathbf{u}|^2 \mathrm{d}\mathbf{x},$$

and symmetry of $\mathbf{C}$ implies the first term on the right is real so that $\int_\Omega |\mathbf{u}|^2 = 0$, i.e. $\mathbf{u} = \mathbf{0}$. By Theorem 2.17 it follows $L_\lambda$ is injective over the spaces $W^{k,p}(\Omega)$, $C^{k,\alpha}(\bar{\Omega})$.

We finally remark that all the above theory applies to much wider classes of linear elliptic boundary value problems as long as the boundary conditions form a Dirichlet system (see Schechter [38] and Lions, Magenes [31]). Also by interpolation, $k$ may be a noninteger $\geq 2$, see Triebel [43].

# Chapter 3

# The Complementing and Agmon's Conditions

We examine the linearized elliptic boundary value problem in elasticity and its relation to the complementing condition. For the pure traction problem this is given by:

$$\begin{aligned} \operatorname{Div} \mathsf{C}[\nabla \mathbf{u}] &= \mathbf{b} && \text{on } \Omega, \\ \mathsf{C}[\nabla \mathbf{u}]\mathbf{n} &= \mathbf{t} && \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Here $\Omega \subset \mathbb{R}^n$ is an open, bounded region that an elastic body occupies in its reference configuration. Its boundary $\partial\Omega$ has outward unit normal $\mathbf{n} : \partial\Omega \to \mathbb{R}^n$. We assume

(H1) $\Omega \subset \mathbb{R}^n$ is open, bounded and $\partial\Omega$ is locally $C^2$-smooth, see Adams [1, p. 67]. Thus $\mathbf{n}(\cdot)$ is $C^1$-smooth on $\partial\Omega$.

(H2) The components $\mathsf{C}(\cdot)_{ijkl}$ of $\mathsf{C}$ are $C^1$-smooth on $\bar{\Omega}$ for all $i$, $j$, $k$, $l$.

(H3) $\mathsf{C}(\cdot)_{ijkl} = \mathsf{C}(\cdot)_{klij}$ on $\bar{\Omega}$ for all $i$, $j$, $k$, $l$ (symmetry) and the $\mathsf{C}$–tensor is elliptic: the $n \times n$ matrix function $\mathbf{M}(\mathbf{x})$ with components $M_{ik}(\mathbf{x}) = \sum_{j,l=1}^{n} \mathsf{C}(\mathbf{x})_{ijkl}\xi_j\xi_l$ has nonzero determinant for each $\mathbf{x} \in \bar{\Omega}$ and each (real) $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \setminus \{0\}$. A stronger form is strong ellipticity

$$\sum_{i,j,k,l=1}^{n} \mathsf{C}(\mathbf{x})_{ijkl}\eta_i\xi_j\eta_k\xi_l > 0$$

for each $\mathbf{x} \in \bar{\Omega}$, $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^n \setminus \{0\}$.

With these assumptions the boundary value problem (3.1) is elliptic and symmetric. We could also consider displacement boundary condition with $\mathbf{u}(\mathbf{x}) = \mathbf{d}(\mathbf{x})$ replacing $(3.1)_2$ where $\mathbf{d} : \partial\Omega \to \mathbb{R}^n$ is a given (displacement) function and we remark on this below.

We review the elliptic estimates corresponding to (3.1). Let $X$ stand for the Sobolev space $W^{k,p}(\Omega)$ with corresponding norm (see Adams [1]), $k \geq 2$ is an integer, $1 < p < \infty$. Let $Y = W^{k-2,p}(\Omega) \times W^{k-1-1/p,p}(\partial\Omega)$ (the latter is a trace space on $\partial\Omega$) and let $Z = W^{k-2,p}(\Omega)$. We define the operator

$$L(\mathbf{u}) = (\operatorname{Div} \mathbf{C}[\nabla \mathbf{u}], \mathbf{C}[\nabla \mathbf{u}]\mathbf{n}),$$

for $\mathbf{u} \in X$. Under (H1), (H2), $L : X \to Y$ is a bounded, linear map. Assuming (H1), (H2), (H3) and the complementing condition (defined below) we have the elliptic estimate: there exists $c > 0$ such that

$$\|\mathbf{u}\|_X \leq c(\|L(\mathbf{u})\|_Y + \|\mathbf{u}\|_Z), \tag{3.2}$$

for all $\mathbf{u} \in X$; $c$ is independent of $\mathbf{u}$ and depends only on $\partial\Omega$, $\|\mathbf{C}(\cdot)\|_{C^1(\bar{\Omega})}$, an ellipticity constant and the complementing condition, as well as $k$, $p$. See, for example, Agmon, Douglis, Nirenberg [4], Lions, Magenes [31], Morrey [33].

**Definition 3.1.** Let $\mathbf{x}_0 \in \partial\Omega$ be fixed and define the open *halfspace*

$$H = \{\mathbf{x} \in \mathbb{R}^n \ : \ (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}_0) < 0\}.$$

($\partial H$ is tangential to $\partial\Omega$ at $\mathbf{x}_0$). Consider the linear boundary value problem on $H$

$$\begin{aligned} \operatorname{Div} \mathbf{C}_0[\nabla \mathbf{v}] &= \mathbf{0} && \text{on } H, \\ B_0(\mathbf{v}) &= \mathbf{0} && \text{on } \partial H, \end{aligned} \tag{3.3}$$

where the unknown $\mathbf{v} : H \to \mathbb{R}^n$ has the particular form

$$\mathbf{v}(\mathbf{x}) = \mathbf{w}(t) e^{i\boldsymbol{\xi}\cdot(\mathbf{x}-\mathbf{x}_0)}, \tag{3.4}$$

for some $\boldsymbol{\xi} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $\boldsymbol{\xi}$ is orthogonal to $\mathbf{n}(\mathbf{x}_0)$, $t = -(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}_0)$ is the normal variable pointing positively into $H$ ($t \geq 0$), $\mathbf{w} : [0, \infty) \to \mathbb{R}^n$ is exponentially decreasing as $t \to +\infty$. Here $\mathbf{C}_0 = \mathbf{C}(\mathbf{x}_0)$ is a constant fourth order tensor (frozen at $\mathbf{x}_0$); and $B_0(\mathbf{v}) = \mathbf{v}$ (for the pure displacement problem) or $B_0(\mathbf{v}) = \mathbf{C}_0[\nabla \mathbf{v}]\mathbf{n}(\mathbf{x}_0)$ (in case $(3.1)_2$). Then *complementing condition* (CC) holds at $\mathbf{x}_0$ iff the only such solution of (3.3) is $\mathbf{v}$ identically zero on $H$ ($\mathbf{w}$ identically zero) for all $\boldsymbol{\xi} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $\boldsymbol{\xi}$ orthogonal to $\mathbf{n}(\mathbf{x}_0)$.

(See Agmon, Douglis, Nirenberg [4], Valent [45].) Note the problem in (3.3) is linear with constant coefficients and when (3.4) is substituted into (3.3) we obtain an equivalent ordinary differential system (in $\mathbf{w}$) of the form

$$\begin{aligned} &\mathbf{C}_0\left[\frac{\mathrm{d}^2\mathbf{w}}{\mathrm{d}t^2} \otimes \mathbf{n}\right]\mathbf{n} + i\mathbf{C}_0\left[\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} \otimes \boldsymbol{\xi}\right]\mathbf{n} \\ &\qquad + i\mathbf{C}_0\left[\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} \otimes \mathbf{n}\right]\boldsymbol{\xi} - \mathbf{C}_0[\mathbf{w} \otimes \boldsymbol{\xi}]\boldsymbol{\xi} = \mathbf{0}, \qquad t \in [0, \infty), \\ &\mathbf{C}_0\left[\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} \otimes \mathbf{n}\right]\mathbf{n} + i\mathbf{C}_0[w \otimes \boldsymbol{\xi}]\mathbf{n} = \mathbf{0}, \qquad \text{at } t = 0, \end{aligned} \tag{3.5}$$

on the half-line $t \geq 0$; here $\mathbf{n} = \mathbf{n}(\mathbf{x}_0)$ and $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$. By ellipticity, half the solutions of the differential equation in (3.5) are exponentially decaying as $t \to$

$\infty$. By substituting suitable exponentials in (3.5) the problem becomes purely algebraic. Then a particular determinant $\Delta(\mathbf{x}_0, \boldsymbol{\xi})$ being nonzero is equivalent to forcing $\mathbf{w}$ to be identically zero; the CC holds at $\mathbf{x}_0$ iff $\Delta(\mathbf{x}_0, \boldsymbol{\xi}) \neq 0$ for all $\boldsymbol{\xi} \in \mathbb{R}^n \setminus \{0\}$, $\boldsymbol{\xi}$ orthogonal to $\mathbf{n}(\mathbf{x}_0)$. The functions $\mathbf{v}$ are oscillatory tangential to $\partial H$ (or $\partial\Omega$) and exponentially decaying into $H$. We note that in Agmon, Douglis, Nirenberg [4] an equivalent formulation is given to (3.3), (3.4) above that is more directly of an algebraic nature; this involves looking at the roots $\tau$ of det $\left[\sum_{j,l=1}^{n} \mathbf{C}(\mathbf{x}_0)_{ijkl}(\boldsymbol{\xi}+\tau\mathbf{n})_j(\boldsymbol{\xi}+\tau\mathbf{n})_l\right]$ (with $\boldsymbol{\xi}$, $\mathbf{n}$ as in (3.4)) in the upper half-plane, $\operatorname{Im}\tau > 0$, and their relation to the boundary operator.

It is shown in Agmon, Douglis, Nirenberg [4, p. 83] that if (3.2) holds then (H3) and CC (complementing condition) follow. The latter is proved as follows. Suppose (3.2) holds but CC fails at $\mathbf{x}_0 \in \partial\Omega$. Then by means of a $C^2$-smooth invertible map we may straighten the boundary of $\Omega$ near $\mathbf{x}_0$ and obtain the estimate (3.2) locally on the half-space $H$ (see definition of CC):

$$\|\mathbf{u}\|_{W^{2,p}(H)} \leq c\left(\|L(\mathbf{u})\|_{L^p(H)\times W^{1-1/p,p}(\partial H)} + \|\mathbf{u}\|_{L^p(H)}\right) \tag{3.6}$$

for all $\mathbf{u} \in W^{2,p}(H)$ with the support of $\mathbf{u}$ contained in a half-ball $B_r = \{\mathbf{x} \in H : |\mathbf{x}-\mathbf{x}_0| < r\}$ — see Agmon, Douglis, Nirenberg [4, Thm. 10.4]. By reducing the radius $r$ of $B_r$ we may replace $L(\mathbf{u})$ with $L_0(\mathbf{u}) = (\operatorname{Div}\mathbf{C}_0[\nabla\mathbf{u}], \mathbf{C}_0[\nabla\mathbf{u}]\mathbf{n})$ where $\mathbf{C}_0 = \mathbf{C}(\mathbf{x}_0)$ (a constant 4-tensor frozen at $\mathbf{x}_0$), $\mathbf{n}$ is the outward unit normal to $\partial H$ at $\mathbf{x}_0$. Here we use that

$$\|(L-L_0)(\mathbf{u})\|_{L^p(H)\times W^{1-1/p,p}(\partial H)} \leq \epsilon(r)\|\mathbf{u}\|_{W^{2,p}(H)},$$

with

$$\epsilon(r) = \sup_{\mathbf{x}\in B_r} |\mathbf{C}(\mathbf{x}) - \mathbf{C}(\mathbf{x}_0)| \to 0 \quad \text{as } r \to 0.$$

Now let $\mathbf{v}$ be a nontrivial solution of (3.3) of the form (3.4) for some $\boldsymbol{\xi}$ and $\mathbf{w}$, and define for $\mu > 0$, $\mathbf{v}_\mu(\mathbf{x}) = \mu^{-2+1/p}\mathbf{v}(\mu\mathbf{x})$, $\mathbf{x} \in H$ and let $\zeta : H \to \mathbb{R}$ be a $C^\infty$ cut-off function with support in $\bar{B}_r$ so that $\zeta$ is zero in a neighborhood of the curved part of $\partial B_r$. By homogeneity the functions $\mathbf{v}_\mu$ satisfy (3.3) on $\overline{H}$ for all $\mu > 0$. Let $\mathbf{u} = \zeta\mathbf{v}_\mu$ on $H$; it is readily seen that

$$\|\mathbf{u}\|_{L^p(H)},\ \|L_0(\mathbf{u})\|_{L^p(H)\times W^{1-1/p,p}(\partial H)} \to 0 \quad \text{as } \mu \to 0,$$

while $\|\mathbf{u}\|_{W^{2,p}(H)}$ remains bounded away from zero[1] as $\mu \to \infty$. Thus (3.6) is violated and therefore also (3.2). We see that if CC is violated then the particular relation between the boundary operator $\mathbf{C}[\nabla\mathbf{u}]\mathbf{n}$ and the differential operator $\operatorname{Div}\mathbf{C}[\nabla\mathbf{u}]$ is such that a certain class of functions localized near the boundary of $\Omega$ causes the estimate (3.2) to fail; this class of functions has arbitrarily high-frequency oscillatory behavior tangential to $\partial\Omega$ ($e^{i\mu\boldsymbol{\xi}\cdot(\mathbf{x}-\mathbf{x}_0)}$) and exponential decay normal to the boundary ($\mathbf{w}(\mu t)$) along with any suitable cut-off function to localize near $\partial\Omega$. We see this phenomenon below also in relation

[1] $L_0(\mathbf{u}) = \zeta L_0(\mathbf{v}_\mu) +$ terms involving lower order derivatives of $\mathbf{v}_\mu$; this also uses the special form (3.4) assumed for $\mathbf{v}$.

to the quadratic form associated to (3.1). Finally, the proof that ellipticity (H3) follows from (3.2) at every point of $\bar{\Omega}$ is very similar (utilizing a similar high-frequency class of functions $\mathbf{u}_\mu$, $\mu \to \infty$) — see Agmon, Douglis, Nirenberg [4].

We note that if the boundary condition is $(3.1)_2$ is replaced with displacement (Dirichlet) type so $\mathbf{u} = \mathbf{d}$ is specified on $\partial\Omega$, $\mathbf{d} : \Omega \to \mathbb{R}^n$ is a given element of $W^{k-1/p,p}(\partial\Omega)$, then (3.3) is modified accordingly so $\mathbf{v} = 0$ on $\partial H$. Then we have that the CC holds at every $\mathbf{x}_0 \in \partial\Omega$. This follows from Garding's inequality

$$c_1\|\mathbf{u}\|_{L^2(\Omega)} + c_2\|\mathbf{u}\|_{W^{1,2}(\Omega)} \le \int_\Omega \nabla\mathbf{u} \cdot \mathbf{C}[\nabla\mathbf{u}]\,d\mathbf{x}, \tag{3.7}$$

for all $\mathbf{u} \in W_0^{1,2}(\Omega)$ ($\mathbf{u} = \mathbf{0}$ on $\partial\Omega$) with $c_2 > 0$, $c_1 \in \mathbb{R}$ constants independent of $\mathbf{u}$. Then a result below concerning the quadratic form on the right in (3.7) yields the CC on $\partial\Omega$. For a proof of (3.7) see Garding [16], Friedman [15].

Next we define now a condition similar to the complementing condition.

**Definition 3.2.** Let $\mathbf{x}_0 \in \partial\Omega$ be fixed and define $H$ as in (3.3). We consider the problem

$$\begin{aligned} \operatorname{Div} \mathbf{C}_0[\nabla\mathbf{v}] &= \alpha\mathbf{v}, &&\text{in } H,\\ \mathbf{C}_0[\nabla\mathbf{v}]\mathbf{n}(\mathbf{x}_0) &= \mathbf{0}, &&\text{on } \partial H, \end{aligned} \tag{3.8}$$

where $\alpha > 0$ and $\mathbf{v}$ is exactly as in (3.4), $\boldsymbol{\xi} \in \mathbb{R}^n \setminus \{0\}$, $\boldsymbol{\xi}$ orthogonal to $\mathbf{n}(\mathbf{x}_0)$. Then we say Agmon's condition holds at $\mathbf{x}_0$ iff all such solutions $\mathbf{v}$ in (3.8) of the form (3.4) are identically zero in $H$ for all such $\boldsymbol{\xi}$ and all $\alpha > 0$.

Again (3.8) can be reduced to an ordinary differential equation in $\mathbf{w}$ on the half-line $t \ge 0$ and this in turn is equivalent to an (algebraic) determinant condition. See Simpson, Spector ([40], [41]), Friedman [15]. We note that Friedman says the *strong complementing condition* holds at $\mathbf{x}_0 \in \partial\Omega$ iff CC and Agmon's conditions hold at $\mathbf{x}_0$.

We note Agmon's condition is related to the dynamic problem

$$\begin{aligned} \operatorname{Div} \mathbf{C}_0[\nabla\mathbf{u}] &= \frac{\partial^2\mathbf{u}}{\partial t^2}, &&\text{in } H,\\ \mathbf{C}_0[\nabla\mathbf{u}]\mathbf{n}(\mathbf{x}_0) &= \mathbf{0}, &&\text{on } \partial H, \end{aligned} \tag{3.9}$$

where we look for solutions of the form

$$\mathbf{u}(\mathbf{x},t) = \mathbf{v}(\mathbf{x})\exp\left(\alpha^{1/2}t\right),$$

and $\mathbf{v}$ is a bounded exponential that decays normal to $\partial H$. If $\mathbf{v}(\mathbf{x})$ is as in (3.4), then $\mathbf{u}(\mathbf{x},t)$ represents a surface oscillation in $\mathbf{x}$ along $\partial H$ propagating in time and is called a Raleigh wave. If Agmon's condition fails then such waves exist with $\alpha > 0$. Thus dynamic instability near $\partial H$ would follow.

It was discovered by Agmon ([2], [3]) that his condition is intimately related to spectral properties of the operator $\mathbf{u} \mapsto \operatorname{Div} \mathbf{C}[\nabla\mathbf{u}]$. We assume (H1), (H2) and the strong ellipticity condition for $\mathbf{C}$ (which implies (H3)). Also assume CC and Agmon's condition hold on $\partial\Omega$. Then as shown in Agmon [3] there exist $c_1, c_2, \epsilon > 0$ such that

$$c_1\|\mathbf{u}\|_{W^{k,p}(\Omega)} \leq \|\mathrm{Div}\,\mathbf{C}[\nabla\mathbf{u}] - \mu\mathbf{u}\|_{W^{k-2,p}(\Omega)} \tag{3.10}$$

for all (complex) $\mathbf{u} \in W^{k,p}(\Omega)$ such that $\mathbf{C}[\nabla\mathbf{u}]\mathbf{n} = \mathbf{0}$ holds on $\partial\Omega$ and all complex $\mu$ with $|\arg\mu| < \frac{\pi}{2} + \epsilon$, $|\mu| > c_2$ ($c_1$, $c_2$, $\epsilon$ independent of $\mathbf{u}$). To prove (3.10), adjoin a new variable $x_{n+1}$ to $\mathbf{x}$ and consider the elliptic operator $M[\mathbf{v}] = \mathrm{Div}\,\mathbf{C}[\nabla\mathbf{v}] + \frac{\partial^2\mathbf{v}}{\partial x_{n+1}^2}$ in $\mathbb{R}^{n+1}$ over a cylindrical domain $\Omega_r = \Omega \times (-r, r) \subset \mathbb{R}^{n+1}$, $r > 0$; $\mathbf{v} : \Omega_r \to \mathbb{R}^n$. Then strong ellipticity, CC and Agmon's condition imply that the extended operator $L' = (M, \mathbf{C}[\nabla\cdot]\mathbf{n})$ is strongly elliptic on $\Omega_r$ and satisfies the CC on $\partial\Omega_r$. Then an estimate like (3.2) applies to $L'$ over $\Omega_r$; in fact, a local estimate (see Agmon, Douglis, Nirenberg [4]) of this form yields

$$\begin{aligned} c_3\|e(x_{n+1})\mathbf{u}(\mathbf{x})\|_{W^{k,p}(\Omega_1)} &\leq \|M[\varphi(x_{n+1})e(x_{n+1})\mathbf{u}(\mathbf{x})]\|_{W^{k-2,p}(\Omega_2)} \\ &\quad + \|\varphi(x_{n+1})e(x_{n+1})\mathbf{u}(\mathbf{x})\|_{L^p(\Omega_2)} \end{aligned} \tag{3.11}$$

where $e(x_{n+1}) = \exp\left(i|\mu|^{1/2}x_{n+1}\right)$, $\varphi : \mathbb{R} \to \mathbb{R}$ is $C^\infty$ with support in $(-2, 2)$ and $\varphi = 1$ on $(-1, 1)$, $\mathbf{u} \in W^{k,p}(\Omega)$ with $\mathbf{C}[\nabla\mathbf{u}]\mathbf{n} = \mathbf{0}$ on $\partial\Omega$, and

$$\mathbf{v}(\mathbf{x}, x_{n+1}) = \varphi(x_{n+1})e(x_{n+1})\mathbf{u}(\mathbf{x}).$$

Then

$$M[\varphi e\mathbf{u}] = \varphi(x_{n+1})e(x_{n+1})(\mathrm{Div}\,\mathbf{C}[\nabla u] - \mu\mathbf{u}) + \text{ lower order terms},$$

and (3.10) follows readily from (3.11) provided $\mu$ is sufficiently large and positive. A similar argument applies for all $\mu$ mentioned below (3.10).

If we consider the operator $A(\mathbf{u}) = \mathrm{Div}\,\mathbf{C}[\nabla\mathbf{u}]$ as acting in $W^{k-2,p}(\Omega)$ with domain $D(A) = \left\{\mathbf{u} \in W^{k,p}(\Omega) : \mathbf{C}[\nabla\mathbf{u}]\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\right\}$ then $A$ is closed (see (3.2)) and by (3.10) it has no eigenvalues $\mu$ in

$$S = \left\{\mu \in \mathbb{C} : |\arg\mu| < \frac{\pi}{2} + \epsilon,\ |\mu| > c_2\right\}.$$

In fact it can be shown that $A$ is Fredholm with index zero so $A - \mu I$ ($I$ = identity operator in $W^{k-2,p}(\Omega)$) is bijective if $\mu \in S$ which yields that $S$ is a subset of the resolvent set of $A$; the spectrum of $A$ lies in a set in $\mathbb{C}$ which includes the negative real axis. Thus $A$'s spectrum is bounded above in real part. See Agmon [3].

We next investigate the interplay between the CC and Agmon's condition and a quadratic form associated with the tensor $\mathbf{C}$ and boundary value problem (3.1). We define the bilinear form

$$Q(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla\bar{\mathbf{u}} \cdot \mathbf{C}[\nabla\mathbf{v}]\,\mathrm{d}\mathbf{x}$$

for $\mathbf{u}, \mathbf{v} \in W^{1,2}(\Omega)$. Here we use the Euclidean scalar product over $\mathbb{R}^{n\times n}$ so that

$$\nabla\bar{\mathbf{u}} \cdot \mathsf{C}[\nabla\mathbf{u}] = \sum_{i,j,k,l=1}^{n} \nabla\bar{\mathbf{u}}(\mathbf{x})_{ij}\mathsf{C}_{ijkl}(\mathbf{x})\nabla\mathbf{v}(\mathbf{x})_{kl}, \qquad \mathbf{x}\in\Omega.$$

The associated quadratic form is $Q(\mathbf{u}) = Q(\mathbf{u},\mathbf{u})$, $\mathbf{u} \in W^{1,2}(\Omega)$. Note that the symmetry of $\mathsf{C}$ (see (H3)) implies that $Q(\mathbf{u},\mathbf{v}) = \overline{Q(\mathbf{v},\mathbf{u})}$. An example of the use of $Q$ is in Garding's inequality (3.7) which holds on $W_0^{1,2}(\Omega)$ (zero trace on $\partial\Omega$).

We say $Q$ is coercive over $W^{1,2}(\Omega)$ iff there exist $c_2 > 0$, $c_1 \in \mathbb{R}$ such that (3.7) holds for all $\mathbf{u} \in W^{1,2}(\Omega)$. In the case $\partial\Omega$ is the union of disjoint parts $\partial\Omega_1 \cup \partial\Omega_2$, and Dirichlet data is specified over $\partial\Omega_1$, we say $Q$ is strictly positive over $W^{1,2}(\Omega)$ iff $Q(\mathbf{u}) > 0$ for all $\mathbf{u} \in W^{1,2}(\Omega) \setminus \{\mathbf{0}\}$ such that $\mathbf{u} = \mathbf{0}$ on $\partial\Omega_1$. We shall also consider weakened versions of (H1), (H2):

(H1)' $\Omega \subset \mathbb{R}^n$ is open, bounded and $\partial\Omega$ is locally $C^1$-smooth.

(H2)' $\mathsf{C}(\cdot)_{ijkl}$ are continuous and $\mathsf{C}(\cdot)_{ijkl} = \mathsf{C}(\cdot)_{klij}$ (symmetry) on $\bar{\Omega}$ for all $i$, $j$, $k$, $l$.

We have some results:

**Theorem 3.3.** *Suppose (H1)', (H2)' hold. Suppose $Q$ is strictly positive over $W^{1,2}(\Omega)$. Then $Q$ is coercive with $c_1 = 0$ iff*

*i) $\mathsf{C}(\mathbf{x})$ is strongly elliptic for each $\mathbf{x} \in \bar{\Omega}$,*

*ii) CC holds on $\partial\Omega_2$.*

**Theorem 3.4.** *Suppose (H1)', (H2)' hold. $Q$ is coercive over $W^{1,2}(\Omega)$ iff*

*i) $\mathsf{C}(\mathbf{x})$ is strongly elliptic for each $\mathbf{x} \in \bar{\Omega}$,*

*ii) CC and Agmon's conditions hold on $\partial\Omega$.*

We say $Q$ is (sequentially) weakly lower semicontinuous on $W^{1,2}(\Omega)$ iff $\liminf_{n\to\infty} Q(\mathbf{u}_n) \geq Q(\mathbf{u})$ whenever $(\mathbf{u}_n)$ is a sequence in $W^{1,2}(\Omega)$ that converges weakly in $W^{1,2}(\Omega)$ to $\mathbf{u} \in W^{1,2}(\Omega)$, $\mathbf{u}_n \rightharpoonup \mathbf{u}$.

**Theorem 3.5.** *Suppose (H1)', (H2)' hold, $\mathsf{C}(\mathbf{x})$ is strongly elliptic for all $\mathbf{x} \in \bar{\Omega}$ and CC holds on $\partial\Omega$. Then the following are equivalent:*

*i) $Q$ is weakly lower semicontinuous on $W^{1,2}(\Omega)$,*

*ii) $Q$ is coercive over $W^{1,2}(\Omega)$,*

*iii) Agmon's condition holds on $\partial\Omega$.*

Next, assuming (H1), (H2) we define the operator

$$A(\mathbf{u}) = \operatorname{Div} \mathbf{C}[\nabla \mathbf{u}], \quad \mathbf{u} \in D(A),$$

where

$$D(A) = \left\{ \mathbf{u} \in W^{2,2}(\Omega) \mid \mathbf{C}[\nabla \mathbf{u}]\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega \right\} \subset L^2(\Omega),$$

is the domain of $A$. Then $A$ is a densely defined, closed operator in $L^2(\Omega)$. It can be shown

**Theorem 3.6.** *Suppose (H1), (H2), (H3) hold, $\mathbf{C}(\mathbf{x})$ is strongly elliptic for all $\mathbf{x} \in \bar{\Omega}$, and CC holds on $\partial\Omega$. Then*

*i) $A$ is self-adjoint,*

*ii) $A$ is Fredholm with index zero,*

*iii) the spectrum of $A$ consists of a countable number of real eigenvalues with no finite accumulation point,*

*iv) the eigenspace corresponding to each eigenvalue is finite dimensional,*

*v) $A$ has a sequence of $L^2(\Omega)$-orthogonal eigenfunctions the closure of whose span equals $L^2(\Omega)$.*

We say the spectrum of $A$ is bounded above iff there exists $c_2 \in \mathbb{R}$ such that $\lambda < c_2$ for all $\lambda$ in the spectrum of $A$. For example, the spectrum of the Laplacian with Neumann boundary operator is bounded above by any positive $c_2$.

**Theorem 3.7.** *Suppose the hypotheses of Theorem 3.6 hold. The following are equivalent:*

*i) the spectrum of $A$ is bounded above,*

*ii) Agmon's condition holds on $\partial\Omega$,*

*iii) $Q$ is coercive over $W^{1,2}(\Omega)$.*

The implication (iii) $\Rightarrow$ (i) is immediate and (ii) $\Rightarrow$ (i) follows directly from (3.10). Also (i) $\Rightarrow$ (iii) follows from a variant of Theorem 3.3 by replacing $A$ with $A - \mu I$ for $\mu$ sufficiently large ($I$ = identity in $L^2(\Omega)$) so $Q$ becomes strictly positive over $W^{1,2}(\Omega)$. Finally (iii) $\Rightarrow$ (ii) follows by a contradiction argument. In fact, just as in the proof that (3.2) implies CC we employ a family of functions localized near a straightened portion of $\partial\Omega$ with arbitrarily high oscillatory frequency tangential to $\partial\Omega$ and exponentially decaying into $\Omega$ normal to $\partial\Omega$. The equivalence (i) $\Leftrightarrow$ (ii) is due to Agmon [3]. Similar arguments apply in portions of Theorems 3.3, 3.4, 3.5. For proofs of Theorems 3.3–3.7, see Simpson, Spector [40]. Thus the failure of certain coercivity and spectral boundedness properties for $Q$ and $A$ respectively is closely related to the failure of either the CC or Agmon's condition in the vicinity of a point of $\partial\Omega$. We note that the eigenfunctions $\mathbf{u}_n$ of $A$ become more oscillatory as $n \to \infty$ (eigenvalues $\lambda_n \to \pm\infty$).

Again we note a relation to the dynamic problem (3.9). If the spectrum of $A$ is *not* bounded above then there exist solutions $\mathbf{u}(\mathbf{x},t) = \mathbf{u}_n(\mathbf{x})e^{\sqrt{\lambda_n}t}$ where $\lambda_n$, $\mathbf{u}_n$ are eigenvalues and eigenfunctions of $A$ with $\lambda_n \to +\infty$. Thus there is dynamic instability with arbitrarily high exponential growth in time representing a most extreme form of instability.

We note some related results in the case of the Lame tensor associated with linear elasticity. We consider the special case $\mathbf{C}[\mathbf{H}] = \mu(\mathbf{H}+\mathbf{H}^t) + \lambda(\operatorname{trace}\mathbf{H})\mathbf{I}$, $\mathbf{H} \in \mathbb{R}^{n\times n}$, for some *constants* $\mu, \lambda \in \mathbb{R}$ (Lame moduli), see Gurtin ([20], [19]). We have

i) $C$ is strongly elliptic iff $\mu > 0$ and $\lambda + 2\mu > 0$,

ii) CC holds iff $\mu + \lambda \neq 0$,

iii) Agmon's condition holds iff $\mu + \lambda \geq 0$,

assuming strong ellipticity in (ii) and (iii). Part (i) is well-known (see Gurtin [20]) and (ii), (iii) are proved by direct means from the definitions in Simpson, Spector [40]. Also, when $n = 2$, $\mu > 0$ and $\mu + \lambda > 0$ iff $\mathbf{C}$ is positive definite (i.e. $\mathbf{H}\cdot\mathbf{C}[\mathbf{H}] > 0$ for all symmetric $\mathbf{H} \in \mathbb{R}^{n\times n}$, $\mathbf{H} \neq \mathbf{0}$). Thus by Theorem 3.7, for general $n$, $Q$ is coercive over $W^{1,2}(\Omega)$ iff $\mu > 0$, $\mu + \lambda > 0$. Finally, we note that when $n = 2$ and $\mu + \lambda = 0$ (CC fails) then (3.1) with $\mathbf{b} = \mathbf{0}$, $\mathbf{t} = \mathbf{0}$ has infinitely many linearly independent solutions of the form $\mathbf{u} = (u_1, u_2)$ where $u_1(x_1,x_2) + iu_2(x_1,x_2)$ is an analytic function of $z = x_1 + ix_2$ on $\Omega$, see Simpson, Spector [39]. Thus $\ker L$ has infinite dimension.

We now relate Agmon's condition to the notion of quasiconvexity at the boundary. We consider a hyperelastic material with (finite-valued) stored energy function $W$ so that $E(\mathbf{f}) = \int_\Omega W(\nabla\mathbf{f}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$ equals the total stored energy in a material when the body in $\Omega$ is deformed by $\mathbf{f} : \Omega \to \mathbb{R}^n$. Then we say $\mathbf{f}_0$ is a strong local minimizer of $E$ iff there exists $\epsilon > 0$ such that $E(\mathbf{f}_0) \leq E(\mathbf{f})$ for all deformations $\mathbf{f}$ such that $\sup_\Omega |\mathbf{f} - \mathbf{f}_0| < \epsilon$. The following is in Ball, Marsden [9]. Here $B$ denotes any ball in $\mathbb{R}^n$.

**Theorem.** *Let $\mathbf{f}_0$ be a strong local minimizer of $E$. Then for every $\mathbf{x}_0 \in \partial\Omega$*

$$\int_{HB} W(\mathbf{F}_0)\,\mathrm{d}\mathbf{x} \leq \int_{HB} W(\mathbf{F}_0 + \nabla\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}, \tag{3.12}$$

*for any $\mathbf{u} \in C^1(\overline{HB})$ such that $\mathbf{u} = \mathbf{0}$ in a neighborhood of the curved part of $\partial(HB)$. Here $\mathbf{F}_0 = \nabla\mathbf{f}_0(\mathbf{x}_0)$ and $HB$ is the half-ball $\{\mathbf{x} \in B \,|\, (\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{n}(\mathbf{x}_0) < 0\}$ ($\mathbf{n}(\mathbf{x}_0)$ is the outward normal to $\partial\Omega$ at $\mathbf{x}_0$).*

The condition (3.12) is called quasiconvexity at the boundary and is a "half-ball version" of quasiconvexity (see Ball [8]). By linearizing (3.12) we see that a necessary condition for $\mathbf{f}_0$ to be a strong local minimizer of the energy $E$ is

$$\int_{HB} \nabla\mathbf{u}\cdot\mathbf{C}_0[\nabla\mathbf{u}]\,\mathrm{d}\mathbf{x} \geq 0 \tag{3.13}$$

for all $\mathbf{u} \in W^{1,2}(HB)$ such that $\mathbf{u} = \mathbf{0}$ on the curved part of $\partial(HB)$. Here $\mathsf{C}_0 = \mathsf{C}(\nabla \mathbf{f}(\mathbf{x}_0))$ and the 4-tensor $\mathsf{C}$ is the second derivative of $W$ (cf. (1.19)). We have

**Theorem 3.8.** *Necessary and sufficient conditions for* (3.13) *to hold are:*

*i)* $\mathsf{C}_0$ *satisfies the Legendre-Hadamard condition:*

$$\sum_{i,j,k,l=1}^{n} (\mathsf{C}_0)_{ijkl} \eta_i \xi_j \eta_k \xi_l \geq 0,$$

*for all* $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^n$*;*

*ii) Agmon's condition holds on the flat part of* $\partial(HB)$*,*

*iii) if* $(\mathbf{a} \otimes \mathbf{n}(\mathbf{x}_0)) \cdot \mathsf{C}_0[\mathbf{a} \otimes \mathbf{n}(\mathbf{x}_0)] = 0$ *for some* $\mathbf{a} \in \mathbb{R}^n$*, then* $\mathsf{C}_0[\mathbf{a} \otimes \mathbf{n}(\mathbf{x}_0)] = \mathbf{0}$.

For a proof see Simpson, Spector [41]. We note CC may fail if (3.13) holds.

Finally, we note that in some sense the failure of the CC is at the "limit" of Agmon's condition also failing.

**Theorem 3.9.** *Let* $\mathsf{C}_t$*,* $t \in [0,1]$*, be a one-parameter family of constant* 4*-tensors, continuous in* $t$*. Suppose, for each* $t$*,* $\mathsf{C}_t$ *is symmetric and strongly elliptic. Let* $H$ *be a fixed half-space with unit normal* $\mathbf{n}$*. Consider Agmon's condition with* $\mathsf{C}_t$ *replacing* $\mathsf{C}_0$ *in* (3.8)*,* $\mathbf{n} = \mathbf{n}(\mathbf{x}_0)$*. If Agmon's condition holds for* $t = 0$ *but fails at* $t = 1$ *then there exists* $\gamma \in [0,1]$ *such that CC fails for* $\mathsf{C}_\gamma$*.*

See Healey, Simpson [26].

# Chapter 4

# Brower and Leray-Schauder Degree

Here we sketch some of the basic ideas and properties of the degree of certain classes of mappings. Again, no attempt is made of a complete treatment. We refer the interested reader to, e. g., [BB], [D], [S], and [Z] for comprehensive coverage of this topic.

Consider a mapping $F : \mathcal{O} \subset \mathrm{X} \to \mathrm{Y}$, where X, Y are real Banach spaces, and the equation

$$F(x) = y. \tag{4.1}$$

In the absence of directly solving (4.1), which is generally too difficult, an important task is to compute the number of solutions of (4.1), denoted here by $N(F, \mathcal{O}, y)$. Unfortunately, $N(F, \mathcal{O}, y)$ is generally not continuous in $F$ or $y$. For example, consider $F(x) = x^2$ in $\mathbb{R}$. Clearly, for $x^2 = y$ $N(F, \mathcal{O}, y) = 2, 1, 0$ for $y > 0, y = 0, y < 0$, respectively.

The difference between the root count $N(F, \mathcal{O}, y)$ and the "degree", denoted $d(F, \mathcal{O}, y)$, is that the latter has the above-mentioned continuity properties, which are desirable for at least two good reasons. First, without continuity, a small error in $F$ or $y$ could lead to a large error in the degree. Second, as will become clear later, it may happen that $d(F_o, \mathcal{O}, y)$ is easy to compute for some specific mapping $F_o$, and then, by continuity, we get $d(F, \mathcal{O}, y)$ for some "nearby" mapping $F$.

## 4.1 Brouwer Degree

Consider (4.1) with $F : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{O}$ open and bounded, and $\mathbf{y} \in \mathbb{R}^n$. We make the following assumptions:

i) $\mathbf{y} \notin F(\partial\mathcal{O}) \equiv \{\mathbf{y} = F(\mathbf{x}) : \mathbf{x} \in \partial\mathcal{O}\}$.

ii) $F \in C^1(\overline{\mathcal{O}}, \mathbb{R}^n)$.

iii) $\mathbf{y}$ is a *regular value* of $F$ on $\mathcal{O}$, i. .e, for any solution $\mathbf{x}_o \in \mathcal{O}$ of (4.1), we have

$$\det(\mathrm{D}F(\mathbf{x}_o)) \neq 0, \tag{4.2}$$

where $F(\mathbf{x}_o+\boldsymbol{\eta}) = F(\mathbf{x}_o)+\mathrm{D}F(\mathbf{x}_o)\boldsymbol{\eta}+\mathrm{o}(|\eta|)$ as $|\boldsymbol{\eta}| \to 0$, i. e., the Jacobian matrix of $F$ at $\mathbf{x}_o$ is non-singular.

In view of (4.2), the inverse function theorem implies that every solution of (4.1) in $\mathcal{O}$ is *isolated*, i. e., it is the only solution of (4.1) when $F$ is restricted to some small neighborhood of that solution point. Since $\overline{\mathcal{O}}$ is compact (being closed and bounded in $\mathbb{R}^n$), we conclude that

$$F^{-1}(\mathbf{y}) \cap \mathcal{O} \text{ is a finite set,} \tag{4.3}$$

where $F^{-1}(\mathbf{y}) \equiv \{\mathbf{x} \in \mathrm{X} : F(\mathbf{x}) = \mathbf{y}\}$. (Otherwise, for an infinite set of distinct solutions, we would have a subsequence of solutions converging to a nonisolated solution of (4.1) in $\mathcal{O}$.)

Given the above assumptions (i)-(iii), we define the *index*, or *local degree*, of $F$ *at a solution* $\mathbf{x}_o \in \mathcal{O}$ of (4.1) via

$$i(F, \mathbf{x}_o, \mathbf{y}) \equiv \mathrm{sign}(\det \mathrm{D}F(\mathbf{x}_o)), \tag{4.4}$$

where $\mathrm{sign}(z) = 1$, if $z > 0$, and $\mathrm{sign}(z) = -1$, if $z < 0$.

The *Brouwer degree* is then defined by

$$\deg(F, \mathcal{O}, \mathbf{y}) = \sum_{\mathbf{x} \in F^{-1}(\mathbf{y}) \cap \mathrm{X}} i(F, \mathbf{x}, \mathbf{y}), \tag{4.5}$$

with the understanding that $\deg(F, \mathcal{O}, \mathbf{y}) \equiv 0$, if $F^{-1}(\mathbf{y}) \cap \mathrm{X} = \emptyset$.

Assumption (i) is needed to insure continuity of $\mathbf{y} \mapsto \deg(F, \mathcal{O}, \mathbf{y})$. As a simple illustration, consider $F(x) = 2x$ on $\mathcal{O} \equiv (-1, 1) \subset \mathbb{R}$. Of course for mappings in $\mathbb{R}$, the index reduces to the sign of the slope. Hence, we have $\deg(F, \mathcal{O}, y) = 1, 0$, for $|y| < 2$, $|y| > 2$, respectively. Observe that $F(\partial\mathcal{O}) = \{\pm 2\}$.

Assumptions (ii) and (iii) can be relaxed. We begin with (iii), i. e., suppose that $\mathbf{y}$ is not a regular value of $F$. According to *Sard's Theorem*, if we let $\mathcal{C} \equiv \{\mathbf{x} \in \mathcal{O} : \det \mathrm{D}F(\mathbf{x}) = 0\}$, the $F(\mathcal{C})$ has $\mathbb{R}^n$–measure zero. Thus, for any $\varepsilon > 0$ we can find a regular value $\mathbf{y}_*$ of $F$ on $\mathcal{O}$ such that $|\mathbf{y} - \mathbf{y}_*| < \varepsilon$. In particular, we must choose $|\mathbf{y} - \mathbf{y}_*| < \inf_{\mathbf{x} \in \partial\mathcal{O}} |F(\mathbf{x}) - \mathbf{y}|$ to insure that $\mathbf{y}_*$ satisfies assumption (i). We then define

$$\deg(F, \mathcal{O}, \mathbf{y}) \equiv \deg(F, \mathcal{O}, \mathbf{y}_*). \tag{4.6}$$

Of course, for (4.6) to be reliable, one must show that the degree is independent of the choice of $\mathbf{y}_*$, i. e., $\deg(F, \mathcal{O}, \mathbf{y}_*) = \deg(F, \mathcal{O}, \hat{\mathbf{y}})$ for all regular values $\mathbf{y}_*, \hat{\mathbf{y}}$ sufficiently close to $\mathbf{y}$ and belonging to the same (connected) component of $\mathbb{R}^n \backslash F(\partial\mathcal{O})$. The latter is readily established via methods of vector integral calculus, cf. , [S].

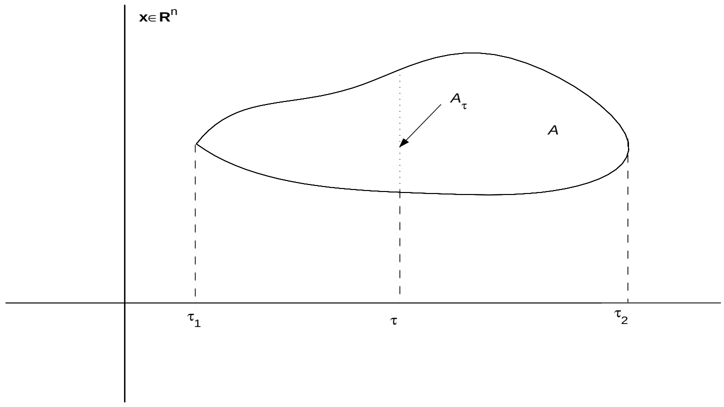


Figure 4.1: Domain of definition of the function $H$ of the homotopy invariance property.

To relax assumption (ii), suppose that $F \in C(\mathcal{O}, \mathbb{R}^n)$. By the Stone–Weierstrass Theorem, there is a mapping $\hat{F} \in C^1(\mathcal{O}, \mathbb{R}^n)$ such that for any $\varepsilon > 0$, we have

$$\max_{\mathbf{x}\in\overline{\mathcal{O}}} |\hat{F}(\mathbf{x}) - F(\mathbf{x})| < \varepsilon. \tag{4.7}$$

Accordingly, for $\varepsilon \leq \inf_{\mathbf{x}\in\partial\mathcal{O}} |F(\mathbf{x}) - \mathbf{y}|$, we define

$$\deg(F, \mathcal{O}, \mathbf{y}) = \deg(\hat{F}, \mathcal{O}, \mathbf{y}). \tag{4.8}$$

Again, one must show that (4.8) is independent of the specific choice of $\hat{F}$. If $\tilde{F}$ is another $C^1$ mapping satisfying (4.7), the fact that $\deg(\tilde{F}, \mathcal{O}, \mathbf{y}) = \deg(\hat{F}, \mathcal{O}, \mathbf{y})$ follows as a special case of *homotopy invariance* (stated below (d2)–set $H(\tau, \mathbf{x}) \equiv (1-\tau)\tilde{F}(\mathbf{x}) + \tau\hat{F}(\mathbf{x})$, $0 \leq \tau \leq 1$).

From our point of view, the two most important properties of the degree are:

**(d1)** (Existence) If $\deg(F, \mathcal{O}, \mathbf{y}) \neq 0$, then (4.1) has at least one solution.

**(d2)** (Homotopy Invariance) $\deg(F, \mathcal{O}, \mathbf{y})$ is continuous (hence, constant in the respective connected components) with respect to its arguments.

A more precise and useful version of (d2) is the following: Let $\mathcal{A} \subset [\tau_1, \tau_2] \times \mathbb{R}^n \subseteq \mathbb{R}^{n+1}$ be open and bounded, and define (see Figure (4.1))

$$\mathcal{A}_\tau \equiv \{\mathbf{x} \in \mathbb{R}^n : (\tau, \mathbf{x}) \in \mathcal{A}\}. \tag{4.9}$$

Let $H : \mathcal{A} :\to \mathbb{R}^n$ be a continuous function and $\mathbf{y} : [\tau_1, \tau_2] \to \mathbb{R}^n$ a continuous curve such that $\mathbf{y}(\tau) \notin H(\tau, \partial\mathcal{A}_\tau)$ $\forall\tau \in [\tau_1, \tau_2]$. Then,

$$\deg(H(\tau, \cdot), \mathcal{A}_\tau, \mathbf{y}(\tau)) = \text{const.}\ \forall\,\tau \in [\tau_1, \tau_2]. \tag{4.10}$$

In particular, if $\mathcal{A}_{\tau_o} = \emptyset$ for some $\tau_o \in [\tau_1, \tau_2]$, then $\deg(H(\tau_o, \cdot), \mathcal{A}_{\tau_o}, \mathbf{y}(\tau_o)) = 0$.

**Remark 4.1.** *The "text–book" version of homotopy invariance is typically stated for "cylindrical" domains* $\Omega = [a,b] \times \mathcal{O}$ *with* $\mathbf{y} \in \mathbb{R}^n$ *such that* $\mathbf{y} \notin H(\tau, \partial\mathcal{O})$ $\forall\, \tau \in [a,b]$.

Other properties of the Brouwer degree are the following:

**(d3)** (Normalization) Let I denote the identity map on $\mathbb{R}^n$. Then

$$\deg(\mathrm{I}, \mathcal{O}, \mathbf{y}) = \begin{cases} 1, & \text{if } \mathbf{y} \in \mathcal{O}, \\ 0, & \text{if } \mathbf{y} \notin \overline{\mathcal{O}}. \end{cases}$$

**(d4)** (Additivity) Let $\mathcal{O}_1$, $\mathcal{O}_2 \subset \mathbb{R}^n$ be open, bounded sets with $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$, $F \in C^0(\overline{\mathcal{O}}_1 \cup \overline{\mathcal{O}}_2, \mathbb{R}^n)$ and $\mathbf{y} \notin F(\partial\mathcal{O}_1 \cup \partial\mathcal{O}_2)$. Then, $\deg(F, \mathcal{O}_1 \cup \mathcal{O}_2, \mathbf{y}) = \deg(F, \mathcal{O}_1, \mathbf{y}) + \deg(F, \mathcal{O}_2, \mathbf{y})$.

**(d5)** (Excision) If $\{\mathbf{x} \in \overline{\mathcal{O}}_1 : F(\mathbf{x}) = \mathbf{y}\} \subset \mathcal{O}_2 \subset \mathcal{O}_1$, then $\deg(F, \mathcal{O}_1, \mathbf{y}) = \deg(F, \mathcal{O}_2, \mathbf{y})$.

## 4.2 Leray-Schauder Degree

Next we consider (4.1) in the case where X is an infinite–dimensional Banach space and X = Y. At this level of generality, it is well known that there is no degree having all the properties of the Brouwer degree in $\mathbb{R}^n$. For example, the group of invertible linear operators, $GL(\mathrm{X})$, is generally connected, cf. , [K] (unlike $GL(\mathbb{R}^n)$ which has two connected components – $GL^+(\mathbb{R}^n)$ and $GL(\mathbb{R}^n)\backslash GL^+(\mathbb{R}^n)$ – this is at the heart of Definitions (4.4) and (4.5)). Also, there exist continuous mappings of the closed unit ball in an infinite–dimensional Banach space into itself having no fixed points, cf. , [B].

Leray and Schauder discovered a generalization of the Brouwer degree to the important class of mappings of the form $F(x) = x + f(x)$, where $f : \overline{\mathcal{O}} \subset \mathrm{X} \to \mathrm{Y}$ is *compact*, i. e., for any bounded sequence $\{x_n\} \subset \mathrm{X}$, the image sequence $\{f(x_n)\}$ has a convergent subsequence. Mappings of this type, viz. , $\mathrm{I} + f$, "identity plus compact", arise naturally in applications when boundary value problems are converted into equivalent integral equations (as we will see in Chapters 5 and 6).

Consider (4.1) with $y = 0$ (temporarily), $F(x) = x + f(x)$, where $f : \overline{\mathcal{O}} \subset \mathrm{X} \to \mathrm{Y}$ is continuos and compact (i. e., *completely continuous*), $\mathcal{O}$ is open and bounded, and $0 \notin F(\partial\mathcal{O})$. The key observation here is that $f|_{\overline{\mathcal{O}}}$ can be uniformly approximated by mappings with finite–dimensional range. That is, for any $\varepsilon > 0$ there is a mapping $f_\varepsilon : \overline{\mathcal{O}} \to \mathrm{X}_{n_\varepsilon}$, with $\dim \mathrm{X}_{n_\varepsilon} = n_\varepsilon < \infty$, such that

$$\max_{x \in \overline{\mathcal{O}}} \|f(x) - f_\varepsilon(x)\|_{\mathrm{X}} < \varepsilon. \tag{4.11}$$

**Example 4.2.** Let $k : [0,1] \times [0,1] \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Let X be the space $C([0,1])$ with the "sup" norm $\|x\| = \max_{s\in[0,1]} |x(s)|$. Consider the integral operator $f : \mathrm{X} \to \mathrm{X}$ defined by

$$f(x) \equiv \int_0^1 k(s,t)g(x(t))\mathrm{d}t.$$

A routine application of the Arzela–Ascoli's Theorem reveals that $f$ is compact, cf. [LS, p.229]. To verify (4.11), observe that by Weierstrass' Theorem, there is a sequence of polynomials $\{p_n(s,t)\}$ converging uniformly to $k(s,t)$ on $[0,1]^2$, i. e., given any $\hat{\varepsilon} > 0$ there is a natural number $N$ such

$$\max_{(s,t)\in[0,1]^2} |k(s,t) - p_n(s,t)| < \hat{\varepsilon},$$

for all $n \geq N$. Define

$$f_{\hat{\varepsilon}}(x) \equiv \int_0^1 p_n(s,t)g(x(t))\mathrm{d}t. \tag{4.12}$$

Given $x \in \mathrm{X}$, observe that the right–hand side of (4.12) is a polynomial in "$s$", i. e., the range of $f_{\hat{\varepsilon}}$ is finite–dimensional. We then have

$$\|f(x) - f_{\hat{\varepsilon}}(x)\|_{\mathrm{X}} = \max_{s\in[0,1]} \left| \int_0^1 [k(s,t) - p_n(s,t)]g(x(t))\mathrm{d}t \right| \leq \hat{\varepsilon}M(\|x\|),$$

where $M(\|x\|) = \max_{s\in[-\|x\|,\|x\|]} |g(s)|$. For $\overline{\mathcal{O}} \subset \mathrm{X}$ bounded, let $\alpha = \max\{\|x\| : x \in \overline{\mathcal{O}}\}$ and define $M = M(\alpha)$. Then (4.11) follows with $\varepsilon = \hat{\varepsilon}M$.

Returning to (4.11) in the general setting, set $\mathcal{O}_\varepsilon \equiv \mathcal{O} \cap \mathrm{X}_{n_\varepsilon}$, and define $F_\varepsilon = \mathrm{I} + f_\varepsilon$. Since $0 \notin F(\partial\mathcal{O})$, we also have $0 \notin F_\varepsilon(\partial\mathcal{O}_\varepsilon)$ for $\varepsilon > 0$ sufficiently small. Hence, the Brouwer degree, $\deg(F_\varepsilon, \mathcal{O}_\varepsilon, 0)$, is well defined. Moreover, we define the *Leray–Schauder* degree via

$$\begin{aligned} \deg(F,\mathcal{O},0) &\equiv \deg(F_\varepsilon,\mathcal{O}_\varepsilon,0), \text{ and} \\ \deg(F,\mathcal{O},y) &\equiv \deg(F-y,\mathcal{O},0), \end{aligned} \tag{4.13}$$

for $\varepsilon > 0$ sufficiently small. Homotopy invariance, cf. (d2), of the Brouwer degree insures that (4.13) is independent of the specific choice $f_\varepsilon : \overline{\mathcal{O}} \to \mathrm{X}_{n_\varepsilon}$. By construction, the Leray–Schauder degree has all of the properties, (d1)–(d5), of the Brouwer degree (with X in place of $\mathbb{R}^n$).

Finally, we need a formula analogous to (4.4) for computing the index or *local degree* in the regular value case, viz. , suppose that $x_o \in \mathcal{O}$ is a solution of (4.1), that $f$ is differentiable at $x_o$, and that $\mathrm{D}F(x_o) = \mathrm{I} + \mathrm{D}f(x_o) \in L(\mathrm{X})$ is bijective. Thus for some $\alpha > 0$, we have that $\|\mathrm{D}F(x_o)\eta\| \geq \alpha\|\eta\|$, $\forall\, \eta \in \mathrm{X}$, which insures that $x_o$ is the only solution of (4.1) in a sufficiently small ball centered at $x_o$, denoted $B_\delta(x_o) \equiv \{x : \|x - x_o\| < \delta\}$. Then for the mapping $H(\tau,x) \equiv (1-\tau)(F(x)-y)+\tau\mathrm{D}F(x_o)(x-x_o)$, it follows that $0 \notin H(\tau,\partial B_\delta(x_o))$,

$0 \leq \tau \leq 1$. Observe that $\mathrm{D}f(x_o) \in L(\mathrm{X})$ inherits complete continuity from that of $f(x)$. Homotopy invariance (d2) then yields

$$i(F, x_o, y) \equiv \deg(F - y, B_\delta(x_o), 0) = \deg(\mathrm{I} + \mathrm{D}f(x_o), B_\delta(x_o), 0). \tag{4.14}$$

We use (4.13), and (4.4) to compute the degree in the last term of (4.14), i. e.,

$$\begin{aligned} \deg(\mathrm{I} + \mathrm{D}f(x_o), B_\delta(x_o), 0) &= \deg(\mathrm{I} + \mathrm{D}_{n_\varepsilon}, B_\delta(x_o)_{n_\varepsilon}, 0) \\ &= \operatorname{sign}(\det((\mathrm{I} + \mathrm{D}_{n_\varepsilon})|_{\mathrm{X}_{n_\varepsilon}})) \\ &= (-1)^m, \end{aligned} \tag{4.15}$$

where $\mathrm{D}_{n_\varepsilon} : \overline{B_\delta(x_o)} \to \mathrm{X}_{n_\varepsilon}$ is finite–range map approximating $\mathrm{D}f(x_o)$ as in (4.11), $B_\delta(x_o)_{n_\varepsilon} = B_\delta(x_o) \cap \mathrm{X}_{n_\varepsilon}$, $\varepsilon > 0$ is sufficiently small, and the number "$m$" denotes the number of real, negative eigenvalues of $(\mathrm{I} + \mathrm{D}_{n_\varepsilon})|_{\mathrm{X}_{n_\varepsilon}}$, counted by algebraic multiplicity. Since (4.14) is independent of $\varepsilon > 0$ sufficiently small, we conclude that "$m$" in (4.15) is also the number of real, negative eigenvalues of $\mathrm{I} + \mathrm{D}f(x_o)$.

**Remark 4.3.** *Henceforth we consider operator equations of the form (4.1) with $y = 0$. Accordingly, we adopt the notation*

$$\deg(F, \mathcal{O}) \equiv \deg(F, \mathcal{O}, 0). \tag{4.16}$$

# Chapter 5

# Global Continuation and Bifurcation

In this chapter we demonstrate how to use the degree to study "global" solutions of operator equations of the form

$$G(\lambda, u) = 0, \tag{5.1}$$

where $G : \mathbb{R} \times \mathrm{X} \to \mathrm{X}$, and $\mathrm{X}$ is a real Banach space. We assume that $G$ is continuous, and if $\mathrm{X}$ is infinite dimensional, we assume that

$$G(\lambda, u) \equiv u - g(\lambda, u), \tag{5.2}$$

with $g$ completely continuous.

We spend more time by far on global continuation, which is in some sense a global version of the implicit function theorem, than on global bifurcation. Indeed we do not have adequate time to devote to local bifurcation bifurcation theory, which plays an important role in the global theory.

For the *continuation* problem, we assume that

$$G(0,0) = 0. \tag{5.3}$$

Suppose that

$$u \mapsto G(0,u) \ \text{ is differentiable at } u = 0, \tag{5.4}$$

and assume that the (Frechét) derivative

$$\mathrm{D}_u G(0,0) \equiv \mathrm{I} + K_o \in L(\mathrm{X}) \ \text{ is bijective.} \tag{5.5}$$

For motivation only, suppose that mapping $G$ is locally $C^1$ in some neighborhood of the known solution point $(0,0)$. Then, the implicit function theorem yields the existence of a unique local solution curve containing $(0,0)$:

$$\Sigma_{\text{loc}} = \{(\lambda, \hat{u}(\lambda)) \, : \, |\lambda| < \delta\}, \tag{5.6}$$

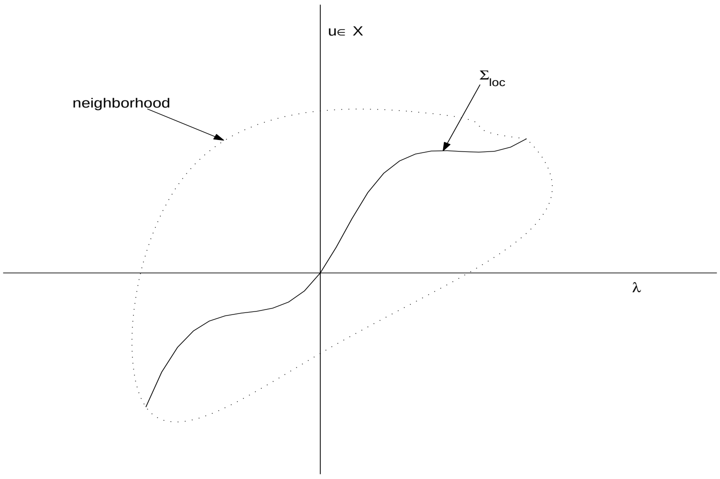


Figure 5.1: Local set $\Sigma_{\text{loc}}$ of solutions of (5.1) given by the curve $\hat{u}(\cdot)$ in a small neighborhood of $(0,0)$.

where $\hat{u}(0) = 0$, and where all solutions of (5.1) in some sufficiently small neighborhood of $(0,0)$ are in $\Sigma_{\text{loc}}$. (See Figure (5.1).)

Although the existence of the *curve* $\Sigma_{\text{loc}}$ is not necessary, in some sense, a "global" result is one characterizing solutions connected to $(0,0)$ "beyond" the small nieghborhood containing $\Sigma_{\text{loc}}$.

Before proceeding, we make a crucial observation for the case when X is infinite dimensional (cf. (5.2)):

**Lemma 5.1.** *Any bounded set of solutions of (5.1) is compact.*

*Proof.* If X is finite dimensional, the result is obvious. In the infinite–dimensional case, let $\{(\lambda_n, u_n)\}$ be a uniformly bounded sequence of solutions of (5.1), i.e., from (5.2)

$$\begin{aligned} u_n &= -g(\lambda_n, u_n) \\ |\lambda_n| + \|u_n\| &\leq M. \end{aligned} \tag{5.7}$$

By the compactness of $g(\cdot)$, $\{g(\lambda_n, u_n)\}$ has a convergent subsequence (not relabelled), $g(\lambda_n, u_n) \to g_*$, and clearly $\{\lambda_n\}$ converges as a subsequence, $\lambda_n \to \lambda_*$. Then, by (5.7) $u_n \to u_* \equiv -g_*$ (as a subsequence). From continuity, $(\lambda_*, u_*)$ is a solution of (5.1). □

**Theorem 5.2 (Leray–Schauder (LR), Rabinowitz (R)).** *Given the hypotheses (5.2)–(5.5), let* $\Sigma \subset \mathbb{R} \times \text{X}$ *denote the maximal connected set of solution pairs* $(\lambda, u)$ *of (5.1) containing the known solution point* $(0,0)$*. Then* $\Sigma$

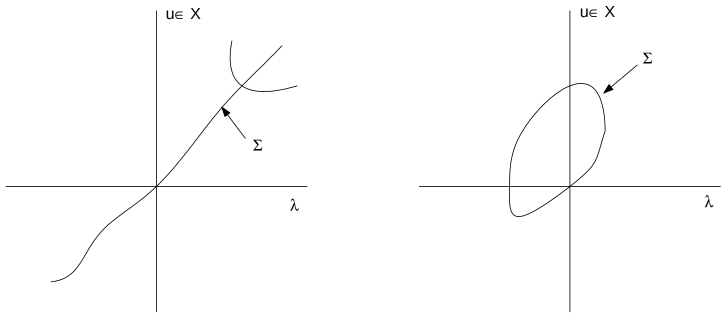


Figure 5.2: A schematic sketch of the two alternatives in Theorem (5.2): $\Sigma$ is unbounded in $\mathbb{R} \times \mathrm{X}$ (left) or $\Sigma \backslash \{(0,0)\}$ is connected (right).

*is characterized by at least one of the following:*

$$\begin{aligned} &\textit{(i)} \quad \Sigma \ \textit{ is unbounded in } \mathbb{R} \times \mathrm{X}, \\ &\textit{(ii)} \quad \Sigma \backslash \{(0,0)\} \ \textit{ is connected.} \end{aligned} \tag{5.8}$$

*Proof.* Suppose that neither (5.8)(i) nor (5.8)(ii) hold. In particular, since $\Sigma$ is bounded, it is compact, cf. Lemma 5.1. Thus, by the separation theorem for compact sets (cf. [Ra], [Z]), there is a bounded open set $\mathcal{A} \subset \mathbb{R} \times \mathrm{X}$ such that $\Sigma \subset \mathcal{A}$ and $\partial\Sigma \cap \mathcal{A} = \emptyset$. Now (5.5) insures that $u = 0$ is an isolated solution of $G(0, u) = 0$ in some sufficiently small open ball of radius $\delta > 0$ centered at $u = 0$, denoted by $B_\delta(0) \subset \mathrm{X}$. Accordingly, we may choose $\mathcal{A}$ such that $\mathcal{A}_o = B_\delta(0)$, where $\mathcal{A}_o$ is as defined in (4.9) (with $\mathrm{X}$ in place of $\mathbb{R}^n$). In view of (4.14), (4.15) and (5.5), we have

$$\deg(\mathrm{I} + g(0, \cdot), \mathcal{A}_o) \neq 0.$$

On the other hand, for $|\lambda|$ sufficiently large, say $\lambda = \lambda_*$, we have

$$\deg(\mathrm{I} + g(\lambda_*, \cdot), \mathcal{A}_{\lambda_*}) = 0,$$

which contradicts homotopy invariance of the degree. □

The *global continuum* of solutions $\Sigma$, given by Theorem 5.2, is called a *global solution branch*. As we shall see in Chapter 6, it is sometimes possible to rule out the characterization (5.8)(ii) of $\Sigma$ by showing that $u = 0$ is the unique solution of $G(0, u) = 0$. This is, for example, if we assume that $g(0, u) \equiv 0$ $\forall\, u \in \mathrm{X}$. In any case, if $u = 0$ is the only solution of (5.1) for $\lambda = 0$, define

$$\begin{aligned} \Sigma^+ &= \Sigma \cap ([0, \infty) \times \mathrm{X}), \\ \Sigma^- &= \Sigma \cap ((-\infty, 0] \times \mathrm{X}). \end{aligned} \tag{5.9}$$

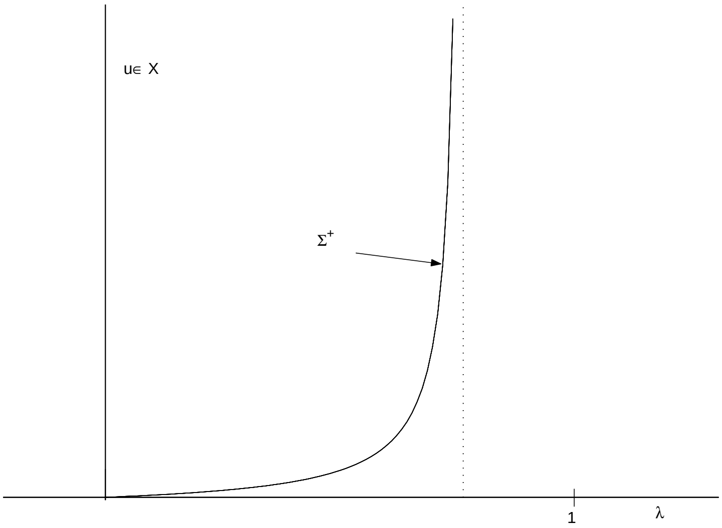


Figure 5.3: The solution branch $\Sigma^+$ blowing–up before reaching $\lambda = 1$.

We then have

$$\Sigma = \Sigma^+ \cup \Sigma^- \quad \text{with } \Sigma^+ \cap \Sigma^- = \{(0,0)\}. \tag{5.10}$$

The same argument employed in the proof of Theorem 5.2 yields:

**Corollary 5.3.** *If* $u = 0$ *is the unique solution of* $G(0,u) = 0$*, then* $\Sigma^+$ *and* $\Sigma^-$ *(cf. (5.9), (5.10)), are each unbounded in* $\mathbb{R} \times \mathrm{X}$.

Related to results like those of Theorem 5.2 and Corollary 5.3, is the Leray–Schauder continuation principle. The basic idea is to consider a one–parameter problem like (5.1), (5.2) for which (5.1) at $\lambda = 0$ is "easy" to solve, e.g., with $g(0,u) \equiv 0$, while the $\lambda = 1$ problem is difficult, but where one hopes to get existence via non–zero degree. Without extra information, this may not be possible – as in Theorem 5.2 or Corollary 5.3, the solution branch could "blow–up" without attaining $\lambda = 1$. (See Figure (5.3).) However, if the solutions of (5.1) for any $\lambda \in [0,1]$ are somehow known to be a–priori bounded, viz.,

$$\|u\| \leq M \;\; \forall (\lambda, u) \in [0,1] \times \mathrm{X} \text{ satisfying (5.1)}, \tag{5.11}$$

then we have:

**Theorem 5.4 (Leray–Schauder continuation principle).** *Assume the a–priori bound (5.11) holds, and suppose that*

$$deg\,(G(0,\cdot), B_{\widetilde{M}}(0)) \neq 0,$$

*where* $B_{\widetilde{M}}(0) \subset \mathrm{X}$ *is an open ball centered at* $u = 0$ *of radius* $\widetilde{M} > M$*. Then,* $G(1,u) = 0$ *has at least one solution.*

*Proof.* (5.11) insures that $0 \notin G(\lambda, \partial B_{\widetilde{M}}(0))\ \forall\ \lambda \in [0,1]$, and the result follows immediately from homotopy invariance. □

**Remark 5.5.** *In the seminal paper of 1934, Leray and Schauder did not discuss solution continua. Rather the topological arguments of Rabinowitz from 1971 (in the context of global bifurcation) show that there is a global solution branch connecting the solutions at $\lambda = 0$ to those at $\lambda = 1$. Also, in the absence of a–priori bounds, Theorem 5.2 or Corollary 5.3 are the "next best thing" to existence.*

**Example 5.6.** Consider the quasilinear elliptic boundary value problem

$$a_{ij}(\mathbf{x}, u, \nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(\mathbf{x}, u\nabla u)\frac{\partial u}{\partial x_i} + c(\mathbf{x}, u\nabla u)u = f(\lambda, \mathbf{x}, u, \nabla u), \text{ in } \Omega, \tag{5.12a}$$

$$u|_{\partial\Omega} = 0. \tag{5.12b}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded $C^3$ domain, $a_{ij}(\cdot)$, $b_i(\cdot)$ and $f(\cdot)$ are continuously differentiable,

$$c(\cdot, u, \mathbf{v}) \le 0 \text{ in } \Omega,\ \forall u \in \mathbb{R},\ \mathbf{v} \in \mathbb{R}^n, \tag{5.13}$$

and

$$f(0, \mathbf{x}, u, \mathbf{v}) \equiv 0. \tag{5.14}$$

In addition, we assume *uniform ellipticity*, i.e., there exist positive constants $c_1$, $c_2$ such that

$$c_1|\boldsymbol{\xi}|^2 \le a_{ij}(\mathbf{x}, u, \mathbf{v})\xi_i\xi_j \le c_2|\boldsymbol{\xi}|^2, \tag{5.15}$$

for all $\mathbf{x} \in \Omega$, $u \in \mathbb{R}$, and $\mathbf{v},\ \boldsymbol{\xi} \in \mathbb{R}^n$.

We first convert (5.12) into an equivalent operator equation of the form (5.1), (5.2). For $0 < \alpha < 1$, let $C^{k,\alpha}(\overline{\Omega})$ denote the class of $k$–times continuously differentiable functions on $\overline{\Omega}$ whose $k$th derivatives are Hölder continuous with exponent $\alpha$ with norm given by (1.4). Let

$$\mathrm{X} = \left\{u \in C^{1,\alpha}(\overline{\Omega}) : u|_{\partial\Omega} = 0\right\},$$

equipped with the $C^{1,\alpha}(\overline{\Omega})$ norm (1.4) with $k = 1$. For any $(\lambda, u) \in \mathbb{R} \times \mathrm{X}$, let $h = -g(\lambda, u)$ denote the solution in $\in C^{2,\alpha}(\overline{\Omega})$ of the *linear* elliptic problem

$$a_{ij}(\mathbf{x}, u, \nabla u)\frac{\partial^2 h}{\partial x_i \partial x_j} + b_i(\mathbf{x}, u, \nabla u)\frac{\partial h}{\partial x_i} + c(\mathbf{x}, u, \nabla u)h = f(\lambda, \mathbf{x}, u, \nabla u), \text{ in } \Omega,$$

$$h|_{\partial\Omega} = 0. \tag{5.16}$$

From (5.13) and the maximum principle [PW] we have that $h = -g(\lambda, u)$ is unique. Hence, any solution of (5.12) satisfies

$$u + g(\lambda, u) = 0, \tag{5.17}$$

and conversely, any solution of (5.17) also satisfies (5.12). We claim that $g : \mathbb{R} \times \mathrm{X} \to \mathrm{X}$ is compact. To see this, suppose that $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times \mathrm{X}$ is uniformly bounded. By the Schauder estimate for (5.16) (cf. (2.9)) we have

$$\|h\|_{2,\alpha} = \||g(\lambda_n, u_n)\|_{2,\alpha} \leq C\|f(\lambda_n, u_n, \nabla u_n)\|_{0,\alpha}\,, \tag{5.18}$$

where $C > 0$ depends upon $\Omega$, $c_1$ and $c_2$ (cf. (5.15), and the $C^{0,\alpha}(\overline{\Omega})$ bounds for the coefficient functions $a_{ij}(\mathbf{x}, u_n, \nabla u_n)$, $b_i(\mathbf{x}, u_n, \nabla u_n)$ and $c(\mathbf{x}, u_n, \nabla u_n)$. Thus, $\{g(\lambda_n, u_n)\}$ is uniformly bounded in $C^{2,\alpha}(\overline{\Omega})$ and, by compact embedding, has a convergent subsequence in X. By similar reasoning it follows that $g(\cdot)$ is continuous.

It remains to verify (5.4) and (5.5). Consider (5.16) at $\lambda = 0$. In view of (5.13), (5.14) and the maximun principle, we conclude that $h = -g(0, u) \equiv 0$ is the unique solution. Clearly (5.4) and (5.5) hold with $\mathrm{D}_u G(0, 0) = \mathrm{I}$ (cf. (5.5)). Finally, we observe that Corollary 5.3 (as Theorem 5.2) is applicable in this example.

**Remark 5.7.** *In many cases related to the above example for second–order quasilinear elliptic pde's, one can obtain a–priori bounds on solutions akin to (5.11) and thus employ Theorem 5.4, c.f. [LS], [LU], [GT].*

In contrast to (5.3), bifurcation problems are typically characterized by the existence of a *trivial solution branch*, i.e.,

$$G(\lambda, 0) \equiv 0 \;\forall\, \lambda \in \mathbb{R}. \tag{5.19}$$

We call $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$ the *trivial line* or the *trivial solution branch*. Of course, for such problems, Theorem 5.2 and Corollary 5.3 are useless, since the trivial line itself is unbounded. Rather it is of interest to characterize *nontrivial* solutions branches.

We say that $(\lambda_o, 0)$ is a *bifurcation point* of (5.1), (5.2) if every neighborhood of $(\lambda_o, 0)$ contains nontrivial solutions. Assume now that $G$ is locally $C^1$ in some (open) neighborhood of $(\lambda_o, 0)$. Then, by the implicit function theorem, we see that a necessary condition for bifurcation is that the derivative $\mathrm{L}(\lambda_o) \equiv \mathrm{D}_u G(\lambda_o, 0)$ is not bijective. By the Riesz–Schauder theory, this is equivalent to the condition that the linear map $\mathrm{L}(\lambda_o)$ has at least one *null solution*, i.e., there exist $\eta \in \mathrm{X}$ such that $\eta \neq 0$ and

$$\mathrm{L}(\lambda_o)\eta = 0. \tag{5.20}$$

**Theorem 5.8 (Rabinowitz).** *Given (5.19) and (5.20), suppose that* $D_u G(\lambda, 0) \equiv L(\lambda) = \mathrm{I} + K(\lambda)$ *is bijective on* $[\lambda_o - \varepsilon, \lambda_o) \cup (\lambda_o, \lambda_o + \varepsilon]$*, for some* $\varepsilon > 0$*. In addition, assume that*

$$deg(L(\lambda_1), B_\delta(0)) \neq deg(L(\lambda_2), B_\delta(0)), \tag{5.21}$$

*for some* $\lambda_1 \in [\lambda_o - \varepsilon, \lambda_o)$ *and* $\lambda_2 \in (\lambda_o, \lambda_o + \varepsilon]$*, where* $B_\delta(0) \subset \mathrm{X}$ *is the open ball centered at* $u = 0$ *of sufficiently small radius* $\delta > 0$*. Let* $S$ *denote the*

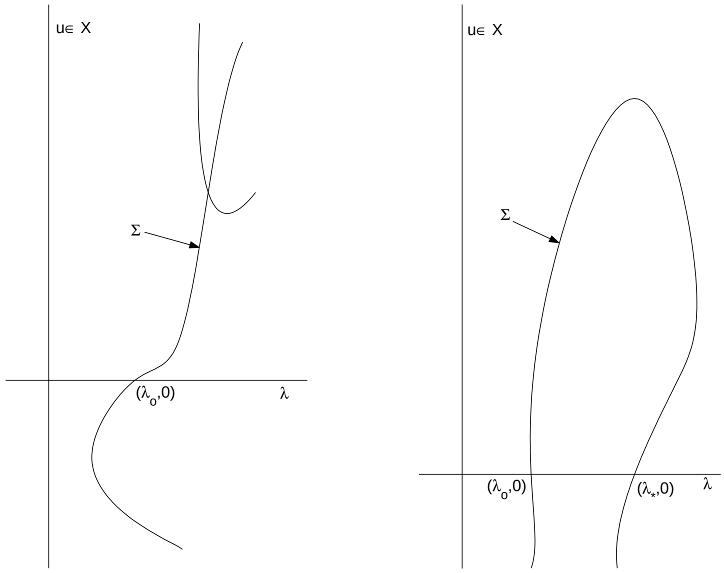


Figure 5.4: A schematic sketch of the two alternatives in Theorem (5.8): $\Sigma$ is unbounded in $\mathbb{R} \times \mathrm{X}$ (left) or $\Sigma$ intersects the trivial branch at $(\lambda_*, 0)$ with $\lambda_* \neq \lambda_o$ (right).

*closure of all nontrivial solutions of (5.1), (5.2), and let $\Sigma$ denote the connected component of $S$ containing $(\lambda_o, 0)$. Then, $\Sigma$ is characterized by at least one of the following:*

$$\begin{aligned} &\textit{(i)} \quad \Sigma \ \textit{ is unbounded in } \mathbb{R} \times \mathrm{X}, \\ &\textit{(ii)} \quad (\lambda_*, 0) \in \Sigma, \quad \textit{for some } \lambda_* \neq \lambda_o. \end{aligned} \tag{5.22}$$

**Remark 5.9.** *On one hand, by virtue of (4.14), we see that (5.21) insures that the local degree of $G(\lambda, \cdot)$ "jumps" along the trivial line. On the other hand, in view of (4.15), (5.21) is the same as saying an odd number of real eigenvalues "crosses" the imaginary axis as $\lambda$ varies along the trivial line through $(\lambda_o, 0)$. The simplest case of this (and the most common) is when a single eigenvalue "strictly" crosses, which also insures the existence of a local path of bifurcating solutions, cf. [CR]. The proof of Theorem 5.8 is similar to that of Theorem 5.2, but requires an additional step involving a "tubular" neighborhood of the trivial line, cf. [R], [Z]. The "jump" in degree again violates homotopy invariance in case neither (i) nor (ii) of (5.22) hold.*

# Chapter 6

# Global Continuation in Displacement Problems

In this chapter we return to nonlinear elasticity. In particular, we consider the *displacement problem* for (1.25), viz., $\partial\Omega_1 = \emptyset$ and $\partial\Omega_2 = \partial\Omega$:

$$\begin{aligned} \mathbf{C}(\mathbf{I} + \nabla\mathbf{u})[\nabla^2\mathbf{u}] + \mathbf{b}(\lambda, \mathbf{u}, \nabla\mathbf{u}) &= \mathbf{0} \text{ in } \Omega\,, \\ \mathbf{u} &= \mathbf{d}(\lambda) \text{ on } \partial\Omega\,. \end{aligned} \tag{6.1}$$

Problem (6.1) looks a lot like problem (5.12), c.f. Example (5.6). However, two difficulties, not present in (5.12), arise in the analysis of (6.1). First, the principal, quasilinear part of (6.1) does not generally define an invertible operator as in the left side of (5.16) leading (5.17). This is a consequence of the fact that strong ellipticity (1.22) is weaker than positive definiteness of the elasticity tensor, c.f. Remark 1.1. A deeper associated with (6.1) is that we do not generally have uniform ellipticity, c.f. (5.15) and Remark 1.2.

In addition to our assumptions in Chapter 1, we also presume a stress–free reference configuration:

$$\mathbf{S}(\mathbf{F}) = \frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\mathbf{I}) = \mathbf{0}. \tag{6.2}$$

We take $W(\cdot)$ and $d(\cdot)$ to be $C^3$, $b(\cdot)$ is presumed to be continuously differentiable, and we suppose that the external fields vanishes when the "control" parameter $\lambda = 0$:

$$\mathbf{b}(0, \cdot) = \mathbf{d}(0, \cdot) \equiv \mathbf{0}. \tag{6.3}$$

For ease of presentation, we treat the case $\mathbf{d}(\lambda) \equiv \mathbf{0}$. The more general case is handled similarly after a change of variables $\mathbf{u} = \mathbf{d}(\lambda) + \mathbf{w}$ (where $\mathbf{d} : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}^3$ of class $C^3$ is presumed), c.f. [21].

Similar to Example 5.6, for $0 < \alpha < 1$, let $C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ denote the class of $k$–times continuously differentiable, vector–valued functions on $\overline{\Omega}$ whose $k$th derivatives are Hölder continuous with exponent $\alpha$. We define $\|\mathbf{u}\|_{k,\alpha}$ as in (1.4), and we define

$$\mathrm{X} \equiv \{\mathbf{u} \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3) \,:\, \mathbf{u}|_{\partial\Omega} = \mathbf{0}\}. \tag{6.4}$$

In view of requirement (1.5), we define

$$\mathcal{A} \equiv \{\mathbf{u} \in \mathrm{X} \,:\, \det(\mathbf{I} + \nabla\mathbf{u}) > 0 \text{ on } \overline{\Omega}\}, \tag{6.5}$$

which is the set of admissible displacements. In view of (6.2) and (6.3), we see that $(\lambda, \mathbf{u}) = (0, \mathbf{0})$ is a solution point for (6.1) (with $\mathbf{d}(\lambda) \equiv 0$). Accordingly, we shall look for solutions in $\mathbb{R} \times \mathcal{O}$, where $\mathcal{O} \subset \mathrm{X}$ is the *component* of $\mathbf{0}$ in $\mathcal{A}$, i.e., the maximal connected set in $\mathcal{A}$ containing $\mathbf{u} = \mathbf{0}$.

To overcome the difficulties discussed at the start of this chapter, we do not work directly in $\mathcal{O}$. For each $\delta > 0$, we define

$$\mathcal{O}_\delta \equiv \{\mathbf{u} \in \mathcal{O} \,:\, \det(\mathbf{I} + \nabla\mathbf{u}) > \delta \text{ on } \overline{\Omega}\}, \tag{6.6}$$

which is open in X. Observe that

$$\mathcal{O} = \bigcup_{\delta > 0} \mathcal{O}_\delta. \tag{6.7}$$

Since $W(\cdot)$ is $C^3$, the elasticity tensor $\mathbf{C}(\cdot)$ is continuously differentiable, cf. (1.18). Accordingly, if $\det \mathbf{F} > \delta$ and $|\mathbf{F}| < M$, then there are constants $c_1$, $c_2 > 0$ (dependent upon $\delta$ and $M$) such that $\mathbf{C}(\mathbf{F})$ is *uniformly elliptic*:

$$c_1|\mathbf{a}|^2|\mathbf{b}|^2 \leq \mathbf{a} \otimes \mathbf{b} \cdot \mathbf{C}(\mathbf{F})[\mathbf{a} \otimes \mathbf{b}] \leq c_2|\mathbf{a}|^2|\mathbf{b}|^2 \tag{6.8}$$

for all $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^3$.

Define

$$\begin{aligned} \mathrm{Z} &\equiv \{\mathbf{u} \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3) \,:\, \mathbf{u}|_{\partial\Omega} = \mathbf{0}\}, \quad \text{and} \\ \mathrm{Y} &\equiv C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^3), \end{aligned} \tag{6.9}$$

each equipped with the usual Hölder norm. For any $\mathbf{u} \in \mathcal{O}$, define the linear differential operator $\mathrm{A}(\mathbf{u}) : \mathrm{Z} \to \mathrm{Y}$ via (cf. (6.1)

$$\mathrm{A}(\mathbf{u})[\mathbf{h}] \equiv \mathbf{C}(\mathbf{I} + \nabla\mathbf{u})[\nabla^2\mathbf{h}] \;\; \forall\, \mathbf{h} \in \mathrm{Z}. \tag{6.10}$$

Unlike the situation in Example 5.6, $\mathrm{A}(\mathbf{u}) \in L(\mathrm{Z}, \mathrm{Y})$ is not generally invertible. Nonetheless, we do have a *spectral estimate*:

**Proposition 6.1.** *For each $\mathbf{u} \in \overline{\mathcal{O}_\delta \cap B_M(\mathbf{0})}$, where $B_M(\mathbf{0})$ denotes the open ball mathrmX of radius "M" centered at $\mathbf{u} = \mathbf{0}$, there exist positive constants $\varepsilon$, $K_1$, $K_2$ (independent of $\mu$, $\mathbf{h}$, $\lambda$, $\mathbf{u}$) such that*

$$\|\mathbf{h}\|_{\mathrm{Z}} \leq K_1|\mu|^{\alpha/2}\|\mathrm{A}(\mathbf{u} - \mu\mathbf{I})[\mathbf{h}]\|_{\mathrm{Y}}\,, \tag{6.11}$$

*for all $\mathbf{h} \in \mathrm{Z}$ and for all $\mu \in \mathbb{C}$ satisfying $|arg(\mu)| \leq \frac{\pi}{2} + \varepsilon$ and $|\mu| \geq K_2$, where $\mathbf{I} \in L(\mathrm{Z}, \mathrm{Y})$ denotes the identity.*

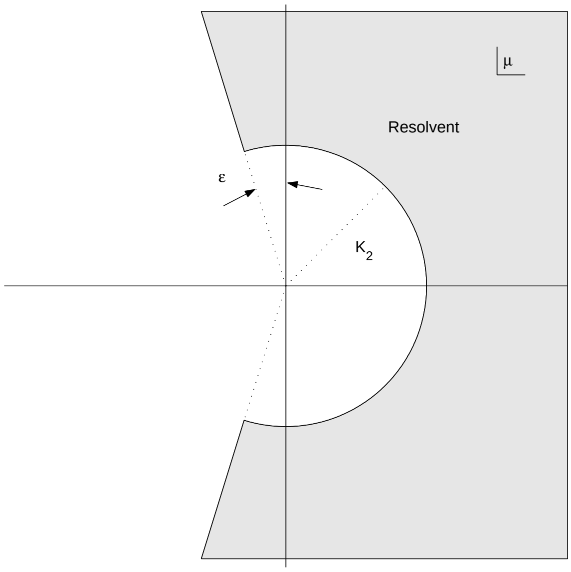


Figure 6.1: The resolvent of the sectorial operator $\mathrm{A}(\mathbf{u})$ given by the region $\{\mu \in \mathbb{C} : |\arg(\mu)| \leq \frac{\pi}{2} + \varepsilon, |\mu| \geq K_2\}$.

Estimate (6.11) is a Hölder–space version of a famous result due to Agmon in the $L^p$ setting [3]. In Chapter 8 we employ a more general version of (6.11), where we also provide an outline of the proof. Suffice it to say for now that uniform ellipticity (6.8) plays a key role in the uniformity of (6.11) (of course, the constants $K_1$, $K_2$, $\varepsilon$ in (6.11) generally depend upon $\delta$ and $M$). The spectral bound (6.11) insures that the operator $\mathrm{A}(\mathbf{u})$ is "sectorial" (see Figure (6.1)). In particular, for any real number $a > K_2$, the operator $\mathrm{A}(\mathbf{u}) - a\mathbf{I} \in L(\mathrm{Z}, \mathrm{Y})$ is injective, for all $\mathbf{u} \in \overline{\mathcal{O}_\delta \cap B_M(\mathbf{0})}$. In fact, $\mathrm{A}(\mathbf{u}) - a\mathbf{I}$ is readily shown to be surjective by standard arguments (e.g., c.f. [**J,sec. 10.3**]) or by the stability of the Fredholm property (of index zero), c.f. Chapter 8. At any rate, we are now in a position to imitate the approach taken in Example 5.6.

For any $(\lambda, \mathbf{u}) \in \mathbb{R} \times \overline{\mathcal{O}_\delta \cap B_M(\mathbf{0})}$, with $\delta > 0$ and $M > 0$ fixed, we consider the linear uniformly elliptic system

$$\begin{aligned} (\mathrm{A}(\mathbf{u}) - a\mathbf{I})[\mathbf{h}] &\equiv \mathbf{C}(\mathbf{I} + \nabla\mathbf{u})[\nabla^2\mathbf{h}] - a\mathbf{h} \\ &= -\mathbf{b}(\lambda, \mathbf{u}, \nabla\mathbf{u}) - a\mathbf{u} \text{ in } \Omega, \\ \mathbf{h}|_{\partial\Omega} &= \mathbf{0}, \end{aligned} \tag{6.12}$$

for some fixed $a > K_2$, as discussed above. Let $\mathbf{h} = -g_a(\lambda, \mathbf{u})$ denote the unique solution of (6.12). Then any solution of (6.1) (with $\mathbf{d}(\lambda) \equiv \mathbf{0}$), $(\lambda, \mathbf{u}) \in \mathbb{R} \times \overline{\mathcal{O}_\delta \cap B_M(\mathbf{0})}$, satisfies

$$\mathbf{u} + g_a(\lambda, \mathbf{u}) = \mathbf{0}, \tag{6.13}$$

and conversely. An argument identical to that in Example 5.6 (based upon the Schauder estimate for (6.12)) shows that $g_a(\cdot)$ is completely continuous.

In order to use Theorem 5.2, we must first verify (5.4) and (5.5). Let $\mathbf{h} = -\mathbf{T}_a\mathbf{u}$ denote the unique solution of the elliptic system

$$\begin{aligned} \mathsf{C}(\mathbf{I} + \nabla\mathbf{u})[\nabla^2\mathbf{h}] - a\mathbf{h} &= -a\mathbf{u} \quad \text{in } \Omega, \\ \mathbf{h}|_{\partial\Omega} &= \mathbf{0}, \end{aligned} \tag{6.14}$$

for all $\mathbf{u} \in \overline{\mathcal{O}_\delta \cap B_M(\mathbf{0})}$. Clearly $\mathbf{T}_a \in L(\mathrm{X})$ is compact. We claim that

$$D_{\mathbf{u}} g_a(0, \mathbf{0}) = \mathbf{T}_a. \tag{6.15}$$

We argue by contradiction, viz., if not, then there is a constant $\gamma > 0$ and a sequence $\{u_n\} \subset B_\varepsilon(\mathbf{0}) \subset \mathrm{X}$ with $\mathbf{u}_n \to \mathbf{0}$ such that

$$\|g_a(0, \mathbf{u}_n) - \mathbf{T}_a\mathbf{u}_n\|_{\mathrm{X}} > \gamma\|\mathbf{u}_n\|_{\mathrm{X}}\,, \tag{6.16}$$

for n sufficiently large. Define $\mathbf{v}_n = -g_a(0, \mathbf{u}_n)/\|\mathbf{u}_n\|_{\mathrm{X}}$ and $\mathbf{w}_n = -\mathbf{T}_a\mathbf{u}_n)/\|\mathbf{u}_n\|_{\mathrm{X}}$, which satisfy (c.f. (6.12), (6.14))

$$\mathsf{C}(\mathbf{I} + \nabla\mathbf{u}_n)[\nabla^2\mathbf{v}_n] - a\mathbf{v}_n = -a\mathbf{u}_n/\|\mathbf{u}_n\|_{\mathrm{X}} \quad \text{in } \Omega,$$

and

$$\begin{aligned} \mathsf{C}(\mathbf{I})[\nabla^2\mathbf{w}_n] - a\mathbf{w}_n &= -a\mathbf{u}_n/\|\mathbf{u}_n\|_{\mathrm{X}} \quad \text{in } \Omega, \\ \mathbf{v}_n|_{\partial\Omega} &= \mathbf{w}_n|\partial\Omega = \mathbf{0}, \end{aligned}$$

respectively. Now $\{\mathbf{u}_n/\|\mathbf{u}_n\|\}$ is uniformly bounded in X, and thus, $\mathbf{u}_n/\|\mathbf{u}_n\| \to \mathbf{e}$, say, in $C^1(\overline{\Omega}, \mathbb{R}^3)$ (as a subsequence). Also, by the Schauder estimates, $v_n \to \mathbf{v}$ and $\mathbf{w}_n \to \mathbf{w}$, say, in $C^2(\overline{\Omega}, \mathbb{R}^3)$ (as subsequences) where $\mathbf{v}$ and $\mathbf{w}$ each satisfy the boundary value problem

$$\begin{aligned} \mathsf{C}(\mathbf{I})[\nabla^2\mathbf{h}] - a\mathbf{h} &= -a\mathbf{e} \quad \text{in } \Omega, \\ \mathbf{h}|_{\partial\Omega} &= \mathbf{0}. \end{aligned}$$

By uniqueness, we conclude that $\mathbf{v} \equiv \mathbf{w}$, and thus

$$\|\mathbf{v}_n - \mathbf{w}_n\|_{\mathrm{X}} = \|g_a(0, \mathbf{u}_n) - \mathbf{T}_a\mathbf{u}_n\|_{\mathrm{X}}/\|\mathbf{u}_n\|_{\mathrm{X}} \to 0$$

which contradicts (6.16).

In consonance with (5.2), define $G_a(\lambda, \mathbf{u}) \equiv \mathbf{u} + g_a(\lambda, \mathbf{u})$. Then, $\mathbf{u}_n \mapsto G_a(0, \mathbf{u})$ is differentiable at $\mathbf{u} = \mathbf{0}$ with

$$D_{\mathbf{u}} G_a(0, \mathbf{0}) = \mathbf{I} + \mathbf{T}_a. \tag{6.17}$$

Now consider

$$
\begin{aligned}
(\mathbf{I}+\mathbf{T}_a)\mathbf{h} &= \mathbf{0} \ \Leftrightarrow & (6.18)\\
\mathbf{h} &= -\mathbf{T}_a\mathbf{h}, & (6.19)
\end{aligned}
$$

which, by virtue of (6.14) is equivalent to

$$
\begin{aligned}
\mathbf{C}(\mathbf{I})[\nabla^2\mathbf{h}] &= \mathbf{0} \quad \text{in } \Omega, & (6.20)\\
\mathbf{h}|_{\partial\Omega} &= \mathbf{0}.
\end{aligned}
$$

Now, $\mathbf{C}(\mathbf{I})$ is uniform (independent of $\mathbf{x}$) and satisfies strong ellipticity. Thus, by a theorem of Van Hove [**VH**], we conclude that $\mathbf{h} \equiv \mathbf{0}$ in (6.18) and (6.20), and the Riesz–Schauder theory insures that $\mathbf{I} + \mathbf{T}_a \in L(\mathrm{X})$ is bijective. With the aid of (4.14)–(4.16), we conclude that

$$
\deg(G_a(0,\cdot), B_\varepsilon(\mathbf{0})) = \deg(\mathbf{I}+\mathbf{T}_a, B_\varepsilon(\mathbf{0})) = \pm 1, \qquad (6.21)
$$

for $\varepsilon > 0$ sufficiently small.

**Remark 6.2.** *The calculation (6.21) corrects an inconsequential error in [21] (where the degree was proported to be "+1"). The theorem of Van Hove holds only for the (linear) displacement problem (6.20) for homogenous materials, c.f [29]. For inhomogeneous materials and/or "mixed" boundary conditions, a stronger assumption on* $\mathbf{C}(\mathbf{I})$ *is needed, c.f. Chapter* (8).

Given (6.21), we are now ready to employ the global continuation Theorem 5.2, except for the fact that we can only compute the degree in bounded open subsets $\mathcal{W} \subset \mathcal{O}_\delta$, i.e., for a given $\delta > 0$, we may have a solution branch $\Sigma$, characterized by neither (i) nor (ii) of Theorem 5.2, with $\Sigma \not\subseteq \mathbb{R} \times \mathcal{O}_\delta$. Accordingly, we conclude:

**Theorem 6.3.** *The maximal connected solution set of (6.1) (with* $\mathbf{d}(\lambda) \equiv \mathbf{0}$*) containing* $(0, \mathbf{0})$*,* $\Sigma \subset \mathbb{R} \times \mathcal{O}$*, is characterized by at least one of properties (i) and (ii) of Theorem 5.2 and/or*

$$
\textbf{(iii)} \quad \Sigma \not\subseteq \mathbb{R} \times \mathcal{O}_\delta \quad \textit{for each } \delta > 0. \qquad (6.22)
$$

The proof of Theorem 6.3 is nearly identical to that of Theorem 5.2. For a given $\delta$, we assume that $\Sigma$ is not characterized by either alternative (i), (ii) or (iii) and argue by contradiction. In view of (6.7), we consider the same argument for all $\delta > 0$. Also note that if characterization (iii) holds, then there is a sequence of solutions $\{(\lambda_n, \mathbf{u}_n)\} \subset \Sigma$ such that

$$
\inf_{\mathbf{x}\in\overline{\Omega}} \det(\mathbf{I} + \nabla\mathbf{u}_n(\mathbf{x})) \to 0^+ \ \text{ as } n \to \infty, \qquad (6.23)
$$

indicating a breakdown in local injectivity. In particular, observe that Theorem 6.3 leaves open the possibility that (6.23) holds without (i) or (ii) being true. In other words, the bounded branch could "terminate" at point $(\lambda_*, \mathbf{u}_*)$, where $(\lambda_n, \mathbf{u}_n) \to (\lambda_*, \mathbf{u}_*)$ in $C^1(\overline{\Omega}, \mathbb{R}^3)$ (by compact imbedding) and

$$\inf_{\mathbf{x}\in\overline{\Omega}} \det(\mathbf{I}+\nabla\mathbf{u}_*(\mathbf{x})) = 0, \tag{6.24}$$

which would seem to contradict (1.9), intuitively speaking. However, it is not generally clear how a bounded branch culminating in (6.24) contradicts anything in our construction. Observe from (1.16) that the first term in (6.1) is the divergence of the stress tensor.

We finish this chapter with more specific, physically reasonable restrictions on the stored energy $W(\cdot)$, enabling sharper results. To begin, we employ a special case of a result due to Knops and Stuart [30] to eliminate characterization (ii) of Theorem 6.3. For this we assume:

$$W(\mathbf{F}) > W(\mathbf{I}) \quad \forall \ \ \mathbf{F}\in \mathrm{GL}^+(\mathbb{R}^3)/\mathrm{SO}(3), \tag{6.25}$$

and

$$\Omega \quad \text{is star–shaped,} \tag{6.26}$$

viz., there is at least one point $\mathbf{x}_o \in \Omega$ such that every ray emanating out of $\mathbf{x}_o$ intersects the boundary $\partial\Omega$ at precisely one point. Condition (6.25) insures the quasiconvexity of $W(\cdot)$ at $\mathbf{F}=\mathbf{I}$, c.f. [25]. The result of [30] then insures that the $\lambda = 0$ problem for (6.1) (c.f. (1.25)), viz.,

$$\begin{aligned} \mathbf{C}(\mathbf{I}+\nabla\mathbf{u})[\nabla^2\mathbf{u}] &= \mathbf{0} \ \text{ in } \Omega, \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{0}, \end{aligned}$$

has the unique solution $\mathbf{u}\equiv\mathbf{0}$. Hence, we have

**Corollary 6.4.** *In addition to the hypothesis of Chapter 1 and those of this Chapter leading up to Theorem 6.3, assume that (6.25) and (6.26) hold. Then the global solution branch $\Sigma$ of Theorem 6.2 admits the decomposition (5.9), (5.10), with each component $\Sigma^\pm$ characterized by property* **(i)** *and/or property* **(iii)** *of Theorem 6.3.*

Next we consider a special class of materials for which (6.23) on a bounded branch is not possible:

$$W(\mathbf{F}) = \Psi(\mathbf{F}) + \Gamma(\det\mathbf{F}), \tag{6.27}$$

with

$$\Psi(\mathbf{F}) \in C^3(\mathrm{GL}^+(\mathbb{R}^3),\mathbb{R}) \cap C^2(\overline{\mathrm{GL}^+(\mathbb{R}^3)},\mathbb{R}) \tag{6.28}$$

and $\Gamma(\cdot)\in C^3(\mathbb{R}^+)$ is such that

$$\begin{aligned} \Gamma(\eta) &\to \infty \ \text{ as } \eta\to 0^+, \\ \Gamma'(\eta) &< 0, \ \text{ for } 0<\eta<\eta_o. \end{aligned} \tag{6.29}$$

We now have (c.f. [25]):

**Theorem 6.5.** *In addition to the hypotheses leading to Theorem 6.3, assume that (6.27)–(6.29) hold. If the global branch* $\Sigma$ *has property* **(iii)***, then property* **(i)** *is also true, i.e. if (6.23) holds, then*

$$|\lambda_n| + \|\mathbf{u}_n\|_{\mathrm{X}} \to \infty \quad \text{as } n \to \infty.$$

*Moreover, for each* $(\lambda, \mathbf{u}) \in \Sigma$*, the associated deformation* $\mathbf{f}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x})$ *is injective on* $\overline{\Omega}$*. Finally, if (6.25) and (6.26) also hold, then each of the components* $\Sigma_\pm$ *(c.f. Corollary 6.4) is unbounded.*

*Proof*: For the first assertion, observe that if $(\lambda, \mathbf{u})$ satisfies (6.1), then the following *balance law* also holds:

$$\boldsymbol{\nabla} \cdot (W(\mathbf{F})\mathbf{I} - \mathbf{F}^{\mathrm{T}}\mathbf{S}) = \mathbf{F}^{\mathrm{T}}\mathbf{b}(\lambda, \mathbf{u}, \boldsymbol{\nabla}\mathbf{u}) \text{ in } \Omega, \tag{6.30}$$

where $\mathbf{F} \equiv \mathbf{I} + \boldsymbol{\nabla}\mathbf{u}$ and $\mathbf{S} = \frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\mathbf{I} + \boldsymbol{\nabla}\mathbf{u})$, c.f. (1.15). ((6.30) is related to a conservation law of the Eshelby type [13]). If we substitute (6.27) into (6.30) we obtain

$$\boldsymbol{\nabla}_{\mathbf{x}}\Phi(J) = \boldsymbol{\nabla} \cdot \mathbf{P}(\mathbf{F}) + \mathbf{F}^{\mathrm{T}}\mathbf{b}(\lambda, \mathbf{u}, \boldsymbol{\nabla}\mathbf{u}) \text{ in } \Omega, \tag{6.31}$$

where $J(\mathbf{x}) \equiv \det \mathbf{F}(\mathbf{x})$,

$$\begin{aligned} \mathbf{P}(\mathbf{F}) &\equiv \mathbf{F}^{\mathrm{T}}\frac{\mathrm{d}\Phi}{\mathrm{d}\mathbf{F}}(\mathbf{F}) - \Phi(\mathbf{F})\mathbf{I}, \\ \Phi(J) &= \Gamma(J) - J\Gamma'(J), \end{aligned} \tag{6.32}$$

and "$\boldsymbol{\nabla}_{\mathbf{x}}\Phi(J)$" refers to the total derivative of the composite function $\mathbf{x} \mapsto \Phi(J())$. We argue by contradiction, viz., suppose that property (iii) of Theorem 6.3 holds with $\Sigma$ bounded in X. Let $\{(\lambda_n, \mathbf{u}_n)\} \subset \Sigma$; by the Schauder estimates for (6.12) $\{(\lambda_n, \mathbf{u}_n)\}$ is uniformly bounded in Z (cf. (6.9)) and hence, converges in $C^2(\overline{\Omega}, \mathbb{R}^3)$ (by compact imbedding). We write $\mathbf{f}_n(\mathbf{x}) = \mathbf{x} + \mathbf{u}_n(\mathbf{x})$, $\mathbf{F}_n = \mathbf{I} + \boldsymbol{\nabla}\mathbf{u}_n$ and $J_n = \det \mathbf{F}_n$, and substitute into (6.31) to find

$$\|\boldsymbol{\nabla}_{\mathbf{x}}\Phi(J_n)\|_\infty \leq \left\|\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{F}}(\mathbf{F}_n)[\boldsymbol{\nabla}^2\mathbf{u}_n]\right\|_\infty + \|\mathbf{F}_n^{\mathrm{T}}\mathbf{b}(\lambda_n, \mathbf{u}_n, \boldsymbol{\nabla}\mathbf{u}_n)\|_\infty, \tag{6.33}$$

where $\|\cdot\|_\infty$ denotes sup norm over $\overline{\Omega}$. In view of (6.28) and (6.32), the first term on the right side of (6.33) has a finite limit as $n \to \infty$. The same is clearly true for the second term on the right side of (6.33). On the other hand, the left side of (6.33) becomes unbounded as $n \to \infty$. To see this, note that (6.24) holds with $\|J_n\|_\infty$ bounded away from zero as $n \to \infty$ (note that $\mathbf{f}_n(\mathbf{x}) \equiv \mathbf{x}\ \forall \mathbf{x} \in \partial\Omega$). Consequently, there are distinct points $\mathbf{x}_o, \mathbf{x}_* \in \overline{\Omega}$ such that $\Phi(J_n(\mathbf{x}_o))$ remains bounded while $\Phi(J_n(\mathbf{x}_*)) \to \infty$ as $n \to \infty$. If we integrate $\boldsymbol{\nabla}_{\mathbf{x}}\Phi(J_n(\mathbf{x}))$ along any path connecting $\mathbf{x}_o$ and $\mathbf{x}_*$, we see, with aid of (6.29) and (6.32), that the left side of (6.33) grows without bound as $n \to \infty$. This is a contradiction, and we have shown that property (iii) is possible only when $\Sigma$ is unbounded.

To see that $\mathbf{f}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x})$ defines an injective map on $\overline{\Omega}$, observe by construction that $(\lambda, \mathbf{u}) \in \Sigma$ implies $\det(\mathbf{I} + \boldsymbol{\nabla}\mathbf{u}) > 0$ on $\overline{\Omega}$. This together with

$\mathbf{u}|_{\partial\Omega} = \mathbf{0}$ gives the desired result by a well–known Brouwer–degree theoretic argument, c.f. [11, Thm. 5.5-2].

The last claim concerning the unboundedness of the components $\Sigma_{\pm}$ is an easy consequence of Corollary 6.4. $\square$

# Chapter 7

# A Generalized Degree for a Class of Nonlinear Fredholm Mappings

In spite of the power and range of applicability of the Leray–Schauder degree, many problems of nonlinear continuum physics with "natural" boundary conditions, like $(1.26)_2$, fall outside its range. A simpler illustration comes from a problem related to Example 5.6. In (5.12), suppose that there is a vector–valued function $\mathbf{q}(u, \mathbf{v})$ such that $a_{ij}(\cdot) = \frac{\partial q_i(\cdot)}{\partial v_j}$ and $b_i(\cdot) = \frac{\partial q_i(\cdot)}{\partial u}$ (with $c(\cdot) \equiv 0$). Instead of $(5.12)_2$, we then impose the "zero–flux" condition

$$\mathbf{q}(u, \boldsymbol{\nabla} u) \cdot \mathbf{n}(\mathbf{x}) = 0 \;\; \forall \; \mathbf{x} \in \partial\Omega, \tag{7.1}$$

where $\mathbf{n}(\cdot)$ denotes the outward unit vector field on $\partial\Omega$. Note that $(5.12)_1$ now reads

$$\begin{aligned} \nabla \cdot (\mathbf{q}(u, \boldsymbol{\nabla} u)) &= a_{ij}(u, \boldsymbol{\nabla} u)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(u, \nabla u)\frac{\partial u}{\partial x_i} \\ &= f(\lambda, \mathbf{x}, u, \boldsymbol{\nabla} u) \quad \text{in } \Omega. \end{aligned} \tag{7.2}$$

For simplicity, assume that $\mathbf{q}(0, \mathbf{0}) = \mathbf{0}$. Observe (from the fundamental theorem of calculus) that

$$\mathbf{q}(u, \boldsymbol{\nabla} u) \cdot \mathbf{n} = \left( \int_0^1 b_i(tu, t\boldsymbol{\nabla} u)\, dt\, n_i \right) u + \left( \int_0^1 a_{ij}(tu, t\boldsymbol{\nabla} u)\, dt\, n_i \right) \frac{\partial u}{\partial x_j},$$

which suggests that we can use the same approach used in Example 5.6. That is, for a given $(\lambda, u) \in \mathbb{R} \times C^{1,\alpha}(\overline{\Omega})$, consider the linear problem $(5.16)_1$ (with $c \equiv 0$) subject to

$$\left(\int_0^1 a_{ij}(tu, t\boldsymbol{\nabla} u)\, dt\, n_i\right) \frac{\partial h}{\partial x_j} + \left(\int_0^1 b_i(tu, t\boldsymbol{\nabla} u)\, dt\, n_i\right) h = 0 \text{ on } \partial\Omega. \quad (7.3)$$

However, there is a problem–the coefficient functions in the first term on the left side (7.3) are elements of $C^{0,\alpha}(\overline{\Omega})$ (regardless of the smoothness of $\mathbf{q}(\cdot)$), whereas in the linear theory of such equations, such coefficient functions need to be in $C^{1,\alpha}(\overline{\Omega})$ in order to insure $C^{2,\alpha}(\overline{\Omega})$ solutions $h$, cf. [**LU,3.3,10.1**]. In fact, natural boundary conditions like (7.1) and $(1.25)_2$ are fully nonlinear.

The starting point for the generalized degree that we present comes from two observations. First, there is an alternative to the construction of the Leray–Schauder degree based upon finite–dimensional approximations (4.11) coupled with the Brouwer degree. That is, the Leray–Schauder degree can be "built from scratch" by immitating the construction of the Brouwer degree, with formula (4.15) in place of (4.4), cf. [**R**]. Second, there are more general classes of operators, e.g., those induced by elliptic boundary value problems, enjoying spectral properties that enable formulas akin to (4.15). Of course, spectral properties alone are not enough– Sard's theorem must be available, and homotopy invariance must be established directly (without the benefit of vector integral calculus, as is the case for the Brouwer degree, cf. [**S**]). Although these ideas are more or less well known (e.g., [**L**], [**EF**], [**F**]), actual details and proofs are provided in the works of Kielhöfer [**K1**], [**K2**] for a general class of Fredholm operators. Nonetheless, even that treatment does not anticipate nonlinear boundary conditions the likes of $(1.25)_2$ and (7.1), which we must accomodate.

To begin, we consider again (4.1). However, with applications like (1.25) and (7.1), (7.2) in mind, we incorporate the nonlinear boundary condition into the mapping $F(\cdot)$ via

$$F(\mathbf{u}) \equiv (F_1(\mathbf{u}), F_2(\mathbf{u}), \quad (7.4)$$

where $F : \mathcal{W} \subset \mathrm{X} \to \mathrm{Y}$ and $\mathrm{Y} = \mathrm{Y}_1 \times \mathrm{Y}_2$. Here, X, $\mathrm{Y}_1$, $\mathrm{Y}_2$ are real Banach spaces with $\mathrm{X} \subset \mathrm{Y}_1$ continuously embedded and $\mathcal{W}$ is open and bounded. For example, (7.2), (7.1) define $F_1(\mathbf{u})$, $F_2(\mathbf{u})$, respectively, with $\mathrm{X} = C^{2,\alpha}(\overline{\Omega})$, $\mathrm{Y}_1 = C^{0,\alpha}(\overline{\Omega})$ and $\mathrm{Y}_2 = C^{1,\alpha}(\overline{\Omega})$.

For technical reasons (use of Sard–Smale theorem) we need

$$F \in C^2(\Upsilon, \mathrm{Y}), \quad \text{where } \overline{\mathcal{W}} \subset \Upsilon. \quad (7.5)$$

Observe that the derivative of $F$ has the form

$$DF(\mathbf{u})[\mathbf{h}] \equiv L(\mathbf{u})[\mathbf{h}], \quad \forall\ \mathbf{h} \in \mathrm{X}, \quad (7.6)$$

which we also express

$$\begin{aligned} DF(\mathbf{u})[\mathbf{h}] &= (DF_1(\mathbf{u})[\mathbf{h}], DF_2(\mathbf{u})[\mathbf{h}]) \\ &\equiv (A(\mathbf{u})[\mathbf{h}], B(\mathbf{u})[\mathbf{h}])\ \forall \mathbf{h} \in \mathrm{X}, \end{aligned} \quad (7.7)$$

where $L(\mathbf{u}) \in L(\mathrm{X}, \mathrm{Y})$, $A(\mathbf{u}) \in L(\mathrm{X}, \mathrm{Y}_1)$ and $B(\mathbf{u}) \in L(\mathrm{X}, \mathrm{Y}_2)$.

For every $\mathbf{u} \in \mathcal{W}$, we make the following hypotheses concerning $L(\mathbf{u})$:

**(L1)** $L(\mathbf{u}) \in L(\mathrm{X}, \mathrm{Y})$ is Fredholm of index zero., i.e.

$$\dim \mathrm{Null}(L(\mathbf{u})) = \mathrm{codim}\, \mathrm{Range}(L(\mathbf{u})) < \infty.$$

**(L2)** $B(\mathbf{u}) \in L(\mathrm{X}, \mathrm{Y}_2)$ is surjective.

**(L3)** $A(\mathbf{u}) : \mathrm{Y}_1 \to \mathrm{Y}_1$, with domain of definition

$$D(A(\mathbf{u})) \equiv \mathrm{Z}_{\mathbf{u}} = \{\mathbf{h} \in \mathrm{X} \,:\, B(\mathbf{u})[\mathbf{h}] = \mathbf{0}\}, \text{ is closed,} \tag{7.8}$$

i.e., essentially $\mathrm{Z}_{\mathbf{u}}$ becomes a Banach space when equipped with the graph norm $\|\mathbf{u}\|_{\mathrm{Z}_{\mathbf{u}}} = \|\mathbf{h}\|_{\mathrm{Y}_1} + \|A(\mathbf{u})[\mathbf{h}]\|_{\mathrm{Y}_1}$, and $A(\mathbf{u}) : \mathrm{Z}_{\mathbf{u}} \to \mathrm{Y}_1$ is then a bounded linear operator, cf. [**Sche**]. With $A(\mathbf{u})$ so defined, recall that the *spectrum* of $A(\mathbf{u})$ is defined by

$$\sigma(A(\mathbf{u})) = \{\mu \in \mathbb{C} \,:\, A(\mathbf{u}) - \mu \mathrm{I} \in L(\mathrm{Z}_{\mathbf{u}}, \mathrm{Y}_1) \text{ is not bijective}\}, \tag{7.9}$$

where $\mathrm{Z}_{\mathbf{u}}$ and $\mathrm{Y}_1$ are complexified in the usual way. A point $\mu \in \sigma(A(\mathbf{u}))$ for which $\mathrm{Null}(A(\mathbf{u}))$ is not trivial is called an eigenvalue. An eigenvalue $\mu$ is said to have finite algebraic multiplicity if $\dim \mathrm{Null}((A(\mathbf{u}) - \mu \mathrm{I})^m) = \dim \mathrm{Null}((A(\mathbf{u}) - \mu \mathrm{I})^{m+1}) < \infty$ for some $m \in \mathbb{N}$, in which case $\dim \mathrm{Null}((A(\mathbf{u}) - \mu \mathrm{I})^m)$ is called the algebraic multiplicity of $\mu$.

**(L4)** There exists an open neighborhood $\eta$ of the ray $\{\mu \in mathbbC \,:\, \mu \in \mathbb{R}, \mu \geq 0\}$ such that $\sigma(A(\mathbf{u})) \cap \eta$ consists of finitely many eigenvalues, each of finite algebraic multiplicity.

**(L5)** The set of eigenvalues in $\sigma(A(\mathbf{u})) \cap \eta$ is uniformly bounded above in the sense that there is a constant $C > 0$ such that

$$\mathrm{Re}(\mu) < C \quad \forall \mu \in \sigma(A(\mathbf{u})) \cap \eta. \tag{7.10}$$

**Remark.** *Assumptions* **(L4)**, **(L5)** *insure, in particular, that* $A(\mathbf{u})$ *has a finite number of positive real eigenvalues (counted by algebraic multiplicity). Of course, the operator "*$-A(\mathbf{u})$*" then has spectral properties reminiscent of that for linear compact vector fields "*$\mathrm{I} + K$*" (*$K$ *compact) leading to formula (4.15), and we could have just as well reversed the inequalities in* **(L4)**, **(L5)**. *However,* **(L4)**, **(L5)** *in their present form are more natural for elliptic probelms like (1.25) and (7.1), (7.2).*

In addition to (7.4) and (7.5), we need another assumption on the nonlinear operator $F(\cdot)$ yielding, in particular, the crucial property in Lemma5.1 for compact vector fields:

$$\begin{aligned} &F(\cdot) \text{ is proper, i.e., for any compact set} \\ &\quad K \subset \mathrm{Y}, \quad F^{-1}(K) \cap \overline{\mathcal{W}} \text{ is compact.} \end{aligned} \tag{7.11}$$

With these assumptions in hand, we are ready to proceed as in the start of Lecture 4 for the Brouwer degree. For convenience, a mapping $F : \mathcal{W} \subset \mathrm{X} \to \mathrm{Y}$ satisfying assumptions (7.4)–(7.7), (7.9) and (7.10) is henceforth said to be *admissible.*

Consider the equation

$$F(\mathbf{u}) = \mathbf{y}, \tag{7.12}$$

with $\mathbf{y} \notin F(\partial\mathcal{W})$. To begin we assume that $y$ is a regular value, i.e. $DF(\mathbf{u}) = L(\mathbf{u})$ is surjective (and hence bijective by the Fredholm property) for all $\mathbf{u} \in \mathcal{W}$ such that (7.12) holds. As discussed in Remark7, hypotheses **(L4)** and **(L5)**, cf. (7.10), insure that $A(\mathbf{u})$ has a finite number of real eigenvalues, denoted $m(\mathbf{u})$, counted by algebraic multiplicity. Now by properness $F^{-1}(y) \cap \mathcal{W}$ is compact, and by the inverse function theorem every solution of (7.12) is isolated. Thus, $F^{-1}(y) \cap \mathcal{W}$ is a finite set. Accordingly, we define the degree by

$$\deg(F, \mathcal{W}, \mathbf{y}) \equiv \sum_{\mathbf{u} \in F^{-1}(\mathbf{y}) \cap \mathcal{W}} i(F, \mathbf{u}, \mathbf{y}), \tag{7.13}$$

where the *index* $i(\cdot)$ is defined by

$$i(F, \mathbf{u}_o, \mathbf{y}) = (-1)^{m(\mathbf{u}_o)}, \tag{7.14}$$

where $\mathbf{u}_o \in F^{-1}(\mathbf{y}) \cap \mathcal{W}$. As before, if $F^{-1}(\mathbf{y}) \cap \mathcal{W} = \emptyset$, we set $\deg(F, \mathcal{W}, \mathbf{y}) = 0$.

Next we suppose that $\mathbf{y}$ in (7.12) is not a regular value (viz., a *critical value*). Recall that a linear map $L \in L(\mathrm{X}, \mathrm{Y})$ is a *Fredholm operator* of index $k \in \mathbb{Z}$ if dim Null$(L)$ and codim Range$(L)$ are each finite, with $k =$ dim Null$(L)$ $-$ codim Range$(L)$. According to the Sard–Smale–Quinn theorem [**Sm**], [**QS**], the set of regular values of ma proper $C^{k+1}$ map $F : \mathcal{W} \subset \mathrm{X} \to \mathrm{Y}$, with $DF(\mathbf{u})$ a Fredholm operator of index $k$, $k \geq 0$, for all $\mathbf{u} \in \mathcal{W}$, is open and dense in Y. Accordingly, as in Lecture 4, by **(L1)** we may choose a regular value $\mathbf{y}_*$ with $\|\mathbf{y}_* - \mathbf{y}\|_{\mathrm{Y}} < \varepsilon$, sufficiently small, and define

$$\deg(F, \mathcal{W}, \mathbf{y}) \equiv \deg(F, \mathcal{W}, \mathbf{y}_*). \tag{7.15}$$

There are two concerns with regard to (7.15). First, we need to be sure that $\mathrm{y}_* \notin F(\partial\mathcal{W})$ for $\varepsilon > 0$ sufficiently small. Indeed, $\mathbf{y} \notin F(\partial\mathcal{W})$, and by properness, we claim that $\mathbf{y}$ has a positive distance "$d$" to $F(\partial\mathcal{W})$, viz., $\|F(\mathbf{u}) - \mathbf{y}\|_{\mathrm{Y}} \geq d > 0 \ \forall \ \mathbf{u} \in \partial\mathcal{W}$. To see this, assume the contrary. Then, there is a sequence $\{\mathbf{u}_j\} \subset \partial\mathcal{W}$ such that $\|F(\mathbf{u}_j) - \mathbf{y}\|_{\mathrm{Y}} \to 0$ as $j \to \infty$. By properness, $\{\mathbf{u}_j\}$ has a convergent subsequence $\mathbf{u}_j \to \mathbf{u}_* \in \partial\mathcal{W}$, and by continuity, $F(\mathbf{u}_*) = \mathbf{y}$, which is contradiction. Clearly $\varepsilon < d$ insures $\mathbf{y}_* \notin F(\partial\mathcal{W})$. Second, we want to show (7.15) to be independent of the choice of $\mathbf{y}_*$, i.e., if $\mathbf{y}_*$ and $\mathbf{y}_o$ are two distinct regular values satisfying the above conditions for $\varepsilon > 0$ sufficiently small, we need that

$$\deg(F, \mathcal{W}, \mathbf{y}_o) = \deg(F, \mathcal{W}, \mathbf{y}_*). \tag{7.16}$$

The proof of (7.16) turns out to be a special case of homotopy invariance, which we take up shortly. We the return to (7.16).

First we need a preliminary result:

**Proposition 7.1.** *For* $F : \mathcal{W} \to \mathrm{Y}$ *admissible,* $\deg(F, \mathcal{W}, \cdot)$ *is locally constant on the set of all regular values, i.e., for any regular value* $\mathbf{y}_*$ *and open ball* $B_\delta(\mathbf{y}_*) \subset \mathrm{Y}$ *(centered at* $\mathbf{y}_*$ *of radius* $\delta > 0$*), we have*

$$\deg(F, \mathcal{W}, \mathbf{y}) = \textit{constant}, \tag{7.17}$$

*for all* $\mathbf{y} \in B_\delta(\mathbf{y}_*)$ *with* $\delta > 0$ *sufficiently small.*

*Proof.* Set $\mathbf{y}_1 = \mathbf{y}_*$ and $F^{-1}(\mathbf{y}_1) = \{\mathbf{u}_1^j : j = 1, 2, \ldots, p\}$. Now, each $\mathrm{D}F(\mathbf{u}_1^j) \in L(\mathrm{X}, \mathrm{Y})$ is bijective. Also $y_2 \in B_\delta(\mathbf{y}_*)$ is a regular value (by the Sard–Smale–Quinn theorem) for $\delta > 0$ sufficiently small. If we apply the implicit function theorem to $F(\mathbf{u}) - \mathbf{y} = \mathbf{0}$ at each $(\mathbf{u}_1^j, \mathbf{y}_1)$, we see that every point $\mathbf{u}_1^j$ has a neighborhood $\mathcal{V}_j \subset \mathcal{W}$ such that each $\mathcal{V}_j$ contains precisely one solution $\mathbf{u}_2^j$ of $F(\mathbf{u}) = \mathbf{y}_2$, for $j = 1, 2, \ldots, p$. We claim that $F^{-1}(\mathbf{y}_2) = \{\mathbf{u}_2^j : j = 1, 2, \ldots, p\}$. To see this, let $\mathcal{V}$ denote the (disjoint) union of the $\mathcal{V}_j$. Then $\overline{\mathcal{W}} \backslash \mathcal{V}$ is closed, and by properness $\mathbf{y}_1 \notin F(\overline{\mathcal{W}} \backslash \mathcal{V})$ implies that $\mathbf{y}_1$ has a positive distance to $F(\overline{\mathcal{W}} \backslash \mathcal{V})$. Thus, for $\delta > 0$ sufficiently small, $\mathbf{y}_2 \notin F(\overline{\mathcal{W}} \backslash \mathcal{V})$, and the sets $F^{-1}(\mathbf{y}_1)$ and $F^{-1}(\mathbf{y}_2)$ have the same cardinality.

We claim that $m(\mathbf{u}_1^j) = m(\mathbf{u}_2^j)(\mathrm{mod}2)$ (for each $j = 1, 2, \ldots, p$), which will complete the proof in view of (7.14), (7.15). In what follows we drop the superscript "$j$", since the argument is the same for each $j = 1, 2, \ldots, p$. We first write (cf. (7.7)) $\mathrm{L}(\mathbf{u}_i) = (\mathrm{A}(\mathbf{u}_i), \mathrm{B}(\mathbf{u}_i))$, $i = 1, 2$. By virtue of (bf L4), (**L5**) (cf. (7.10)) the set of eigenvalues of $\mathrm{A}(\mathbf{u}_i)$ contained in $\sigma(\mathrm{A}(\mathbf{u}_i)) \cap \eta$ can be enclosed in a simple closed curve $\Gamma$ in the complex $\mu$–plane and thus constitutes a finite set of eigenvalues in the sense of Kato [**Ka, IV, secc. 3.5**], for $i = 1, 2$. Although we do not go into the details here, cf. [**HS, Prop. A.2**], we point out that properties **(L1)-(L3)** play a crucial role in eigenvalue perturbation results, which insure that the total number of eigenvalues (counted by algebraic multiplicity) contained in $\Gamma$ is the same for $i = 1$ and $i = 2$, for $\delta > 0$ sufficiently small. In particular, **(L2)** is needed to show that the "gap", [**Kg**], between the closed operators $\mathrm{A}(\mathbf{u}_1)$ and $\mathrm{A}(\mathbf{u}_2)$ can be made arbitrary small for $\delta > 0$ sufficiently small. Also, since $y_1$ and $y_2$ are regular values, **(L1)** insures that neither $\mathrm{A}(\mathbf{u}_1)$ nor $\mathrm{A}(\mathbf{u}_2)$ has a zero eigenvalue. Finally, since $\mathrm{A}(\mathbf{u})$ is a real linear operator, all complex eigenvalues occur in complex–conjugate pairs. Hence, $m(\mathbf{u}_1) = m(\mathbf{u}_2)(\mathrm{mod}2)$ □

We are now ready for homotopy invariance of $\deg(\cdot)$:

**Theorem 7.2.** *Let* $\mathcal{A} \subset \mathbb{R} \times \mathrm{X}$ *be open, with* $\mathcal{A}_\tau$ $\{\mathbf{u} \in \mathrm{X} : (\tau, \mathbf{u}) \in \mathcal{A}\}$ *uniformly bounded (possibly empty) for* $\tau$ *in finite intervals, cf. page 45 bf ( NOTA: ESTA CITA HAY QUE ARREGLARLA.). Assume that* $\mathrm{H} \in C^2(\tilde{\mathcal{A}}, \mathrm{Y})$*, where* $\overline{\mathcal{A}} \subset \tilde{\mathcal{A}}$*, with* $\tilde{\mathcal{A}} \subset \mathbb{R} \times \mathcal{Y}$ *open. We further presume that* $\mathcal{H} : \overline{\mathcal{A}} \to \mathrm{Y}$ *is proper (cf. (7.11)) and that* $\mathrm{H}(\tau, \cdot) : \overline{\mathcal{A}} \to \mathrm{Y}$ *is admissible for each* $\tau \in \mathbb{R}$*. Finally, let* $\mathbf{y} : \mathbb{R} \to \mathrm{Y}$ *be a* $C^2$ *curve such that* $\mathbf{y}(\tau) \neq \mathrm{H}(\tau, \mathbf{u})$ *for all* $(\tau, \mathbf{u}) \in \partial\mathcal{A}$*. Then,*

$$\deg(\mathrm{H}(\tau,\cdot),\mathcal{A}_\tau,\mathbf{y}(\tau)) = const.\, \forall \tau \in \mathbb{R}.$$

*In particular, if* $\mathcal{A}_{\tau_o} = \emptyset$ *for some* $\tau_o \in \mathbb{R}$*, then* $\deg(\mathrm{H}(\tau_o,\cdot),\mathcal{A}_{\tau_o},\mathbf{y}(\tau_o)) = 0$.

*Proof.* (Sketch) For simplicity we take $\mathbf{y}(\tau) \equiv \mathbf{y} \in \mathrm{Y}$ (otherwise, set $\tilde{\mathrm{H}}(\tau,\mathbf{u}) \equiv \mathrm{H}(\tau,\mathbf{u}) - \mathbf{y}(\tau)$). We fix some open bounded, "noncylindrical" domain $\mathcal{A}_{a,b} \equiv \mathcal{A} \cap ((a,b) \times \mathrm{X})$. Suppose that $\mathbf{y}$ is a regular value of both $\mathrm{H}(a,\cdot)$ and $\mathrm{H}(b,\cdot)$. We claim that

$$\deg(\mathrm{H}(a,\cdot),\mathcal{A}_a,\mathbf{y}) = \deg(\mathrm{H}(b,\cdot),\mathcal{A}_b,\mathbf{y}), \tag{7.18}$$

which is the heart of the argument. We outline the steps.

First observe that $\mathrm{H}(\tau,\cdot)$ admissible for each $\tau \in [a,b]$ implies $\mathrm{DH}(\tau,\mathbf{u}) \in \mathrm{L}(\mathbb{R}\times\mathrm{X},\mathrm{Y})$ is Fredholm of index $k = 1$. Now, by Proposition 7.1, $\deg(\mathrm{H}(a,\cdot),\mathcal{A}_a,\mathbf{y}')$ and $\deg(\mathrm{H}(b,\cdot),\mathcal{A}_b,\mathbf{y})$ are independent of $\mathbf{y}' \in \mathrm{B}_\delta(\mathbf{y})$, for $\delta > 0$ sufficiently small. Also, by the Sard–Smale–Quinn theorem, $\mathrm{B}_\delta(\mathbf{y})$ contains a regular value $\tilde{\mathbf{y}}$ for $\mathrm{H}(\cdot)$, i.e., $\mathrm{DH}(\tau,\mathbf{u})$ is surjective for all $(\tau,\mathbf{u}) \in \mathrm{H}^{-1}(\tilde{\mathbf{y}}) \cap \mathcal{A}_{a,b}$. (Here is the first and only place where we use $\mathrm{H} \in C^2$.) Accordingly we may assume, without lost of generality, that $\mathbf{y}$ is also a regular value of $\mathrm{H} : \mathcal{A}_{a,b} \to \mathrm{Y}$.

Next, since H is proper and $\mathbf{y}$ is a regular value, it can be shown that $\mathrm{F}^{-1}(\mathbf{y}) \cap \overline{\mathcal{A}}_{a,b} \equiv \mathcal{M}$ is a $C^2$, compact one–dimensional manifold with boundary comprising the disjoint union $\partial\mathcal{M}_a \cup \partial\mathcal{M}_b$, with $\partial\mathcal{M}_a \subset \{a\} \times \mathcal{A}_a$ and $\partial\mathcal{M}_b \subset \{b\} \times \mathcal{A}_b$. The components of $\mathcal{M}$ are finite in number, each of which is diffeomorphic to either a circle or a compact interval, e.g., cf. [**Ke**], [**K2**], [**GP**].

Choose a curve $m_j$ that starts at some point $(a,\mathbf{u}_a) \in \partial\mathcal{M}_a$. The the curve $m_j$ terminates either at some other point $(a,\tilde{\mathbf{u}}_a) \in \partial\mathcal{M}_a$ or at $(b,\mathbf{u}_b) \in \partial\mathcal{M}_b$. In the former case, we claim that (cf. (7.14))

$$i(\mathrm{H}(a,\cdot),\mathbf{u}_a,\mathbf{y}) = -i(\mathrm{H}(a,\cdot),\tilde{\mathbf{u}}_a,\mathbf{y}), \tag{7.19}$$

while in the latter case, we have

$$i(\mathrm{H}(a,\cdot),\mathbf{u}_a,\mathbf{y}) = i(\mathrm{H}(b,\cdot),\mathbf{u}_b,\mathbf{y}) \tag{7.20}$$

There is an obvious analogue of (7.19) for curves "starting" and "stopping" on $\partial\mathcal{M}_b$. Accordingly, from (7.13) only curves connecting $\partial\mathcal{M}_a$ to $\partial\mathcal{M}_b$ contribute to the degree, yielding (7.18). (Closed curves $m_j$ in $\mathcal{A}_{a,b}$ clearly do not contribute to either side of (7.18).)

There are two main ingredients in obtaining (7.19) and (7.20). First, by arguments similar to those involved in the proof of Proposition 7.1, it is shown that the index $i(\cdot)$ is constant along relatively open segments of $m_j$ for which $\mathrm{D}_{\mathbf{u}}\mathrm{H}(\tau, \mathbf{u})$ is surjective (and hence, bijective by the Fredholm property). (Note that all curves $m_j$ –except for the possible closed curves– "start" and "stop" in this fashion, i.e., by assumption, $\mathbf{y}$ is a regular value of both $\mathrm{H}(a, \cdot)$ and $\mathrm{H}(b, \cdot)$.)

Observe that if we parametrize $m_j$, viz., $\{(\tau(t), \mathbf{u}(t)) \ : \ \alpha \leq t \leq \beta\}$, then we readily see that $\dot{\tau}(t) \neq 0$ whenever $\mathrm{D}_{\mathbf{u}}\mathrm{H}(\tau(t), \mathbf{u}(t))$ is surjective. Similarly, $\dot{\tau}(t) = 0$ when $\mathrm{D}_{\mathbf{u}}\mathrm{H}(\tau(t), \mathbf{u}(t))$ is not surjective (i.e., at turning points).

Now the set $\mathrm{T} = \{(\tau, \mathbf{u}) \in m_j \ : \ \mathrm{D}_{\mathbf{u}}\mathrm{H}(\tau, \mathbf{u}) \text{is not surjective}\}$ is compact and thus possesses a finite open cover. By virtue of a careful one-dimensional Liapunov-Schmidt reduction on any such open set, one obtains the following [**K2**]:

$$i(\mathrm{H}(\tau(t), \cdot), \mathbf{u}(t), \mathbf{y}) \mathrm{sign}(\dot{\tau}(t)) = \text{const.},$$

for all $t \in [\alpha, \beta]$ with $\dot{\tau}(t) \neq 0$. (Note that $i(\cdot)$ exists if $\dot{\tau}(t) \neq 0$.) This gives (7.19), (7.20) and thus, (7.18).

To finish things, we return to the proof (7.17), which will take care of the case when $\mathbf{y}$ is not a regular value. Define $\overline{\mathrm{A}}_{a,b} \equiv [0,1] \times \overline{\mathcal{W}}$, and set $\mathrm{H}(\tau, \mathbf{u}) = \mathrm{F}(\mathbf{u}) - \tau \mathbf{y}_* - (1-\tau)\mathbf{y}_o$. Observe that $\mathbf{y} = \mathbf{0}$ is a regular value of both $\mathrm{H}(0, \cdot)$ and $\mathrm{H}(1, \cdot)$. Returning to (7.15), recall that the critical value $\mathbf{y}$ has positive distance "$d$" to $\mathrm{F}(\partial \mathrm{W})$. Clearly, for $\mathbf{y}_*$ and $\mathbf{y}_o$ sufficiently small, e.g., $\|\mathbf{y}_*\| < d/2$ and $\|\mathbf{y}_o\| < d/2$, we have $\|\tau \mathbf{y}_* + (1-\tau)\mathbf{y}_o\| < d$, and thus $\mathbf{0} \notin \mathrm{H}([0,1] \times \partial \mathcal{W})$. We conclude that (7.17 is a special case of (7.18). □

From this point it is fairly routine to verify that our generalized degree presented in this Chapter enjoys all of the other properties, (d1), (d3)–(d5), of the Brouwer and the Leray-Schauder degrees.

# Chapter 8

# Global Continuation in the Presence of Boundary Traction

In this chapter we return to the "mixed" boundary value problem

$$\begin{aligned}\mathbf{C}(\mathbf{I}+\nabla\mathbf{u})[\nabla^2\mathbf{u}]+\mathbf{b}(\lambda,\mathbf{u},\nabla\mathbf{u}) &= \mathbf{0} \text{ in } \Omega\,, &(8.1a)\\ \mathbf{S}(\mathbf{I}+\nabla\mathbf{u})\mathbf{n} &= \boldsymbol{\tau}(\lambda,\mathbf{u}) \text{ on } \partial\Omega_1\,, &(8.1b)\\ \mathbf{u} &= \mathbf{d}(\lambda) \text{ on } \partial\Omega_2\,. &(8.1c)\end{aligned}$$

To avoid difficulties with regularity at the boundary (which cannot presently be overcome even for local analysis based on the implicit function theorem, cf., e.g., [32]), we assume that $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$. We take $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ to be $C^3$. In addition to our hypotheses from Chapter (1), we also presume (6.2) and (6.3), and to the later we also append

$$\boldsymbol{\tau}(0,\cdot) = \mathbf{0}. \tag{8.2}$$

As in Chapter 6, we take $\mathbf{d}(\lambda) = \mathbf{0}$ for ease of presentation.

As discussed at the beginning of Chapter 7, problem (8.1) is apparently beyond the reach of the Leray–Schauder degree, and we turn to our generalized degree. Recall that our use of the Sard–Smale–Quinn theorem dictates $C^2$ smoothness of the nonlinear operator. Accordingly, we need to strengthen the differentiability hypotheses of Chapter 6. We assume

$$W(\cdot) \text{ is } C^5, \quad \mathbf{b}(\cdot) \text{ is } C^3, \quad \boldsymbol{\tau}(\cdot) \text{ is } C^4, \quad \mathbf{d}(\cdot) \text{ is } C^3. \tag{8.3}$$

Recall in Chapter 6, that the linearized problem (6.20) has the unique solution $\mathbf{h} = \mathbf{0}$, which is crucial in the calculation of the degree, c.f. (6.21). As mentioned in Remark (6.2) we need a strengthened hypothesis for $\mathbf{C}(\mathbf{I})$, the elasticity tensor at the reference configuration, which is in consonance with classical linear elasticity [18]:

$$\mathbf{H}\cdot\mathbf{C}(\mathbf{I})[\mathbf{H}] > 0, \quad \mathbf{H}\in L(\mathbb{R}^3), \quad \mathbf{H}^T = \mathbf{H}, \quad \mathbf{H}\neq\mathbf{0}, \tag{8.4}$$

i.e., $\mathbf{C}(\mathbf{I})$ is positive definite on nonzero, symmetric tensors.

Referring to the formulation in Chapter 7, we choose the Banach spaces:

$$\begin{aligned} X &= \left\{\mathbf{u}\in C^{2,\alpha}(\overline{\Omega},\mathbb{R}^3) : \mathbf{u}|_{\partial\Omega_2} = \mathbf{0}\right\}, \quad \|\cdot\|_X = \|\cdot\|_{2,\alpha}, \\ Y_1 &= C^{0,\alpha}(\overline{\Omega},\mathbb{R}^3), \quad \|\cdot\|_{Y_1} = \|\cdot\|_{0,\alpha}, \\ Y_2 &= C^{1,\alpha}(\partial\Omega_1,\mathbb{R}^3), \quad \|\cdot\|_{Y_2} = \|\cdot\|_{1,\alpha,\partial\Omega_1}. \end{aligned} \tag{8.5}$$

Define

$$\mathcal{U} = \left\{\mathbf{u}\in X : \det(\mathbf{I}+\nabla\mathbf{u}) > 0 \text{ on } \overline{\Omega}\right\}. \tag{8.6}$$

We define a nonlinear operator $G : \mathbb{R}\times\mathcal{U}\to Y = Y_1\times Y_2$ via:

$$\begin{aligned} &G_1(\lambda,\mathbf{u}) \equiv \mathbf{C}(\mathbf{I}+\nabla\mathbf{u})[\nabla^2\mathbf{u}] + \mathbf{b}(\lambda,\mathbf{u},\nabla\mathbf{u}), \quad G_1 : \mathbb{R}\times\mathcal{U}\to Y_1, \\ &G_2(\lambda,\mathbf{u}) \equiv \mathbf{S}(\mathbf{I}+\nabla\mathbf{u})\mathbf{n} - \boldsymbol{\tau}(\lambda,\mathbf{u}), \quad G_2 : \mathbb{R}\times\mathcal{U}\to Y_2, \\ &G(\lambda,\mathbf{u}) \equiv (G_1(\lambda,\mathbf{u}), G_2(\lambda,\mathbf{u})). \end{aligned} \tag{8.7}$$

The problem (8.1) is given abstractly by

$$G(\lambda,\mathbf{u}) = \mathbf{0}. \tag{8.8}$$

In view of (6.2), (6.3) and (8.1), we have that

$$G(0,\mathbf{0}) = \mathbf{0}. \tag{8.9}$$

Using (8.3), it can also be shown that $G : \mathbb{R}\times\mathcal{U}\to Y$ is $C^2$. Thus in contrast to the situation in Chapter 6, where the use of the Leray–Schauder degree demands merely $C^0$ smoothness of the nonlinear operator, we have more than enough differentiability to employ the implicit function theorem. A routine calculation yields

$$\mathrm{D}_{\mathbf{u}}G(0,\mathbf{0})[\mathbf{h}] \equiv (\mathbf{C}(\mathbf{I})[\nabla^2\mathbf{h}], \mathbf{C}(\mathbf{I})[\nabla\mathbf{h}]\mathbf{n}). \tag{8.10}$$

We claim that $\mathrm{D}_{\mathbf{u}}G(0,\mathbf{0})\in L(X,Y)$ is bijective. Injectivity is tantamount to the demonstration that the linear problem

$$\begin{aligned} \mathbf{C}(\mathbf{I})[\nabla^2\mathbf{h}] = \mathrm{Div}\,(\mathbf{C}(\mathbf{I})[\nabla\mathbf{h}]) &= \mathbf{0} \ \text{ in } \Omega, && \text{(8.11a)} \\ \mathbf{C}(\mathbf{I})[\nabla\mathbf{h}]\mathbf{n} &= \mathbf{0} \ \text{ on } \partial\Omega_1, && \text{(8.11b)} \\ \mathbf{h} &= \mathbf{0} \ \text{ on } \partial\Omega_2, && \text{(8.11c)} \end{aligned}$$

has the unique solution $\mathbf{h} = \mathbf{0}$. That this is indeed the case, follows from (8.4) and Kirchhoff's classical theorem [29]. Surjection turns out to be a special case of an argument that we give shortly. Since $\mathcal{U}\subset X$ is open, the implicit function theorem yields:

**Proposition 8.1.** *There exists a local branch of solutions to* (8.8),

$$\Sigma_{loc} = \{(\lambda, \tilde{\mathbf{u}}(\lambda)) \,:\, |\lambda| < \varepsilon\} \subset \mathbb{R} \times \mathcal{U}, \tag{8.12}$$

*for* $\varepsilon > 0$ *sufficiently small, such that* $\lambda \mapsto \tilde{\mathbf{u}}(\lambda)$ *is* $C^2$, $\tilde{\mathbf{u}}(0) = \mathbf{0}$ *and all local solutions of* (8.8) *in a sufficiently small neighborhood of* $(0, \mathbf{0})$ *belong to* $\Sigma_{loc}$.

Before proceeding with our global analysis, we recall a preliminary result from [26], for which (8.4) is crucial.

**Proposition 8.2.** *Let* $\mathbf{n}(\cdot)$ *denote the outward unit normal on* $\partial\Omega_1$. *The pair* $(\mathbf{C}(\mathbf{I}+\mathbf{A}), \mathbf{n})$ *satisfies the strong complementing condition (complementing condition and Agmon's condition, c.f. Chapter* (3)*) for all* $\mathbf{x} \in \partial\Omega$, *and for all* $\mathbf{A} \in L(\mathbb{R}^3)$ *such that* $|\mathbf{A}|$ *is sufficiently small.*

In particular, Proposition (8.2) implies that

$$\begin{aligned}&(\mathbf{C}(\mathbf{I}), \mathbf{n}) \text{ satisfies the strong complementing condition}\\ &\text{for all } \mathbf{x} \in \partial\Omega_1.\end{aligned} \tag{8.13}$$

We now define a set of admissible solutions, appropriate for our global analysis:

$$\begin{aligned}\mathcal{Y} \;=\; &\{\mathbf{u} \in X \,:\, \det(\mathbf{I} + \nabla\mathbf{u}) > 0 \text{ in } \overline{\Omega},\\ &(\mathbf{C}(\mathbf{I} + \nabla\mathbf{u}), \mathbf{n}) \text{ satisfies CC on } \partial\Omega_1\}.\end{aligned} \tag{8.14}$$

Similar to Chapter (6), we are interested in the maximal connected component set in $\mathcal{Y}$ containing $\mathbf{u} = \mathbf{0}$, i.e.,

$$\mathcal{O} \equiv \text{comp}\,\{\mathbf{0}\} \in \mathcal{Y}. \tag{8.15}$$

Also, we do not work directly in $\mathcal{O}$; for each $\delta > 0$, we define

$$\begin{aligned}\mathcal{O}_\delta \;=\; &\{\mathbf{u} \in \mathcal{O} \,:\, \det(\mathbf{I} + \nabla\mathbf{u}) > \delta \text{ in } \overline{\Omega}, \text{ and}\\ &|\Delta(\mathbf{I} + \nabla\mathbf{U}, \mathbf{x}, \boldsymbol{\xi})| > \delta,\ \boldsymbol{\xi}\cdot\mathbf{n}(\mathbf{x}) = 0,\ |\boldsymbol{\xi}| = 1, \text{ on } \partial\Omega_1\},\end{aligned} \tag{8.16}$$

where "$\Delta$" refers to the determinant whose nonvanishing occurs if and only if the complementing condition holds (c.f. Chapter (3)). Again, we observe that

$$\overline{\mathcal{O}}_\delta \subset \mathcal{O} \text{ for each } \delta > 0, \text{ and } \mathcal{O} = \cup_{\delta>0}\mathcal{O}_\delta. \tag{8.17}$$

Moreover, the continuity of $\mathbf{C}(\cdot)$ and $\Delta(\cdot)$ insure that $\mathcal{O}_\delta$ is open for each $\delta > 0$. Accordingly $\mathcal{O}$ itself is open. By (8.13) and Proposition (8.2), we see that $\mathcal{U} \cap \mathcal{O} \neq \emptyset$ is open, and $\mathbf{0} \in \mathcal{U} \cap \mathcal{O}$. Thus the local branch belongs to the admissible set, i.e., $\Sigma_{\text{loc}} \subset \mathcal{O}$, for $\varepsilon > 0$ sufficiently small, c.f. (8.12).

Now $G : \mathbb{R} \times \mathcal{O} \to \mathcal{Y}$ is also $C^2$, and returning to the setup of Chapter 7, we express $\mathrm{D}_{\mathbf{u}}G(\lambda, \mathbf{u}) \equiv L(\lambda, \mathbf{u})$ as in (7.6):

$$L(\lambda, \mathbf{u})[\mathbf{h}] = (A(\lambda, \mathbf{u})[\mathbf{h}], B(\lambda, \mathbf{u})[\mathbf{h}]), \tag{8.18}$$

where the principal parts of the operators $A(\lambda,\mathbf{u}) \in L(X,Y_1)$ and $B(\lambda,\mathbf{u}) \in L(X,Y_2)$ are given by

$$\begin{aligned} A(\lambda,\mathbf{u}) &= \mathbf{C}(\mathbf{I}+\nabla\mathbf{u})[\nabla^2\mathbf{h}]+\cdots \quad \text{in } \Omega, && (8.19a)\\ B(\lambda,\mathbf{u}) &= \mathbf{C}(\mathbf{I}+\nabla\mathbf{u})[\nabla\mathbf{h}]\mathbf{n}+\cdots \quad \text{on } \partial\Omega_1. && (8.19b)\end{aligned}$$

With properties (L1)–(L5) (c.f. (7.7), (7.9)) in mind, we establish:

**Proposition 8.3.** *For each* $(\lambda,\mathbf{u}) \in \mathbb{R}\times\overline{(\mathcal{O}_\delta\cap B_M(\mathbf{0}))}$ *with* $M,\delta>0$ *fixed, where* $B_M(\mathbf{0})\subset X$ *is an open ball (c.f. Proposition* (6.1)*), there are positive constants* $\varepsilon, c_1, c_2$ *(independent of* $\mu,\mathbf{h},\lambda,\mathbf{u}$*) such that*

$$\|\mathbf{h}\|_X \le c_1\left[|\mu|^{\alpha/2}\,\|(A(\lambda,\mathbf{u})-\mu\mathbf{I})[\mathbf{h}]\|_{Y_1} + |\mu|^{(1+\alpha)/2}\,\|B(\lambda,\mathbf{u})[\mathbf{h}]\|_{Y_2}\right], \quad (8.20)$$

*for all* $\mathbf{h}\in X$*, and all* $\mu\in\mathbb{C}$ *satisfying that* $|arg(\mu)|\le\pi/2+\varepsilon$ *and* $|\mu|\ge c_2$.

*Proof*: [**Sketch**] The first important observation here is that $B(\lambda,\mathbf{u})$ or $(\mathbf{C}(\mathbf{I}+\nabla\mathbf{u}),\mathbf{n})$ satisfies the strong complementing condition even though $\mathbf{u}\in\mathcal{O}_\delta\subset\mathcal{O}$ merely insures that the complementing condition holds. This follows from (8.13), Theorem (3.9) and the fact that $\mathbf{u}$ is connected to $\mathbf{0}$ in $\mathcal{O}$.

For $(\lambda,\mathbf{u})\in\mathbb{R}\times\overline{(\mathcal{O}_\delta\cap B_M(\mathbf{0}))}$, observe that (6.8) holds, i.e., $L(\lambda,\mathbf{u})$ is uniformly elliptic. Following Agmon [3], we introduce the differential operator

$$\mathcal{L}(\lambda,\mathbf{u}) \equiv L(\lambda,\mathbf{u}) + \mathrm{e}^{\mathrm{i}\theta}\frac{\partial^2}{\partial t^2},$$

on the cylinder $\Gamma\equiv\Omega\times(-\infty,\infty)$, which is elliptic if $|\theta|<\pi/2+\varepsilon$, for some small $\varepsilon>0$ (depending upon the constants in (6.8)). Define

$$\partial\Gamma_j = \partial\Omega_j\times(-\infty,\infty), \quad j=1,2.$$

On $\partial\Gamma$ we impose

$$B(\lambda,\mathbf{u})[\mathbf{v}]=\mathbf{0} \;\text{ on } \partial\Gamma_1, \quad \mathbf{v}=\mathbf{0} \;\text{ on } \partial\Gamma_2,$$

where $\mathbf{v}\in C^{2,\alpha}(\overline{\Omega}\times(-\infty,\infty),\mathbb{R}^3)$. Satisfaction of the strong complementing condition for $(\mathbf{C}(\mathbf{I}+\nabla\mathbf{u}),\mathbf{n})$ on $\partial\Omega_1$ insure that $B(\lambda,\mathbf{u})$ satisfies the complementing condition on $\partial\Gamma_1$, c.f. [3]. Accordingly, we may write down the apriori Schauder estimate for $(\mathcal{L}(\lambda,\mathbf{u}),B(\lambda,\mathbf{u}))$, which leads to (skipping a few steps): there are constants $\varepsilon, K_1>0$ such that

$$\begin{aligned}\|e(t)\mathbf{h}(\mathbf{x})\|_{2,\alpha\,;\,\Gamma_1} \le\; & K_1\big[\,\|\phi(t)e(t)(A(\lambda,\mathbf{u})-\mu\mathbf{I})[\mathbf{h}(\mathbf{x})]\|_{0,\alpha\,;\,\Gamma_2}\\ &+\|\phi(t)e(t)B(\lambda,\mathbf{u})[\mathbf{h}(\mathbf{x})]\|_{1,\alpha\,;\,\partial\tilde\Gamma_1}\\ &+|\mu|^{1/2}\,\|\phi_1(t)e(t)\mathbf{h}(\mathbf{x})\|_{0,\alpha\,;\,\Gamma_2}\\ &+\|\phi_2(t)e(t)\mathbf{h}(\mathbf{x})\|_{0,\alpha\,;\,\Gamma_2}\,\big], \qquad (8.21)\end{aligned}$$

for all $\mathbf{h}\in X$, $\mu\in\mathbb{C}$ such that $|\arg(\mu)|<\pi/2+\varepsilon$. Here $\tilde\Gamma_r=\Omega\times[-r,r]$, $\partial\tilde\Gamma_1=\partial\Omega_1\times[-2,2]$, $e(t)=\exp(\mathrm{i}\,|\mu|^{1/2}\,t)$, and $\phi,\phi_1,\phi_2$ are real–valued $C^\infty$ functions with support in $[-2,2]$, with $\phi\equiv1$ on $[-1,1]$.

For $|\mu| \geq c_2$ sufficiently large, we obtain the following inequalities: there is a constant $K_2 > 0$ such that

$$\begin{aligned}
\|\phi(t)e(t)(A(\lambda,\mathbf{u}) - \mu\mathbf{I})[\mathbf{h}]\|_{0,\alpha\,;\,\Gamma_2} &\leq K_2\,|\mu|^{\alpha/2}\,\|(A(\lambda,\mathbf{u}) - \mu\mathbf{I})[\mathbf{h}]\|_{Y_1}\,,\\
\|\phi(t)e(t)B(\lambda,\mathbf{u})[\mathbf{h}]\|_{1,\alpha\,;\,\partial\tilde{\Gamma}_1} &\leq K_2\,|\mu|^{(1+\alpha)/2}\,\|B(\lambda,\mathbf{u})[\mathbf{h}]\|_{Y_2}\,,\\
|\mu|^{1/2}\,\|\phi_1(t)e(t)\mathbf{h}\|_{0,\alpha\,;\,\Gamma_2} &\leq K_2\,|\mu|^{(1+\alpha)/2}\,\|\mathbf{h}\|_{Y_1}\,,\\
\|\phi_2(t)e(t)\mathbf{h}\|_{0,\alpha\,;\,\Gamma_2} &\leq K_2\,|\mu|^{\alpha/2}\,\|\mathbf{h}\|_{Y_1}\,,\\
\|e(t)\mathbf{h}\|_{2,\alpha\,;\,\Gamma_1} &\geq K_2\big[\,\|\mathbf{h}\|_X + |\mu|\,\|\mathbf{h}\|_{Y_1}\,\big],
\end{aligned} \tag{8.22}$$

for all $\mathbf{h} \in X$. Combining (8.21) with (8.22), and again using $|\mu| \geq c_2$ sufficiently large, we arrive at (8.20). □

With Proposition (8.3) in hand, we can now verify the assumptions (L1)–(L5) of Chapter 7, c.f. (7.7) and (7.9). In what follows $(\lambda,\mathbf{u}) \in \mathbb{R} \times \overline{(\mathcal{O}_\delta \cap B_M(\mathbf{0}))}$, c.f. Proposition (8.3). Consider the operator

$$L_\gamma = (A - \gamma\mathbf{I}, B), \tag{8.23}$$

where we have adopted the simplified notation $A = A(\lambda,\mathbf{u})$, $B = B(\lambda,\mathbf{u})$ and $L_\gamma = L_\gamma(\lambda,\mathbf{u})$. Inequality (8.20) implies that $\mathrm{Null}(L_\gamma)$ is trivial for any $\gamma \in \mathbb{R}$ such that $\gamma > c_2$. We now claim:

**Proposition 8.4.** *$L = L(\lambda,\mathbf{u}) \in L(X,Y)$ fulfills assumptions (L1)–(L5) of Chapter 7.*

*Proof*: Consider $L_\gamma$ for $\gamma > c_2$. The Schauder estimate implies that $L_\gamma$ is semi–Fredholm (c.f. Chapter 2). To show that $L_\gamma$ is Fredholm of index zero, we consider a one–parameter family of operators

$$L_t = (1-t)L_0 + tL_\gamma, \quad 0 \leq t \leq 1, \tag{8.24}$$

where $L_0 = (A_0, B_0) \in L(X,Y)$ is defined by

$$\begin{aligned}
A_0[\mathbf{h}] &= \Delta\mathbf{h} = (\Delta h_1, \Delta h_2, \Delta h_3), \quad \textit{in } \Omega,\\
B_0[\mathbf{h}] &= (\nabla\mathbf{h})\mathbf{n} = \left(\frac{\partial h_1}{\partial x_j}n_j, \frac{\partial h_2}{\partial x_j}n_j, \frac{\partial h_3}{\partial x_j}n_j\right), \quad \text{on } \partial\Omega_1.
\end{aligned}$$

Now $L_0$ is bijective, c.f. [LU, Sec. 3.3] and thus Fredholm of index zero. The Schauder estimate for (8.24) (uniform in $t$) and the consequent stability of the Fredholm index [27, Chap. 4] imply that $L_\gamma$ is Fredholm of index zero for all $\gamma \in \mathbb{R}$ including $\gamma = 0$. Thus (L1) is verified. Also for $\gamma > c_2$, $L_\gamma$ is bijective. Thus, $B$ is surjective, and (L2) holds.

For (L3) of (7.7), we note that

$$D(A) = Z_{\lambda,\mathbf{u}} = \{\mathbf{h} \in X \;:\; B[\mathbf{h}] = \mathbf{0}\}\,.$$

It can be shown that (L3) is a direct consequence of the Schauder estimate (c.f. Chapter (2)):

$$\|\mathbf{h}\|_X \le C\left[\|A[\mathbf{h}]\|_{Y_1} + \|\mathbf{h}\|_{Y_1}\right], \tag{8.25}$$

for all $\mathbf{h} \in Z_{\lambda,\mathbf{u}}$, $C > 0$ (independent of $\mathbf{h}$), c.f. [Fried, Section 1.1].

For (L4) and (L5), recall from above that $L_{\gamma_0} \in L(X,Y)$ is bijective for $\gamma_0 > c_2$. Clearly $A - \gamma_0\mathbf{I}$ also fulfills (L3), and $A - \gamma_0\mathbf{I} : Z_{\lambda,\mathbf{u}} \to Y_1$ is bijective. Hence, $(A - \gamma_0\mathbf{I})^{-1}$ is compact (by the compact embedding $X \to Y_1$ – this is essentially the same argument used in Chapters (4) and (5), c.f. (5.18)). Thus by the Riesz–Schauder Theory, we find that $\sigma(A)$:

i) comprises only isolated eigenvalues, each of finite algebraic multiplicity;

ii) has no finite accumulation points;

iii) possesses finitely many eigenvalues in the sector $|\arg(\mu)| \le \pi/2 + \varepsilon$.

Clearly (L4) and (L5) are fulfilled. □

Finally we need to verify (7.10) (actually, the slightly stronger version – including parameter dependence – in the hypothesis of Theorem 7.3).

**Proposition 8.5.** *For each fixed $\delta > 0$, $G$ is proper on $\mathbb{R}\times\overline{\mathcal{O}_\delta}$, i.e., $G^{-1}(K)\cap\overline{D}$ is compact for each bounded set $D \in \mathbb{R}\times\mathcal{O}_\delta$ and compact set $K \subset Y$.*

*Proof*: [**Sketch**] Let $\{\mathbf{y}_j\} \subset K \subset Y$ be a convergent sequence, and let $\{(\lambda_j, \mathbf{u}_j)\} \subset \overline{D} \subset \mathbb{R}\times\overline{\mathcal{O}_\delta}$ satisfy

$$G(\lambda_j, \mathbf{u}_j) = \mathbf{y}_j, \quad j = 1, 2, \ldots. \tag{8.26}$$

Our goal is to show that $\{(\lambda_j, \mathbf{u}_j)\}$ has a convergent subsequence in $\mathbb{R}\times X$. In the notation of (8.18), (8.19a), we write

$$\begin{aligned} \hat{A}(\mathbf{u})[\mathbf{u}] &\equiv \mathbf{C}(\mathbf{I}+\nabla\mathbf{u})[\nabla^2\mathbf{u}], \\ G_1(\lambda,\mathbf{u}) &\equiv \hat{A}(\mathbf{u})[\mathbf{u}] + \mathbf{g}(\lambda,\mathbf{u}), \end{aligned} \tag{8.27}$$

where $\mathbf{g} : \mathbb{R}\times\mathcal{O} \to Y_1$ is defined by $\mathbf{g}(\lambda,\mathbf{u}) \equiv \mathbf{b}(\lambda,\mathbf{u},\nabla\mathbf{u})$. Now $\{y_j\} = \{(y_j^1, y_j^2)\}$ converges in $Y = Y_1 \times Y_2$, and by compact embedding, $\mathbf{u}_j \to \mathbf{u}_*$ in $C^1(\overline{\Omega}, \mathbb{R}^3)$ and $\lambda_j \to \lambda_*$ (as subsequences – not relabeled). By continuity, we then have

$$\begin{aligned} \mathbf{g}(\lambda_j,\mathbf{u}_j) &\to \mathbf{g}(\lambda_*,\mathbf{u}_*) \quad \text{in } Y_1, \\ \boldsymbol{\tau}(\lambda_j,\mathbf{u}_j) &\to \boldsymbol{\tau}(\lambda_*,\mathbf{u}_*) \quad \text{in } Y_2. \end{aligned} \tag{8.28}$$

Moreover, the coefficient functions comprising $\mathbf{C}(\mathbf{I}+\nabla\mathbf{u}_j)$ are equicontinuous in $j$, and since $\{\mathbf{u}_j\} \subset X$ is bounded, we have

$$\hat{A}(\mathbf{u}_j)[\mathbf{u}_j] - \hat{A}(\mathbf{u}_k)[\mathbf{u}_j] \to \mathbf{0}, \quad \text{as } j,k \to \infty. \tag{8.29}$$

Next we define

$$\begin{aligned} \hat{B}(\mathbf{u})[\mathbf{h}] &\equiv \mathbf{C}(\mathbf{I}+\nabla\mathbf{u})[\nabla\mathbf{h}]\mathbf{n}, \\ \hat{L}(\mathbf{u})[\mathbf{h}] &\equiv (\hat{A}(\mathbf{u})[\mathbf{h}], \hat{B}(\mathbf{u})[\mathbf{h}]), \end{aligned} \tag{8.30}$$

where $\hat{L}(\mathbf{u}) \in L(X,Y)$ for each $\mathbf{u} \in \overline{\mathcal{O}_\delta}$. Since $\{\mathbf{u}_j\} \in \overline{\mathcal{O}_\delta}$ is bounded, we have the uniform estimate (c.f. Chapter (2))

$$\|\mathbf{h}\|_X \le C\left[\left\|\hat{L}(\mathbf{u}_j)[\mathbf{h}]\right\|_Y + \|\mathbf{h}\|_{Y_1}\right], \tag{8.31}$$

where $C > 0$ is independent of $\mathbf{h}$ and $j$, In particular, for $\mathbf{h} = \mathbf{u}_k - \mathbf{u}_m$ we get that

$$\|\mathbf{u}_k - \mathbf{u}_m\|_X \le C\left[\left\|\hat{L}(\mathbf{u}_j)[\mathbf{u}_k - \mathbf{u}_m]\right\|_Y + \|\mathbf{u}_k - \mathbf{u}_m\|_{Y_1}\right]. \tag{8.32}$$

Clearly the second term on the right side of (8.32) approaches zero as $k, m \to \infty$. Using (8.7) and (8.26) we get that

$$\begin{aligned}
\hat{L}(\mathbf{u}_j)[\mathbf{u}_k - \mathbf{u}_m] &= (\hat{A}(\mathbf{u}_j)[\mathbf{u}_k - \mathbf{u}_m], \hat{B}(\mathbf{u}_j)[\mathbf{u}_k - \mathbf{u}_m]), \\
&= (\hat{A}(\mathbf{u}_j)[\mathbf{u}_k - \mathbf{u}_m] - \hat{A}(\mathbf{u}_k)[\mathbf{u}_k] - \mathbf{g}(\lambda_k, \mathbf{u}_k) + \mathbf{y}_k^1 \\
&\quad + \hat{A}(\mathbf{u}_m)[\mathbf{u}_m] + \mathbf{g}(\lambda_m, \mathbf{u}_m) - \mathbf{y}_m^1, \\
&\quad \hat{B}(\mathbf{u}_j)[\mathbf{u}_k - \mathbf{u}_m] - \mathbf{S}(\mathbf{I} + \nabla\mathbf{u}_k)\mathbf{n} + \boldsymbol{\tau}(\lambda_k, \mathbf{u}_k) + \mathbf{y}_k^2 \\
&\quad + \mathbf{S}(\mathbf{I} + \nabla\mathbf{u}_m)\mathbf{n} - \boldsymbol{\tau}(\lambda_m, \mathbf{u}_m) - \mathbf{y}_m^2) \\
&= ((\hat{A}(\mathbf{u}_j) - \hat{A}(\mathbf{u}_k))[\mathbf{u}_k] + (\hat{A}(\mathbf{u}_m) - \hat{A}(\mathbf{u}_j))[\mathbf{u}_m] \\
&\quad - \mathbf{g}(\lambda_k, \mathbf{u}_k) + \mathbf{g}(\lambda_m, \mathbf{u}_m) + \mathbf{y}_k^1 - \mathbf{y}_m^1, \\
&\quad \hat{B}(\mathbf{u}_j)[\mathbf{u}_k - \mathbf{u}_m] - \mathbf{S}(\mathbf{I} + \nabla\mathbf{u}_k)\mathbf{n} + \mathbf{S}(\mathbf{I} + \nabla\mathbf{u}_m)\mathbf{n} \\
&\quad + \boldsymbol{\tau}(\lambda_k, \mathbf{u}_k) - \boldsymbol{\tau}(\lambda_m, \mathbf{u}_m) + \mathbf{y}_k^2 - \mathbf{y}_m^2).
\end{aligned} \tag{8.33}$$

Now it can be shown that

$$\left\|\hat{B}(\mathbf{u}_j)[\mathbf{u}_k - \mathbf{u}_m] - \mathbf{S}(\mathbf{I} + \nabla\mathbf{u}_k)\mathbf{n} + \mathbf{S}(\mathbf{I} + \nabla\mathbf{u}_m)\mathbf{n}\right\|_{Y_2} \to 0, \tag{8.34}$$

as $j, k, m \to \infty$, then this together with the convergence of $\{\mathbf{y}_j\}$, (8.28) and (8.29) implies that the first term on the right side of (8.32) also approaches zero as $j, k, m \to \infty$. Thus $\{\mathbf{u}_j\} \subset X$ is a Cauchy sequence, and thus has a convergent subsequence in $X$.

To show (8.34), we note first that $\mathbf{u}_j \to \mathbf{u}_*$ in $C^1(\overline{\Omega}, \mathbb{R}^3)$ implies that (8.34) holds in the $C^1(\overline{\Omega}, \mathbb{R}^3)$ norm. To complete the argument involves taking tangential derivatives of the terms inside the norm in (8.34) and using an argument like (8.29), c.f. [26, Thm. 4.6]. □

We are now in a position to use the degree of Chapter 7, in particular, Theorem 7.3, to obtain a result similar to Theorem (6.3).

**Theorem 8.6.** *Let $\Sigma \subset \mathbb{R} \times \mathcal{O}$ denote the maximal connected solution set of* (1.25) *containing* $(0, \mathbf{0})$. *Then $\Sigma$ is characterized by at least one of the properties (i) and (ii) of Theorem* (5.2) *or*

$$\textit{(iii)} \quad \Sigma \not\subseteq \mathbb{R} \times \mathcal{O}_\delta \textit{ for each } \delta > 0. \tag{8.35}$$

The proof of Theorem (8.6) is identical to the proof of Theorem (6.3) (c.f. also Theorem (5.2)), except that we now use our generalized degree in place of the Leray–Schauder degree.

In view of (8.14)–(8.16), in the case of characterization (iii) of $\Sigma$ in (8.35), then there is a sequence of solutions $\{(\lambda_j, \mathbf{u}_j)\} \subset \Sigma$ such that (6.23) and/or

$$\inf_{\substack{\mathbf{x}\in\partial\Omega_1 \\ \boldsymbol{\xi}\perp\mathbf{n}(\mathbf{x}),\ \|\boldsymbol{\xi}\|=1}} \Delta(\mathbf{I}+\nabla\mathbf{u}_j, \mathbf{x}, \boldsymbol{\xi}) \to 0, \quad \text{as } j\to\infty, \tag{8.36}$$

indicating a potential breakdown in the complementing condition. As in Chapter (6), we also have:

**Corollary 8.7.** *In addition to the hypotheses of Theorem* (8.6)*, assume the constitutive hypotheses* (6.27)–(6.29)*. If* $\Sigma$ *has property* (8.35)*, and* (8.36) *does not occur, then property (i) of Theorem* (5.2) *holds.*

Unfortunately, the Knops–Stuart result does not hold for mixed problems. Worse than that, there is no way to rule out (8.36) for $\Sigma$ bounded. Indeed there are known examples where this occurs.

# Chapter 9

# Global Bifurcation in Nonlinear Elasticity

In this chapter we summarize the typical setup for bifurcation problems in nonlinear elasticity, that are amenable to our approach. In particular, what we have in mind here are a class of problems involving bifurcation from a homogeneous state. In most cases, only very special types of boundary conditions, so–called sliding conditions, together with traction–free conditions on another part of the boundary admit a homogeneous solution for all values of the loading parameter. We refer to the books [10], [35] for examples where only the linearized analysis is carried out. See [42] and [24] for the nonlinear analysis of barrelling and buckling of compress elastic cylinders. In particular, the later presents a global analysis, which is a model for the material in this chapter.

Let $\mathbf{H}(\lambda) \in \mathrm{GL}^+(\mathbb{R}^3)$ denote a smooth curve for $\lambda \in \mathbb{R}$ such that $\mathbf{H}(0) = \mathbf{I}$. Note that $\mathbf{f}_\lambda(\mathbf{x}) = \mathbf{H}(\lambda)\mathbf{x}$ is a one–parameter family of homogeneous deformations. In what follows, we let

$$\mathbf{u}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \mathbf{f}_\lambda(\mathbf{x}),$$

i.e., $\mathbf{u}(\cdot)$ is the displacement field from the homogeneous deformed *trivial* state. Going back through the assumptions in Chapter (8), we adopt (6.2), $(8.3)_1$, and (8.4).

Typically such problems have a lot of symmetry although sometimes "hidden" by the boundary conditions, c.f. [42], [24]. We assume that such symmetry has been fully exploited, leading to aa problem of the form

$$G(\lambda, \mathbf{u}) = (\mathbf{C}(\mathbf{H}(\lambda) + \nabla\mathbf{u})[\nabla^2\mathbf{u}], \mathbf{S}(\mathbf{H}(\lambda) + \nabla\mathbf{u})\mathbf{n}) = \mathbf{0}, \tag{9.1}$$

where

$$\mathbf{C}(\mathbf{H}(\lambda) + \nabla\mathbf{u})[\nabla^2\mathbf{u}] = \mathbf{0} \quad \text{in } \Omega, \tag{9.2}$$

$$\mathbf{S}(\mathbf{H}(\lambda) + \nabla\mathbf{u})\mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega_1, \tag{9.3}$$

and other conditions, say, on $\partial\Omega\backslash\partial\Omega_1$ are automatically satisfied by some specified symmetry condition. Abstractly, we have

$$G : \mathcal{V} \subset \mathbb{R} \times X \to Y = Y_1 \times Y_2,$$

where $X, Y_1, Y_2$ are closed subspaces of $C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$, $C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^3)$, $C^{1,\alpha}(\partial\Omega, \mathbb{R}^3)$ respectively, as set by the exploitation of symmetry. Here

$$\begin{aligned} \mathcal{V} \quad = \quad & \big\{\mathbb{R} \times X \ : \ \det(\mathbf{H}(\lambda) + \nabla \mathbf{u}) > 0 \text{ in } \overline{\Omega}, \\ & (\mathbf{C}(\mathbf{H}(\lambda) + \nabla \mathbf{u}), \mathbf{n}) \text{ satisfies CC on } \partial\Omega_1\big\} \end{aligned} \tag{9.4}$$

We let

$$\Phi = \text{comp}(0, \mathbf{0} \quad \text{in } \mathcal{V}, \tag{9.5}$$

and, as in Chapter (8), for $\delta > 0$ we define

$$\begin{aligned} \Phi_\delta \quad = \quad & \big\{(\lambda, \mathbf{u}) \in \Phi \ : \ \det(\mathbf{H}(\lambda) + \nabla \mathbf{u}) > \delta \text{ in } \overline{\Omega}, \text{ and} \\ & |\Delta(\mathbf{H}(\lambda) + \nabla \mathbf{u}, \mathbf{x}, \boldsymbol{\xi})| > \delta \text{ on } \partial\Omega_1, \\ & \forall\ \boldsymbol{\xi} \cdot \mathbf{n}(\mathbf{x}) = 0, \ |\boldsymbol{\xi}| = 1\big\}. \end{aligned} \tag{9.6}$$

As before, we have

$$\overline{\Phi_\delta} \subset \Phi,$$

and

$$\Phi = \cup_{\delta > 0} \Phi_\delta. \tag{9.7}$$

Since $\Phi_\delta$ is open for each $\delta > 0$, $\Phi$ is also open.

We assume that (9.3) is satisfied when $\mathbf{u} = \mathbf{0}$ for all $\lambda \in \mathbb{R}$. Then

$$G(\lambda, \mathbf{0}) = \mathbf{0}, \quad \lambda \in \mathbb{R}, \tag{9.8}$$

i.e., we have the trivial line of solutions as in (5.19). As in Chapter (5), let

$$D_{\mathbf{u}} G(\lambda, \mathbf{0}) \equiv L(\lambda) \in L(X, Y),$$

and suppose that (5.20) holds, i.e., for some $\lambda_0 \in \mathbb{R}$,

$$L(\lambda_0)\boldsymbol{\eta} = \mathbf{0}, \quad \boldsymbol{\eta} \neq \mathbf{0}. \tag{9.9}$$

Now $G(\cdot)$ as in Chapter (8) fulfills all of the properties needed to use the degree from Chapter 7. Accordingly, we have:

**Theorem 9.1.** *Given* (9.8) *and* (9.9)*, suppose that* $L(\lambda) \in L(X, Y)$ *is injective (and bijective by the Fredholm property) on* $[\lambda_0 - \varepsilon, \lambda_0) \cup (\lambda_0, \lambda_0 + \varepsilon]$ *for some* $\varepsilon > 0$. *In addition, suppose that there are numbers* $\lambda_1 \in [\lambda_0 - \varepsilon, \lambda_0)$ *and* $\lambda_2 \in (\lambda_0, \lambda_0 + \varepsilon]$ *such that*

$$i(G(\lambda_1, \cdot), \mathbf{0}, \mathbf{0}) \neq i(G(\lambda_2, \cdot), \mathbf{0}, \mathbf{0}), \tag{9.10}$$

*i.e.,*

$$(-1)^{m(\lambda_1)} \neq (-1)^{m(\lambda_2)}, \tag{9.11}$$

*where* $m(\lambda)$ *denotes the number of real positive eigenvalues of* $L(\lambda)$*, counted by algebraic multiplicity, c.f. (L4), (L5), (7.9). Let* $\mathcal{S}$ *denote the closure of all nontrivial solutions of* (9.1)*, and let* $\Sigma = comp(\lambda_0, \mathbf{0})$ *in* $\mathcal{S}$*. Then* $\Sigma$ *is characterized by at least one of the properties (i), (ii) of Theorem* (5.8) *or*

$$\textit{(iii)} \quad \Sigma \nsubseteq \Phi_\delta \text{ for all } \delta > 0. \tag{9.12}$$

The proof is nearly identical to Rabinowitz's proof [37]. Once we assume that neither (i), (ii) nor (iii) hold, we argue by contradiction, c.f. [24]. If (9.12) holds, we are in the same situation as discussed at the end of Chapter (8), viz., there is a sequence of solutions $\{(\lambda_j, \mathbf{u}_j)\} \subset \Sigma$ such that (6.23) and/or (8.36) hold. We can get the analogue of Corollary (8.7). Interestingly enough, this requires a stronger growth condition: we replace $(6.29)_2$ with

$$\eta\Gamma'(\eta) \to -\infty \quad \text{as } \eta \searrow 0. \tag{9.13}$$

**Corollary 9.2.** *In addition to the hypotheses leading to Theorem* (9.1)*, assume the constitutive hypotheses* (6.27), (6.28), $(6.29)_1$*, and* (9.13)*. If* $\Sigma$ *has property* (9.12)*, then property (i) is also true.*

The growth condition (9.13) rules out the possibility that

$$J_j(\mathbf{x}) \equiv \det(\mathbf{H}(\lambda) + \nabla\mathbf{u}_j(\mathbf{x})) \searrow 0,$$

identically on $\overline{\Omega}$, which is required in the proof of Theorem (6.5). It can be shown that (9.13) insures that the traction on $\partial\Omega_1$ "blows up" in such a case, which violates (9.3), c.f. [24]. Again (8.36) on a bounded branch is entirely possible.

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