The Nonlinear Brachistochrone Problem with Friction

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Abstract

The brachistochrone problem is posed as a problem of the calculus of variations with differential side constraints, among smooth parametrized curves satisfying appropriate initial and boundary conditions. In this paper we consider several generalizations of the classical brachistochrone problem in which friction is considered as an arbitrary nonlinear function of either the normal component of force or the speed of the particle. We emphasize on the computational aspects of calculating the resulting curves of minimum descend.

Key words: brachistochrone, calculus of variations, differential constraints.

1 Introduction

The brachistochrone problem consists of finding the curve, joining two (non–vertical) given points, along which a bead of given mass falls under the influence of gravity in the minimum time. This problem was first posed by Johann Bernoulli in 1696 and solved that same year by Newton, Leibniz, the Bernoulli brothers Johann and Jacob, and de L'Hôpital. In 1744, Euler solved a variation of the brachistochrone problem in which friction is included as a nonlinear function of the square of the speed of the bead. Although his solution was not explicit, it showed that the curve of minimum descend is no longer a cycloid as in the problem without friction. Ashby, Brittin, Love, and Wyss [1] and Lipp [5] have considered variations of the problem in which friction is either a linear function or the absolute value, of the component of the force acting on the particle

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that is normal to the curve (including the term corresponding to the acceleration in the direction of the normal).

In this paper we consider several generalizations of these problems emphasizing on the computational aspects of calculating the resulting curves of minimum descend. We formulate the problem as one of the calculus of variations with the equation of motion of the mass along the tangential direction to the curve now as a differential side constraint. Our presentation generalizes those in [1] and [5] in the sense that we consider the friction as an arbitrary nonlinear function of either the normal component of force (*kinetic friction*) or the speed of the particle (*drag friction*).

First in Section (2) we formulate the basic problem of finding the curve of minimum descend as a problem of the calculus of variations with differential side conditions. In Section (2.1) we consider the special case in which the frictional force is a nonlinear function of the component of the force acting on the particle normal to the curve but not including the component of the acceleration in the direction of the normal. In Section (2.2) we consider the full problem in which the frictional force is a nonlinear function of the component of the force acting on the particle normal to the curve including the component of the acceleration in the direction of the normal. Finally in Section (3) we derive once again the result of Euler [3] in which friction is taken as a nonlinear function of the speed of the bead.

2 The Nonlinear Brachistochrone with Friction

We let a particle slide from $(0, 0)$ to (a, b) (where $a > 0$ and $b \leq 0$) along the curve $(x(t), y(t))$ under the influence of gravity and subject to a frictional force. Let

- $\theta(t)$ be the angle of inclination of the tangent to the curve from the horizontal;
- $v(t)$, $\vec{a}(t)$ the speed and acceleration respectively of the particle;
- $\vec{T}(t)$ the unit tangent vector to the curve;
- $\vec{N}(t)$ the unit normal vector to the curve.

It follows now that

$$\vec{a}(t) = v(t)\dot{\theta}(t)\vec{N}(t) + \dot{v}(t)\vec{T}(t).$$

The forces acting on the particle are:

- $\vec{W}$ the weight of the mass m;
- $\vec{F}_n$ (reaction) force normal to the curve;
- $\vec{F}_f$ frictional force.

We can write these forces in terms of the quantities defined above as:

$$\begin{aligned}\vec{W} &= -mg\vec{\imath} = -mg[\sin\theta(t)\vec{T}(t) + \cos\theta(t)\vec{N}(t)],\\ \vec{F}_n &= n(t)\vec{N}(t),\\ \vec{F}_f &= f(t)\vec{T}(t).\end{aligned}$$

From Newton's second law of motion we get that:

$$m\vec{a}(t) = \vec{W} + \vec{F}_f + \vec{F}_n.$$

That is

$$\begin{aligned}m[v(t)\dot{\theta}(t)\vec{N}(t) + \dot{v}(t)\vec{T}(t)] &= -mg[\sin\theta(t)\vec{T}(t) + \cos\theta\vec{N}(t)]\\ &\quad + n(t)\vec{N}(t) + f(t)\vec{T}(t).\end{aligned}$$

If we equate the terms in the normal and tangential directions respectively, we get that

$$\begin{aligned}mv(t)\dot{\theta}(t) &= -mg\cos\theta(t) + n(t),\\ m\dot{v}(t) &= -mg\sin\theta(t) + f(t). \qquad (1)\end{aligned}$$

From the first of these equations we obtain the following expression for the normal component of force:

$$n(t) = mg\cos\theta(t) + mv(t)\dot{\theta}(t).$$

Note that the second term in $n(t)$ corresponds to the mass times the component of the acceleration in the direction of the normal to the curve.

We take the frictional force to be a nonlinear function of the normal component of force to the curve. That is

$$f(t) = \hat{K}(n(t)) = \hat{K}(mg\cos\theta(t) + mv(t)\dot{\theta}(t)),$$

The angle, speed, and velocity of the particle are related by:

$$\begin{aligned}\dot{x}(t) &= v(t)\cos\theta(t), \qquad (2a)\\ \dot{y}(t) &= v(t)\sin\theta(t). \qquad (2b)\end{aligned}$$

The equation of motion (1) along the tangential direction can be written now as:

$$\dot{v}(t) = -g\sin\theta(t) + K(g\cos\theta(t) + v(t)\dot{\theta}(t)), \qquad (3)$$

where

$$K(N) = \frac{1}{m}\hat{K}(mN). \qquad (4)$$

We can eliminate $\theta(t)$ from (2) and (3) upon recalling that

$$v(t) = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}, \quad \dot{\theta} = \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{v(t)^2}. \qquad (5)$$

Thus we now have that

$$v(t) = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}, \tag{6a}$$

$$v(t)\dot{v}(t) = -g\dot{y}(t) + v(t)K\left(\frac{\dot{x}(t)(g+\ddot{y}(t)) - \dot{y}(t)\ddot{x}(t)}{v(t)}\right). \tag{6b}$$

We let $t = \hat{t}(\tau)$ be a reparametrization of the curve in terms of a parameter τ where we assume that:

$$\hat{t}'(\tau) = \frac{\mathrm{d}\hat{t}(\tau)}{\mathrm{d}\tau} > 0.$$

If we let $\bar{x}(\tau) = x(\hat{t}(\tau))$, then it follows now that

$$\hat{t}'(\tau)\dot{x}(\hat{t}(\tau)) = \bar{x}'(\tau), \quad \hat{t}'(\tau)^2\ddot{x}(\hat{t}(\tau)) = \bar{x}''(\tau) - \hat{t}''(\tau)\dot{x}(\hat{t}(\tau)).$$

Using these expressions we can write the second equation in (5) as:

$$\bar{\theta}'(\tau) = \frac{\bar{x}'(\tau)\bar{y}''(\tau) - \bar{y}'(\tau)\bar{x}''(\tau)}{\hat{t}'(\tau)^2\bar{v}(\tau)^2}, \tag{7}$$

and (6a), (6b) as:

$$\hat{t}'(\tau)\bar{v}(\tau) = \sqrt{\bar{x}'(\tau)^2 + \bar{y}'(\tau)^2}, \tag{8a}$$

$$\bar{v}(\tau)\bar{v}'(\tau) = -g\bar{y}'(\tau) + \hat{t}'(\tau)\bar{v}(\tau)K\left(\frac{g\bar{x}'(\tau)}{\hat{t}'(\tau)\bar{v}(\tau)} + \frac{\bar{x}'(\tau)\bar{y}''(\tau) - \bar{y}'(\tau)\bar{x}''(\tau)}{\hat{t}'(\tau)^3\bar{v}(\tau)}\right). \tag{8b}$$

The problem now is to minimize the time integral:

$$\int_{\tau_1}^{\tau_2} \hat{t}'(\tau)\mathrm{d}\tau, \tag{9}$$

subject to the differential constraints (8a), (8b), and to the boundary conditions:

$$\bar{x}(\tau_1) = \bar{y}(\tau_1) = 0, \quad \bar{x}(\tau_2) = a\ ,\ \bar{y}(\tau_2) = b, \tag{10a}$$

$$\bar{x}'(\tau_1) = x_0', \quad \bar{y}'(\tau_1) = y_0', \tag{10b}$$

$$\hat{t}(\tau_1) = 0. \tag{10c}$$

2.1 No centrifugal acceleration included

This is the case where the function K depends only on $g\cos\theta(t)$. Thus we have to minimize the time integral (9) subject to (10a), (10c), the differential constraints (8), and $\bar{v}(\tau_1) = v_0$ instead of (10b). Introducing the Lagrange multipliers $\lambda(t), \sigma(t)$ we get that the first order necessary conditions for this problem are equivalent to those for the functional:

$$\begin{aligned} I(\bar{x},\bar{y},\bar{v},\hat{t},\bar{\sigma},\bar{\lambda}) &= \int_{\tau_1}^{\tau_2} \Bigg[\hat{t}'(\tau) + \bar{\sigma}(\tau)\left(\hat{t}'(\tau)\bar{v}(\tau) - \sqrt{\bar{x}'(\tau)^2 + \bar{y}'(\tau)^2}\right) \\ &+ \bar{\lambda}(\tau)\Bigg(\bar{v}(\tau)\bar{v}'(\tau) + g\bar{y}'(\tau) - \hat{t}'(\tau)\bar{v}(\tau)K\left(\frac{g\bar{x}'(\tau)}{\hat{t}'(\tau)\bar{v}(\tau)}\right)\Bigg)\Bigg]\mathrm{d}\tau. \end{aligned} \tag{11}$$

The Euler–Lagrange equations for I are given by (8) and

$$\begin{aligned}
1+\bar\sigma(\tau)\bar v(\tau) &= c_1+\bar\lambda(\tau)\bar v(\tau)\left[K(\bar h(\tau))-\frac{g\bar x'(\tau)}{\hat t'(\tau)\bar v(\tau)}K'(\bar h(\tau))\right], &(12a)\\
\bar v(\tau)\bar\lambda'(\tau) &= \hat t'(\tau)\left(\bar\sigma(\tau)+\bar\lambda(\tau)\left[\bar h(\tau)K'(\bar h(\tau))-K(\bar h(\tau))\right]\right), &(12b)\\
\frac{\bar\sigma(\tau)\bar x'(\tau)}{\hat t'(\tau)\bar v(\tau)} &= c_2-g\bar\lambda(\tau)K'(\bar h(\tau)), &(12c)\\
-\frac{\bar\sigma(\tau)\bar y'(\tau)}{\hat t'(\tau)\bar v(\tau)} &= c_3-g\bar\lambda(\tau), &(12d)
\end{aligned}$$

where now

$$\bar h(\tau)=\frac{g\bar x'(\tau)}{\hat t'(\tau)\bar v(\tau)}.$$

and c_1, c_2, c_3 are constants of integration. The boundary conditions are given by (10a), (10c), $\bar v(\tau_1)=v_0$, and

$$\bar\lambda(\tau_2)=0,\quad 1+\bar\sigma(\tau)\bar v(\tau)-\bar\lambda(\tau)\bar v(\tau)\left[K(\bar h(\tau))-\frac{g\bar x'(\tau)}{\hat t'(\tau)\bar v(\tau)}K'(\bar h(\tau))\right]\Bigg|_{\tau=\tau_2}=0. \tag{13}$$

The second boundary condition in (13) implies that $c_1=0$.

In terms of the parameter τ we have that (2a) and (2b) can be written as:

$$\bar x'(\tau)=\hat t'(\tau)\bar v(\tau)\cos\bar\theta(\tau),\quad \bar y'(\tau)=\hat t'(\tau)\bar v(\tau)\sin\bar\theta(\tau). \tag{14}$$

It follows now that (8a) is satisfied and that (8b) and (12) reduce to:

$$\begin{aligned}
\bar v'(\tau) &= \hat t'(\tau)(-g\sin\hat\theta(\tau)+K(g\cos\bar\theta(\tau))) &(15a)\\
1+\bar\sigma(\tau)\bar v(\tau) &= \bar\lambda(\tau)\bar v(\tau)\left[K(g\cos\bar\theta(\tau))-g\cos\bar\theta(\tau)K'(g\cos\bar\theta(\tau))\right], &(15b)\\
\bar v(\tau)\bar\lambda'(\tau) &= \hat t'(\tau)\big(\bar\sigma(\tau)\\
&\quad +\bar\lambda(\tau)\left[g\cos\bar\theta(\tau)K'(g\cos\bar\theta(\tau))-K(g\cos\bar\theta(\tau))\right]\big), &(15c)\\
\bar\sigma(\tau)\cos\bar\theta(\tau) &= c_2-g\bar\lambda(\tau)K'(g\cos\bar\theta(\tau)), &(15d)\\
-\bar\sigma(\tau)\sin\bar\theta(\tau) &= c_3-g\bar\lambda(\tau). &(15e)
\end{aligned}$$

If we treat (15d) and (15e) as a linear system for $\bar\sigma, \bar\lambda$ we get that

$$\begin{aligned}
\bar\sigma(\tau) &= \frac{c_2-c_3K'(g\cos\bar\theta(\tau))}{\cos\bar\theta(\tau)+\sin\bar\theta(\tau)K'(g\cos\bar\theta(\tau))} &(16a)\\
\bar\lambda(\tau) &= \frac{c_2\sin\bar\theta(\tau)+c_3\cos\bar\theta(\tau)}{g\left(\cos\bar\theta(\tau)+\sin\bar\theta(\tau)K'(g\cos\bar\theta(\tau))\right)}. &(16b)
\end{aligned}$$

This motivates us to take θ as the parameter and rewrite (16) as

$$
\begin{aligned}
\bar{\sigma}(\theta) &= \frac{c_2 - c_3 K'(g\cos\theta)}{\cos\theta + \sin\theta K'(g\cos\theta)} && \text{(17a)}\\
\bar{\lambda}(\theta) &= \frac{c_2\sin\theta + c_3\cos\theta}{g\left(\cos\theta + \sin\theta K'(g\cos\theta)\right)}. && \text{(17b)}
\end{aligned}
$$

We can use (15b) to get that

$$\bar{v}(\theta) = -1/\left(\bar{\sigma}(\theta) - \bar{\lambda}(\theta)(K(g\cos\theta) - g\cos\theta K'(g\cos\theta))\right). \tag{18}$$

Also, using (15b), we have that (15c) simplifies to:

$$\hat{t}'(\theta) = -\bar{v}^2(\theta)\,\bar{\lambda}'(\theta), \tag{19}$$

where $\bar{\lambda}'(\theta)$ can be computed from (17). It follows now that (15a) is equivalent to:

$$\bar{v}'(\theta) = \bar{v}^2(\theta)\bar{\lambda}'(\theta)(g\sin\theta - K(g\cos\theta)). \tag{20}$$

We have now:

Lemma 2.1. *Equation* (20) *follows from* (17), (18), *and* (19).

Proof: First note that from (18) we get that

$$
\begin{aligned}
\bar{v}'(\theta) &= \bar{v}^2(\theta)\left[\bar{\sigma}'(\theta) - \bar{\lambda}'(\theta)(K(g\cos\theta) - g\cos\theta K'(g\cos\theta))\right.\\
&\quad \left. - g^2\sin\theta\cos\theta\bar{\lambda}(\theta)K''(g\cos\theta)\right], && (21)
\end{aligned}
$$

where the prime denotes now differentiation with respect to θ. But (15d), (15e) or equivalently (17) imply that

$$
\begin{aligned}
\bar{\sigma}'(\theta) &= \frac{g\bar{\lambda}'(\theta) - \bar{\sigma}(\theta)\cos\theta}{\sin\theta}, && (22)\\
\bar{\lambda}'(\theta) &= \frac{\bar{\sigma}(\theta) + g^2\bar{\lambda}(\theta)\sin^2\theta K''(g\cos\theta)}{g\left(\cos\theta + \sin\theta K'(g\cos\theta)\right)} && (23)
\end{aligned}
$$

We can write (23) as:

$$g^2\bar{\lambda}(\theta)\sin^2\theta K''(g\cos\theta) = g\bar{\lambda}'(\theta)\left(\cos\theta + \sin\theta K'(g\cos\theta)\right) - \bar{\sigma}(\theta).$$

Using this expression and (22) we can write the term multiplying $\bar{v}^2(\theta)$ in (21) as

$$
\begin{aligned}
\frac{g\bar{\lambda}'(\theta) - \bar{\sigma}(\theta)\cos\theta}{\sin\theta} &- \bar{\lambda}'(\theta)(K(g\cos\theta) - g\cos\theta K'(g\cos\theta))\\
&- \left(g\bar{\lambda}'(\theta)\left(\cos\theta + \sin\theta K'(g\cos\theta)\right) - \bar{\sigma}(\theta)\right)\cot\theta\\
&= \bar{\lambda}'(\theta)\left(g\csc\theta - K(g\cos\theta) - g\cos\theta\cot\theta\right)\\
&= \bar{\lambda}'(\theta)(g\sin\theta - K(g\cos\theta)),
\end{aligned}
$$

from which the result follows. $\square$

It follows now from the first boundary condition in (13) and equation (17b) that

$$c_2 \sin\theta_2 + c_3 \cos\theta_2 = 0, \tag{24}$$

Since $\bar{v}(\theta_1) = v_0$ is given, we get from (18) that

$$v_0 = -1/\left(\bar{\sigma}(\theta_1) - \bar{\lambda}(\theta_1)(K(g\cos\theta_1) - g\cos\theta_1 K'(g\cos\theta_1))\right). \tag{25}$$

Equations (24) and (25) can be written as:

$$\begin{pmatrix} g - A(\theta_1)\sin\theta_1 & -gK'(g\cos\theta_1) - A(\theta_1)\cos\theta_1 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \hat{c}_2 \\ \hat{c}_3 \end{pmatrix} = \begin{pmatrix} -gD(\theta_1) \\ 0 \end{pmatrix}, \tag{26}$$

where the functions $A(\theta)$ and $D(\theta)$ are given by

$$\begin{aligned} A(\theta) &= K(g\cos\theta) - g\cos\theta K'(g\cos\theta), && (27) \\ D(\theta) &= \cos\theta + \sin\theta K'(g\cos\theta), && (28) \end{aligned}$$

and

$$\hat{c}_2 = c_2 v_0, \quad \hat{c}_3 = c_3 v_0. \tag{29}$$

From (10a) we get that:

$$\begin{aligned} a &= \int_{\theta_1}^{\theta_2} \bar{x}'(\theta)\,\mathrm{d}\theta = \int_{\theta_1}^{\theta_2} \hat{t}'(\theta)\bar{v}(\theta)\cos\theta\,\mathrm{d}\theta, && (30a) \\ b &= \int_{\theta_1}^{\theta_2} \bar{y}'(\theta)\,\mathrm{d}\theta = \int_{\theta_1}^{\theta_2} \hat{t}'(\theta)\bar{v}(\theta)\sin\theta\,\mathrm{d}\theta, && (30b) \end{aligned}$$

Combining equations (18), (19), and (23) we get that

$$\begin{aligned} \hat{t}'(\theta)\bar{v}(\theta) = {} & g^2 v_0^2 D(\theta)\left[(1 + g\sin^3\theta K''(g\cos\theta))\hat{c}_2 + \right. \\ & \left.(-K'(g\cos\theta) + g\sin^2\theta\cos\theta K''(g\cos\theta))\hat{c}_3\right] \times \\ & \left[(g - A(\theta)\sin\theta)\hat{c}_2 - (gK'(g\cos\theta) + A(\theta)\cos\theta)\hat{c}_3\right]^{-3}. \end{aligned} \tag{31}$$

It follows now that the left hand side of the following equation

$$\frac{\int_{\theta_1}^{\theta_2} \hat{t}'(\theta)\bar{v}(\theta)\sin\theta\,\mathrm{d}\theta}{\int_{\theta_1}^{\theta_2} \hat{t}'(\theta)\bar{v}(\theta)\cos\theta\,\mathrm{d}\theta} - \frac{b}{a} = 0. \tag{32}$$

is a function of θ_2 only, since $\hat{c}_2, \hat{c}_3$ are functions of θ_2 only, which can then be solved for θ_2.[1] We can compute the value of v_0 using (31) with either (30a) or (30b). It follows

[1] In practice we compute θ_2 by a fixed point iteration. Namely, given an approximate value of θ_2, we solve (26) for $\hat{c}_2, \hat{c}_3$, With these values of $\hat{c}_2, \hat{c}_3$, we solve (32) for θ_2, an then repeat the process.

now from (16) and (29) that $\bar{\sigma}(\cdot)$ and $\bar{\lambda}(\cdot)$ are completely determined from $\hat{c}_2, \hat{c}_3, v_0$. Also from (18) and (19) we get that $\bar{v}(\cdot)$ and $\hat{t}(\cdot)$ are completely determined from $\bar{\sigma}(\cdot)$ and $\bar{\lambda}(\cdot)$. Finally we can construct $\bar{x}(\cdot)$ and $\bar{y}(\cdot)$ using $\bar{v}(\cdot)$, $\hat{t}(\cdot)$ in (14).

Equation (25) acts as a *compatibility condition* between the initial angle of inclination θ_1 and the initial speed v_0. The process described in the previous paragraph is for the problem in which θ_1 is specified and v_0 is adjusted to comply with the compatibility condition. We describe now the problem in which v_0 is prescribed and θ_1 must be adjusted to meet the compatibility condition. In this case we solve the full system (26), (30) for the unknowns $\hat{c}_2, \hat{c}_3, \theta_1, \theta_2$ by a blocked fixed point iteration. Namely, given θ_1, θ_2 one solves (26) for $\hat{c}_2, \hat{c}_3$; with these values of $\hat{c}_2, \hat{c}_3$ we solve (30) to get new values for θ_1, θ_2, and repeat the process. Let

$$g_1(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \gamma(\theta) \cos\theta \, \mathrm{d}\theta - a, \tag{33a}$$

$$g_2(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \gamma(\theta) \sin\theta \, \mathrm{d}\theta - b, \tag{33b}$$

where we defined

$$\gamma(\theta) = \hat{t}'(\theta)\bar{v}(\theta).$$

Also, let $\boldsymbol{\theta} = (\theta_1, \theta_2)$, and $\mathbf{g}(\boldsymbol{\theta}) = (g_1(\boldsymbol{\theta}), g_2(\boldsymbol{\theta}))$. Then

$$\mathrm{D}_{\boldsymbol{\theta}}\mathbf{g}(\boldsymbol{\theta}) = \begin{pmatrix} -\gamma(\theta_1)\cos\theta_1 & \gamma(\theta_2)\cos\theta_2 \\ -\gamma(\theta_1)\sin\theta_1 & \gamma(\theta_2)\sin\theta_2 \end{pmatrix},$$

the inverse of which is given by the formula:

$$\mathrm{D}_{\boldsymbol{\theta}}^{-1}\mathbf{g}(\boldsymbol{\theta}) = \frac{1}{\gamma(\theta_1)\gamma(\theta_2)\sin(\theta_1 - \theta_2)} \begin{pmatrix} \gamma(\theta_2)\sin\theta_2 & -\gamma(\theta_2)\cos\theta_2 \\ \gamma(\theta_1)\sin\theta_1 & -\gamma(\theta_1)\cos\theta_1 \end{pmatrix}.$$

These formulae can be used now for a Newton–type iteration for the solution of (33) as part of the blocked fixed point iteration described above.

We implemented this iteration in MATLAB using for the nonlinear frictional force $K(N)$ the following function:

$$K(N) = -\mu N + \beta N^3, \tag{34}$$

where $\mu > 0$ and $\beta \in \mathbb{R}$ are given constants. When $\beta < 0$ we get the equivalent of a *hard–type spring* frictional force, where as for $\beta > 0$ we get a *soft–type* model. We show in Figure (1) the corresponding brachistochrone curves for $a = 3$, $b = -9$, $g = 9.8$, initial speed $v_0 = 35$, with units consistent with those of a, b, and for $\mu = 5, 10, 15$, $\beta = 0.02$. We see that as the linear frictional coefficient μ increases, the optimal curve of descend opens further to the left. This has the effect that the particle picks up a larger speed at the beginning of the curve, because of free falling, to compensate for the dissipation of

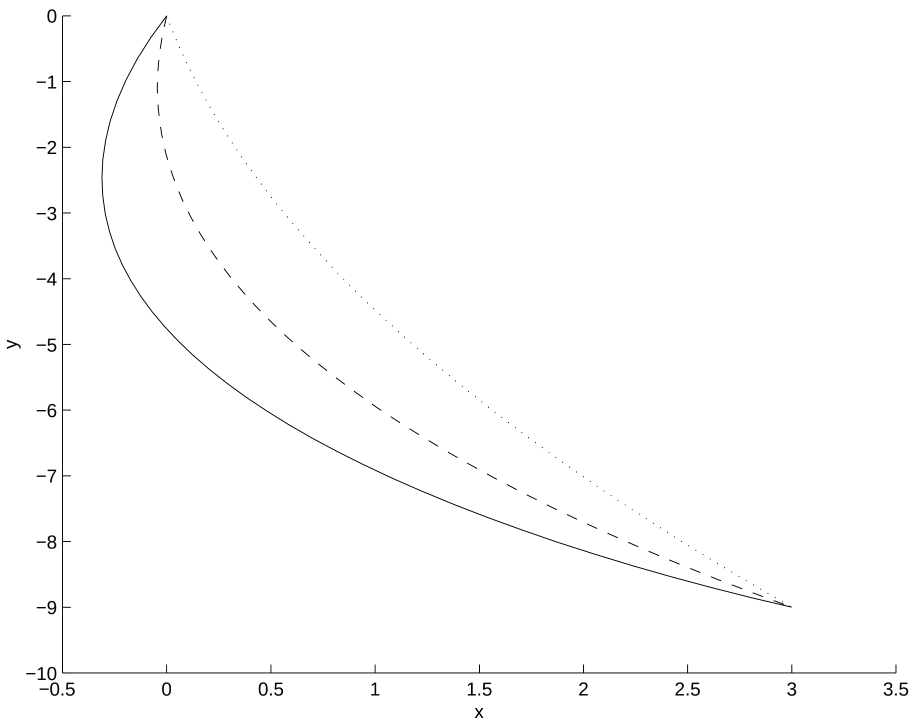

Figure 1: Graphs of the brachistochrone curves for initial speed $v_0 = 35$ and for $\mu = 5, 10, 15,$ (dotted, dashed, and solid respectively) and $\beta = 0.02$ in (34).

energy due to friction and to get to the target point in minimum time. Note that the brachistochrone corresponding to $\mu = 15$ in this example is a genuine curve in the plane, that is, not a function x. The corresponding times of descend where respectively 0.27559, 0.28919, 0.30236.

In Figure (2) we compare the effect of the model been hard versus soft by graphing the curves corresponding to $\mu = 15$ and $\beta = \pm 0.02$ again with initial speed $v_0 = 35$. We see that for the hard frictional force the effect of curving to the left is less pronounced than for the soft response. This is because the model with the hard response dissipates quicker the initial *excess* kinetic energy. Even though the curve for the hard model is shorter than the the corresponding one for the soft response, the time of descend is 0.30567 for the hard model versus 0.30236 for the soft.

Finally in Table (1) we collect the computed values of θ_1, θ_2 with the corresponding total times of descend for different values of μ, β and $v_0 = 35$.

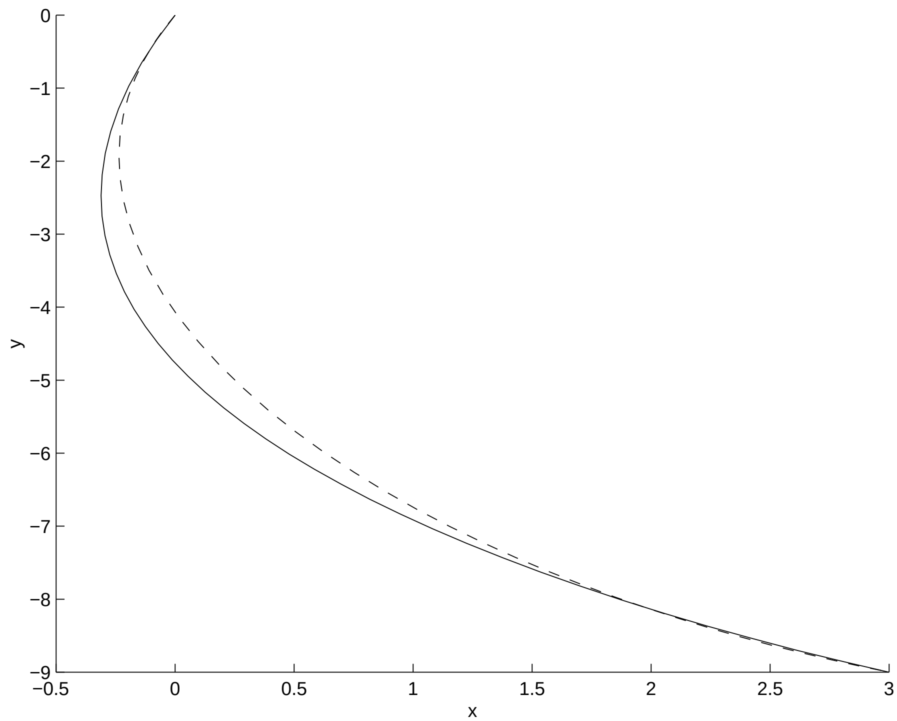

Figure 2: Graphs of the brachistochrone curves for initial speed $v_0 = 35$ and for 15, and $\beta = -0.02, 0.02$ (dashed and solid respectively) in (34).

μ	β	θ_1	θ_2	Total Time
5	0.02	-1.4539	-1.0669	0.27559
5	-0.02	-1.4361	-1.0349	0.27655
10	0.02	-1.6525	-0.85172	0.28919
10	-0.02	-1.622	-0.81621	0.29076
15	0.02	-1.8136	-0.62964	0.30236
15	-0.02	-1.8292	-0.57491	0.30567

Table 1: Computed values of θ_1, θ_2, and total times of descend for different values of μ, β and $v_0 = 35$ for the model response function (34).

2.2 The General Case

Again with the Lagrange multipliers $\lambda(t), \sigma(t)$ we get that the first order necessary conditions for this problem are equivalent to those for the functional

$$
\begin{aligned}
I(\bar{x}, \bar{y}, \bar{v}, \hat{t}, \bar{\sigma}, \bar{\lambda}) \;=\; & \int_{\tau_1}^{\tau_2} \Bigg[\hat{t}'(\tau) + \bar{\sigma}(\tau) \left(\hat{t}'(\tau)\bar{v}(\tau) - \sqrt{\bar{x}'(\tau)^2 + \bar{y}'(\tau)^2} \right) \\
& + \bar{\lambda}(\tau) \Bigg(\bar{v}(\tau)\bar{v}'(\tau) + g\bar{y}'(\tau) \\
& - \hat{t}'(\tau)\bar{v}(\tau) K \left(\frac{g\bar{x}'(\tau)}{\hat{t}'(\tau)\bar{v}(\tau)} + \frac{\bar{x}'(\tau)\bar{y}''(\tau) - \bar{y}'(\tau)\bar{x}''(\tau)}{\hat{t}'(\tau)^3 \bar{v}(\tau)} \right) \Bigg) \Bigg] \mathrm{d}\tau
\end{aligned}
$$

The Euler–Lagrange equations for I are given by (8) and:

$$
\begin{aligned}
1 + \bar{\sigma}(\tau)\bar{v}(\tau) \;=\; & c_1 + \bar{\lambda}(\tau)\bar{v}(\tau) \Bigg[K(\bar{h}(\tau)) - \Bigg(\frac{g\bar{x}'(\tau)}{\hat{t}'(\tau)\bar{v}(\tau)} \\
& + \frac{3(\bar{x}'(\tau)\bar{y}''(\tau) - \bar{y}'(\tau)\bar{x}''(\tau))}{\hat{t}'(\tau)^3 \bar{v}(\tau)} \Bigg) K'(\bar{h}(\tau)) \Bigg], \qquad \text{(35a)}
\end{aligned}
$$

$$
\begin{aligned}
\bar{v}(\tau)\bar{\lambda}'(\tau) \;=\; & \hat{t}'(\tau) \big(\bar{\sigma}(\tau) + \bar{\lambda}(\tau) \times \\
& \big[\bar{h}(\tau) K'(\bar{h}(\tau)) - K(\bar{h}(\tau)) \big] \big), \qquad \text{(35b)}
\end{aligned}
$$

$$
\begin{aligned}
\bar{v}(\tau) \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\bar{\lambda}(\tau) K'(\bar{h}(\tau)) \bar{y}'(\tau)}{\hat{t}'(\tau)^2} \right) \;=\; & c_2 \bar{v}(\tau) - \bar{\sigma}(\tau)\bar{x}'(\tau)/\hat{t}'(\tau) \\
& - \bar{\lambda}(\tau)\bar{v}(\tau) K'(\bar{h}(\tau)) (g + \bar{y}''(\tau)/\hat{t}'(\tau)^2), \qquad \text{(35c)}
\end{aligned}
$$

$$
\begin{aligned}
\bar{v}(\tau) \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\bar{\lambda}(\tau) K'(\bar{h}(\tau)) \bar{x}'(\tau)}{\hat{t}'(\tau)^2} \right) \;=\; & c_3 \bar{v}(\tau) + \bar{\sigma}(\tau)\bar{y}'(\tau)/\hat{t}'(\tau) - g\bar{\lambda}(\tau)\bar{v}(\tau) \\
& - \bar{\lambda}(\tau)\bar{v}(\tau) K'(\bar{h}(\tau)) \bar{x}''(\tau)/\hat{t}'(\tau)^2, \qquad \text{(35d)}
\end{aligned}
$$

where we have set

$$
\bar{h}(\tau) = \frac{g\bar{x}'(\tau)}{\hat{t}'(\tau)\bar{v}(\tau)} + \frac{\bar{x}'(\tau)\bar{y}''(\tau) - \bar{y}'(\tau)\bar{x}''(\tau)}{\hat{t}'(\tau)^3 \bar{v}(\tau)},
$$

and c_1, c_2, c_3 are constants. The boundary conditions are given by (10),

$$
\bar{\lambda}(\tau_2) = 0, \qquad \text{(36)}
$$

and

$$
\begin{aligned}
& 1 + \bar{\sigma}(\tau)\bar{v}(\tau) - \bar{\lambda}(\tau)\bar{v}(\tau) \Bigg[K(\bar{h}(\tau)) \\
& \quad - \left(\frac{g\bar{x}'(\tau)}{\hat{t}'(\tau)\bar{v}(\tau)} + \frac{3(\bar{x}'(\tau)\bar{y}''(\tau) - \bar{y}'(\tau)\bar{x}''(\tau))}{\hat{t}'(\tau)^3 \bar{v}(\tau)} \right) K'(\bar{h}(\tau)) \Bigg] \Bigg|_{\tau=\tau_2} = 0. \qquad \text{(37)}
\end{aligned}
$$

This last condition implies that $c_1 = 0$ in (35).

We take now $\hat{t}(\tau) = \tau$, i.e., $t = \tau$. It follows now that (8a), (8b) reduce again to (6a), (6b), and that (35a)–(35d) become:

$$\begin{aligned}
1 + \sigma(t)v(t) &= \lambda(t)v(t)\Bigg[K(h(t)) - \Bigg(\frac{g\dot{x}(t)}{v(t)} \\
&\quad + \frac{3(\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t))}{v(t)}\Bigg)K'(h(t))\Bigg], && (38a) \\
v(t)\dot{\lambda}(t) &= \sigma(t) + \lambda(t)\left[h(t)K'(h(t)) - K(h(t))\right], && (38b) \\
v(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda(t)K'(h(t))\dot{y}(t)\right) &= c_2 v(t) - \sigma(t)\dot{x}(t) \\
&\quad -\lambda(t)v(t)K'(h(t))(g + \ddot{y}(t)), && (38c) \\
v(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda(t)K'(h(t))\dot{x}(t)\right) &= c_3 v(t) + \sigma(t)\dot{y}(t) \\
&\quad -\lambda(t)v(t)K'(h(t))\ddot{x}(t) - g\lambda(t)v(t), && (38d)
\end{aligned}$$

where now

$$h(t) = \frac{\dot{x}(t)(g + \ddot{y}(t)) - \dot{y}(t)\ddot{x}(t)}{v(t)} = g\cos\theta(t) + v(t)\dot{\theta}(t). \tag{39}$$

Using (2a) and (2b) we can write the above equations as:

$$\begin{aligned}
1 + \sigma(t)v(t) &= \lambda(t)v(t)\big[K(h(t)) \\
&\quad -\big(g\cos\theta(t) + 3v(t)\dot{\theta}(t)\big)K'(h(t))\big], && (40a) \\
v(t)\dot{\lambda}(t) &= \sigma(t) + \lambda(t)\left[h(t)K'(h(t)) - K(h(t))\right], && (40b) \\
\lambda(t)v(t)\sin\theta(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(K'(h(t))\right) &= c_2 - \sigma(t)\cos\theta(t) - g\lambda(t)K'(h(t)) \\
&\quad -2\lambda(t)K'(h(t))(\dot{v}(t)\sin\theta(t) + v(t)\dot{\theta}(t)\cos\theta(t)) \\
&\quad -\dot{\lambda}(t)v(t)\sin\theta(t)K'(h(t)), && (40c) \\
\lambda(t)v(t)\cos\theta(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(K'(h(t))\right) &= c_3 + \sigma(t)\sin\theta(t) - g\lambda(t) - \dot{\lambda}(t)v(t)\cos\theta(t)K'(h(t)) \\
&\quad -2\lambda(t)K'(h(t))(\dot{v}(t)\cos\theta(t) - v(t)\dot{\theta}(t)\sin\theta(t)), && (40d)
\end{aligned}$$

We can now use (40a) and (40b) to eliminate $\sigma(t)$ and $\dot{\lambda}(t)$ respectively from (40c) and

(40d) to get that

$$\begin{aligned}
\lambda(t)v(t)^2\sin\theta(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(K'(h(t))\right) &= c_2v(t) - g\lambda(t)v(t)K'(h(t)) \\
&\quad -2\lambda(t)v(t)[\dot{v}(t) - v(t)\dot{\theta}(t)K'(h(t))]K'(h(t))\sin\theta(t) \\
&\quad +\lambda(t)v(t)[h(t)K'(h(t)) - K(h(t))]\cos\theta(t) \\
&\quad +\cos\theta(t) + \sin\theta(t)K'(h(t)), \qquad \text{(41a)} \\
\lambda(t)v(t)^2\cos\theta(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(K'(h(t))\right) &= c_3v(t) - g\lambda(t)v(t) \\
&\quad -2\lambda(t)v(t)[\dot{v}(t) - v(t)\dot{\theta}(t)K'(h(t))]K'(h(t))\cos\theta(t) \\
&\quad -\lambda(t)v(t)[h(t)K'(h(t)) - K(h(t))]\sin\theta(t) \\
&\quad -\sin\theta(t) + \cos\theta(t)K'(h(t)). \qquad \text{(41b)}
\end{aligned}$$

Multiplying the first of these equations by $\cos\theta(t)$, the second by $\sin\theta(t)$, and subtracting the resulting equations, we get an expression for $\lambda(t)$ in terms of $v(t)$ and $\theta(t)$:

$$\lambda(t) = \frac{c_3\sin\theta(t) - c_2\cos\theta(t) - 1/v(t)}{g\sin\theta(t) - K(h(t)) + v(t)\dot{\theta}(t)K'(h(t))}. \tag{42}$$

If we now multiply (41a) by $\sin\theta(t)$, (41b) by $\cos\theta(t)$, and add the resulting equations, we get that

$$\begin{aligned}
\lambda(t)v(t)^2\frac{\mathrm{d}}{\mathrm{d}t}\left(K'(h(t))\right) &= v(t)[c_2\sin\theta(t) + c_3\cos\theta(t)] + K'(h(t)) \\
&\quad -g\lambda(t)v(t)[\cos\theta(t) + \sin\theta(t)K'(h(t))] \\
&\quad -2\lambda(t)v(t)[\dot{v}(t) - v(t)\dot{\theta}(t)K'(h(t))]K'(h(t)). \qquad \text{(43)}
\end{aligned}$$

Using (42) and the equation of motion (3) to eliminate $\dot{v}(t)$, we can write this equation as

$$\begin{aligned}
v(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(K'(h(t))\right) &= \left[\frac{v(t)\left(c_2\sin\theta(t) + c_3\cos\theta(t)\right) + K'(h(t))}{v(t)\left(c_3\sin\theta(t) - c_2\cos\theta(t)\right) - 1} + 2K'(h(t))\right]\times \\
&\quad \Big(g\sin\theta(t) - K(h(t)) + v(t)\dot{\theta}(t)K'(h(t))\Big) \\
&\quad -g\left(\cos\theta(t) + \sin\theta(t)K'(h(t))\right). \qquad \text{(44)}
\end{aligned}$$

We can eliminate $\dot{\theta}(t)$ from this equation using (39), introduce $h(t)$ as a dependent variable, and with the equation of motion (3) we arrive at the following system of equations for $h(t), \theta(t), v(t)$:

$$\begin{aligned}
v(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(K'(h(t))\right) &= \left[\frac{v(t)\left(c_2\sin\theta(t)+c_3\cos\theta(t)\right)+K'(h(t))}{v(t)\left(c_3\sin\theta(t)-c_2\cos\theta(t)\right)-1}+2K'(h(t))\right]\times \\
&\quad \left(g(\sin\theta(t)-\cos\theta(t)K'(h(t)))+h(t)K'(h(t))-K(h(t))\right) \\
&\quad -g\left(\cos\theta(t)+\sin\theta(t)K'(h(t))\right), &\text{(45a)}\\
\dot{\theta}(t) &= \frac{h(t)-g\cos\theta(t)}{v(t)}, &\text{(45b)}\\
\dot{v}(t) &= -g\sin\theta(t)+K(h(t)). &\text{(45c)}
\end{aligned}$$

We need to assume that $h(0)$ is given. Since $\theta(0), v(0)$ are given, this is equivalent to the specification of the normal component of the acceleration $v(0)\dot{\theta}(0)$. We take this component of acceleration initially to be zero which gives that

$$h(0) = g\cos\theta(0). \tag{46}$$

Thus given c_2, c_3, the system (45) can be solved in principle to get $h(t), \theta(t), v(t)$. The boundary conditions (30a), (30b), (36) and equation (42), imply that

$$\begin{aligned}
c_3\sin\theta(t_f)-c_2\cos\theta(t_f)-1/v(t_f) &= 0, &\text{(47a)}\\
\int_0^{t_f} v(t)\cos\theta(t)\,\mathrm{d}t &= a, &\text{(47b)}\\
\int_0^{t_f} v(t)\sin\theta(t)\,\mathrm{d}t &= b, &\text{(47c)}
\end{aligned}$$

where t_f is the final minimum unknown time. The system (47) together with (45) yields a nonlinear system of equations for c_2, c_3, t_f. The solution of this system maybe done by means of a blocked fixed point iteration: given a value of t_f, we solve (47b), (47c) for c_2, c_3 using Newton's method for systems; with these values of c_2, c_3 we adjust t_f using equation (47a), and repeat the process.

We discuss briefly the solution of (47b), (47c) for c_2, c_3 using Newton's method, given t_f. Let $(h(t;c_2,c_3), \theta(t;c_2,c_3), v(t;c_2,c_3))$ denote the solution of (45) given c_2, c_3 and the boundary conditions for $t=0$. Let

$$\mathbf{y}(t;c_2,c_3) = (h(t;c_2,c_3), \theta(t;c_2,c_3), v(t;c_2,c_3)),$$

and we represent (45) as:

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t}(t;c_2,c_3) = \mathbf{f}(\mathbf{y}(t;c_2,c_3);c_2,c_3), \tag{48}$$

where $\mathbf{f}$ is given by the right hand side of (45). Let

$$\begin{aligned}
g_1(c_2,c_3) &= \int_0^{t_f} v(t;c_2,c_3)\cos\theta(t;c_2,c_3)\,\mathrm{d}t - a,\\
g_2(c_2,c_3) &= \int_0^{t_f} v(t;c_2,c_3)\sin\theta(t;c_2,c_3)\,\mathrm{d}t - b.
\end{aligned}$$

We have now that

$$\begin{aligned}
\frac{\partial g_1}{\partial c_j}(c_2,c_3) &= \int_0^{t_f}\left[\frac{\partial v}{\partial c_j}(t)\cos\theta(t) - v(t)\sin\theta(t)\frac{\partial\theta}{\partial c_j}(t)\right]\mathrm{d}t, \quad j=2,3,\\
\frac{\partial g_2}{\partial c_j}(c_2,c_3) &= \int_0^{t_f}\left[\frac{\partial v}{\partial c_j}(t)\sin\theta(t) + v(t)\cos\theta(t)\frac{\partial\theta}{\partial c_j}(t)\right]\mathrm{d}t, \quad j=2,3,
\end{aligned}$$

where for simplicity we omitted the dependence on c_2, c_3 from v, θ. The partial derivatives under the integral signs can be computed differentiating (48) with respect to c_2, c_3. More specifically, differentiating with respect to c_j in (48) we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{y}}{\partial c_j}(t)\right) = \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{y}(t))\frac{\partial \mathbf{y}}{\partial c_j}(t) + \frac{\partial \mathbf{f}}{\partial c_j}(\mathbf{y}(t)).$$

Given $\mathbf{y}(t)$, this system of ordinary differential equations together with the initial condition

$$\frac{\partial \mathbf{y}}{\partial c_j}(0) = \mathbf{0},$$

can be solved for $\partial\mathbf{y}/\partial c_j$.

3 Friction as a nonlinear function of speed

We consider the case of a drag frictional force which is given as a nonlinear function of the speed, that is $K(v)$. It follows now that (8b) becomes:

$$\bar{v}(\tau)\bar{v}'(\tau) = -g\bar{y}'(\tau) + \hat{t}'(\tau)\bar{v}(\tau)K(\bar{v}(\tau)). \tag{49}$$

The first order necessary conditions for a minimizer of (9) are equivalent to those for the functional:

$$\begin{aligned}
I(\bar{x},\bar{y},\bar{v},\hat{t},\bar{\sigma},\bar{\lambda}) &= \int_{\tau_1}^{\tau_2}\Bigg[\hat{t}'(\tau) + \bar{\sigma}(\tau)\left(\hat{t}'(\tau)\bar{v}(\tau) - \sqrt{\bar{x}'(\tau)^2+\bar{y}'(\tau)^2}\right)\\
&\quad +\bar{\lambda}(\tau)\Big(\bar{v}(\tau)\bar{v}'(\tau) + g\bar{y}'(\tau) - \hat{t}'(\tau)\bar{v}(\tau)K(\bar{v}(\tau))\Big)\Bigg]\mathrm{d}\tau.
\end{aligned}$$

The Euler–Lagrange equations for this functional are given by (8a), (49), and:

$$\begin{aligned}
1+\bar{\sigma}(\tau)\bar{v}(\tau) - \bar{\lambda}(\tau)\bar{v}(\tau)K(\bar{v}(\tau)) &= c_1, &\text{(50a)}\\
\frac{\bar{\sigma}(\tau)\bar{x}'(\tau)}{\sqrt{\bar{x}'(\tau)^2+\bar{y}'(\tau)^2}} &= c_2, &\text{(50b)}\\
-\frac{\bar{\sigma}(\tau)\bar{y}'(\tau)}{\sqrt{\bar{x}'(\tau)^2+\bar{y}'(\tau)^2}} + g\bar{\lambda}(\tau) &= c_3, &\text{(50c)}\\
\bar{\sigma}(\tau)\hat{t}'(\tau) - \bar{\lambda}'(\tau)\bar{v}(\tau) - \bar{\lambda}(\tau)\hat{t}'(\tau)(K(\bar{v}(\tau)) + \bar{v}(\tau)K'(\bar{v}(\tau))) &= 0, &\text{(50d)}
\end{aligned}$$

with boundary conditions (10a), (10c), $\bar{v}(\tau_1) = v_0$, and

$$\bar{\lambda}(\tau_2) = 0, \quad 1 + \bar{\sigma}(\tau)\bar{v}(\tau) - \bar{\lambda}(\tau)\bar{v}(\tau)K(\bar{v}(\tau))\Big|_{\tau=\tau_2} = 0, \tag{51}$$

the second of which implies that $c_1 = 0$.

Using (8a) and (14) we get from (50b), (50c) that:

$$\begin{aligned} \bar{\sigma}(\tau) &= \frac{c_2}{\cos\bar{\theta}(\tau)}, &(52a)\\ \bar{\lambda}(\tau) &= \frac{1}{g}\left[c_3 + c_2\tan\bar{\theta}(\tau)\right]. &(52b)\end{aligned}$$

This motivates us to take θ as the parameter and rewrite the above equations as:

$$\bar{\sigma}(\theta) = \frac{c_2}{\cos\theta}, \quad \bar{\lambda}(\theta) = \frac{1}{g}\left[c_3 + c_2\tan\theta\right]. \tag{53}$$

We now have the following:

Lemma 3.1. *Equation* (50a) *follows from* (53)*,* (49)*, and* (50d)*.*

Proof: From (50d) we get that

$$\frac{\mathrm{d}}{\mathrm{d}v}(vK(v))\Bigg|_{v=\bar{v}(\theta)} = \frac{\bar{\sigma}(\theta)\hat{t}'(\theta) - \bar{\lambda}'(\theta)\bar{v}(\theta)}{\bar{\lambda}(\theta)\hat{t}'(\theta)}.$$

Thus

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta}(\bar{\sigma}(\theta)\bar{v}(\theta) - \bar{\lambda}(\theta)\bar{v}(\theta)K(\bar{v}(\theta))) &= \bar{\sigma}'(\theta)\bar{v}(\theta) + \bar{\sigma}(\theta)\bar{v}'(\theta)\\ &\quad -\bar{\lambda}'(\theta)\bar{v}(\theta)K(\bar{v}(\theta)) - \bar{\lambda}(\theta)\bar{v}'(\theta)\,\frac{\mathrm{d}}{\mathrm{d}v}(vK(v))\Bigg|_{v=\bar{v}(\theta)},\\ &= \frac{\bar{v}(\theta)}{\hat{t}'(\theta)}\left(\hat{t}'(\theta)\bar{\sigma}'(\theta) + (\bar{v}'(\theta) - \hat{t}'(\theta)K(\bar{v}(\theta)))\bar{\lambda}'(\theta)\right).\end{aligned}$$

Using (49) we get that

$$\bar{v}'(\theta) - \hat{t}'(\theta)K(\bar{v}(\theta)) = -g\hat{t}'(\theta)\sin\theta.$$

With this the previous equation simplifies to:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\bar{\sigma}(\theta)\bar{v}(\theta) - \bar{\lambda}(\theta)\bar{v}(\theta)K(\bar{v}(\theta))) = \bar{v}(\theta)\left(\bar{\sigma}'(\theta) - g\sin\theta\bar{\lambda}'(\theta)\right).$$

From (53) one gets that:

$$\bar{\lambda}'(\theta) = \frac{1}{g}\,\bar{\sigma}(\theta)\sec\theta, \quad \bar{\sigma}'(\theta) = \bar{\sigma}(\theta)\tan\theta.$$

Using this in the previous expression gives that,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\bar{\sigma}(\theta)\bar{v}(\theta) - \bar{\lambda}(\theta)\bar{v}(\theta)K(\bar{v}(\theta))) = 0,$$

from which the result follows. □

The first boundary condition in (51) and $\bar{v}(\theta_1) = v_0$ gives the following system for c_2, c_3:

$$\begin{pmatrix} v_0(g - K(v_0)\sin\theta_1) & -v_0K(v_0)\cos\theta_1 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -g\cos\theta_1 \\ 0 \end{pmatrix}. \tag{54}$$

From (49) we have:

$$\bar{v}'(\theta) = [-g\sin\theta + K(\bar{v}(\theta))]\,\hat{t}'(\theta). \tag{55}$$

From (50d) we get that:

$$\hat{t}'(\theta) = \frac{\bar{v}(\theta)\bar{\lambda}'(\theta)}{\bar{\sigma}(\theta) - \bar{\lambda}(\theta)(\bar{v}(\theta)K'(\bar{v}(\theta)) + K(\bar{v}(\theta))}, \tag{56}$$

which can be used in the equation above for $\bar{v}(\theta)$. Using these two equations we can compute both integrands in (30). Thus (30) and (54) gives a system of four equations in the variables v_0, θ_1, θ_2, c_2, and c_3.

In general one can not specify both v_0 and θ_1 as the first equation in (54) act as a compatibility condition on these variables. Thus we can consider two problems: one in which v_0 is specified and θ_1, θ_2, c_2, and c_3 are determined; or θ_1 is specified and v_0, θ_2, c_2, and c_3 are determined. We discuss only the former problem. To determine the solution curve in this case, we would solve (54) for c_2, c_3 as functions of θ_1, θ_2. With these values of c_2, c_3 we get $\bar{\sigma}(\cdot), \bar{\lambda}(\cdot)$ from (53). The differential equation (55) can then be integrated for $\bar{v}(\cdot)$ as a function of θ_1, c_2, c_3. Finally, equation (32) can now be solved for θ_1, θ_2. In practice, this process is done by means of a blocked fixed point iteration. Namely, given values of θ_1, θ_2 one solves (54) for c_2, c_3; with these values of c_2, c_3 we solve (32), (55), (56) for θ_1, θ_2, and repeat the process. We describe in details now this last part of the blocked fixed point iteration.

Using (53) and (56) one can easily show that

$$\hat{t}'(\theta) = F(\bar{v}(\theta), \theta),$$

where

$$F(v, \theta) = \frac{c_2 v \sec\theta}{gc_2 - (vK'(v) + K(v))(c_2\sin\theta + c_3\cos\theta)},$$

where for simplicity we have omitted the dependence on c_2, c_3 from F. Let $\bar{v}(\theta;\theta_1)$ be the solution of the initial value problem

$$\bar{v}'(\theta) = (-g\sin\theta + K(\bar{v}(\theta)))\,F(\bar{v}(\theta), \theta), \quad \bar{v}(\theta_1) = v_0. \tag{57}$$

Let

$$\begin{aligned}
g_1(\theta_1,\theta_2) &= \int_{\theta_1}^{\theta_2} F(\bar v(\theta;\theta_1),\theta)\bar v(\theta;\theta_1)\cos\theta\,\mathrm{d}\theta - a,\\
g_2(\theta_1,\theta_2) &= \int_{\theta_1}^{\theta_2} F(\bar v(\theta;\theta_1),\theta)\bar v(\theta;\theta_1)\sin\theta\,\mathrm{d}\theta - b.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial g_1}{\partial\theta_1} &= \int_{\theta_1}^{\theta_2}\left[\frac{\partial F}{\partial v}(\bar v(\theta),\theta)\bar v(\theta) + F(\bar v(\theta),\theta)\right]\frac{\partial\bar v}{\partial\theta_1}(\theta)\cos\theta\,\mathrm{d}\theta\\
&\quad -F(\bar v(\theta_1),\theta_1)\bar v(\theta_1)\cos\theta_1,\\
\frac{\partial g_1}{\partial\theta_2} &= F(\bar v(\theta_2),\theta_2)\bar v(\theta_2)\cos\theta_2,\\
\frac{\partial g_2}{\partial\theta_1} &= \int_{\theta_1}^{\theta_2}\left[\frac{\partial F}{\partial v}(\bar v(\theta),\theta)\bar v(\theta) + F(\bar v(\theta),\theta)\right]\frac{\partial\bar v}{\partial\theta_1}(\theta)\sin\theta\,\mathrm{d}\theta\\
&\quad -F(\bar v(\theta_1),\theta_1)\bar v(\theta_1)\sin\theta_1,\\
\frac{\partial g_2}{\partial\theta_2} &= F(\bar v(\theta_2),\theta_2)\bar v(\theta_2)\sin\theta_2,
\end{aligned}$$

where for simplicity we omitted the dependence on θ_1 from $\bar v$. From (57) we get that if

$$w(\theta;\theta_1) = \frac{\partial\bar v}{\partial\theta_1}(\theta;\theta_1),$$

then

$$\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}\theta}(\theta;\theta_1) &= \Bigg[(-g\sin\theta + K(\bar v(\theta;\theta_1)))\frac{\partial F}{\partial v}(\bar v(\theta;\theta_1),\theta)\\
&\qquad +K'(\bar v(\theta;\theta_1))F(\bar v(\theta;\theta_1),\theta)\Bigg]w(\theta;\theta_1),\\
w(\theta_1;\theta_1) &= -\bar v'(\theta_1;\theta_1),\\
&= F(\bar v(\theta_1;\theta_1),\theta_1)\left[g\sin\theta_1 - K(\bar v(\theta_1;\theta_1))\right],\\
&= F(v_0,\theta_1)\left[g\sin\theta_1 - K(v_0)\right].
\end{aligned}$$

This is a linear initial value problem which upon integration, has solution:

$$w(\theta;\theta_1) = F(v_0,\theta_1)\left[g\sin\theta_1 - K(v_0)\right]\exp\left[\int_{\theta_1}^{\theta} H(\bar v(\xi;\theta_1),\xi)\,\mathrm{d}\xi\right],$$

where

$$H(v,\theta) = (-g\sin\theta + K(v))\frac{\partial F}{\partial v}(v,\theta) + K'(v)F(v,\theta).$$

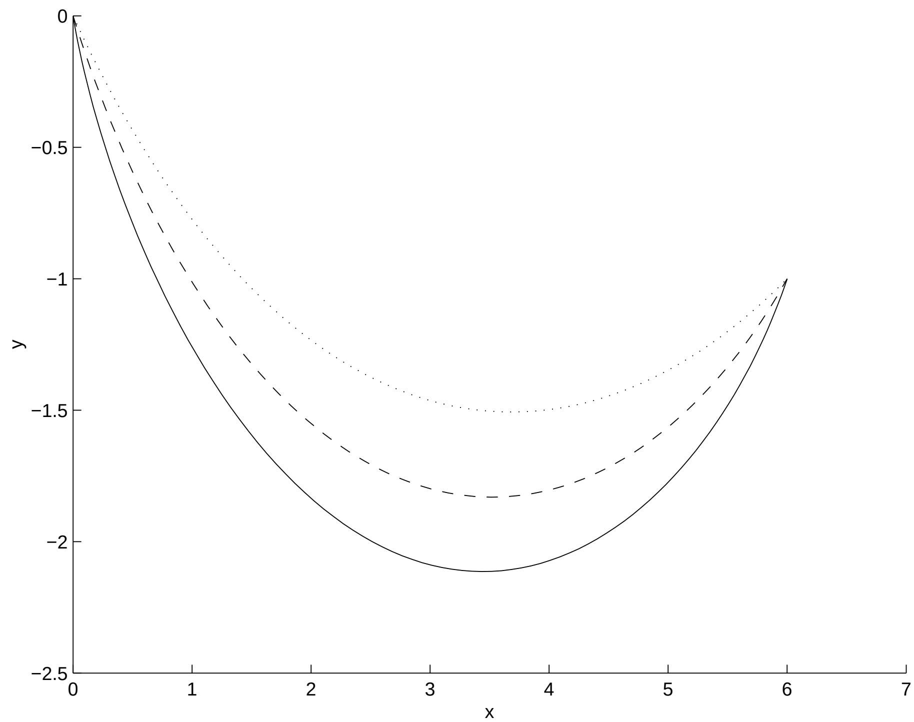

Figure 3: Graphs of the brachistochrone curves with $\mu = 0.3$, $\beta = 0.004$ in (58) and initial speeds $v_0 = 2, 4, 6$ (solid, dashed, and dotted respectively).

We used the above formulae to implemented a blocked fixed point iteration in MATLAB in which the system

$$g_1(\theta_1, \theta_2) = 0, \quad g_2(\theta_1, \theta_2) = 0,$$

is solved for any given c_2, c_3 using Newton's method. We employ for the frictional response function the model equation (34) but with N replaced by the speed v, that is:

$$K(v) = -\mu v + \beta v^3. \tag{58}$$

For these simulations we used $a = 6, b = -1$ and $g = 9.8$. In Figure (3) we show the calculated curves of minimum descend for $\mu = 0.3$, $\beta = 0.004$ and initial speeds $v_0 = 2, 4, 6$. The times of descend are respectively $1.4688, 1.1561, 0.88833$. In Figure (4) we present the drag force profile (left) and the speed profile (right) in each case. We can see clearly the non–monotonicity with respect to v of the force due to the cubic term which has the effect of diminishing the drag force for this regime of speeds. The corresponding profiles withe the same initial speeds but with $\mu = 0.1$ and $\beta = -0.002$ (hard–type frictional force) are shown in Figure (5).

The last simulation presented compares the *soft–type* vs *hard–type* frictional responses. We present in Figure (6) the computed curves for $v_0 = 2$, $\mu = 0.1$ and

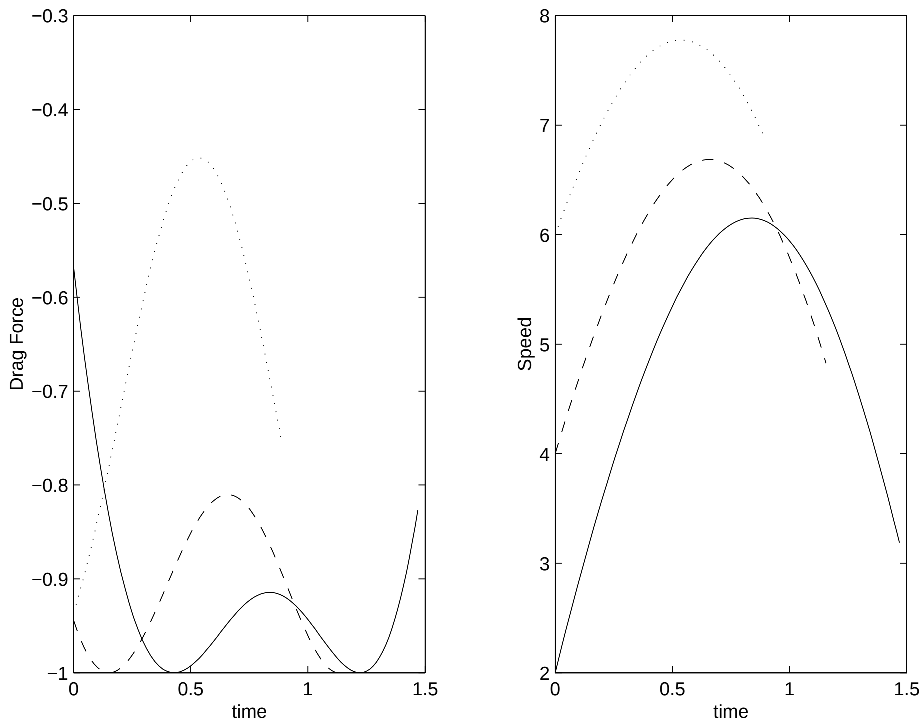

Figure 4: Graphs of the drag force (left) and speed (right) profiles for the curves of minimum descend for $\mu = 0.3$, $\beta = 0.004$ in (58) and initial speeds $v_0 = 2, 4, 6$ (solid, dashed, and dotted respectively).

$\beta = 0.002, -0.002$ (solid and dotted respectively). The corresponding times of descend are 1.3456 (solid curve) and 1.4328 (dotted curve). Note the even though the solid curve is longer, the descend time for the corresponding soft model is lower because the particle can achieve a higher speed profile because the frictional force is not as large as for the hard–type model. We show in Figure (7) the corresponding drag force and speed profiles.

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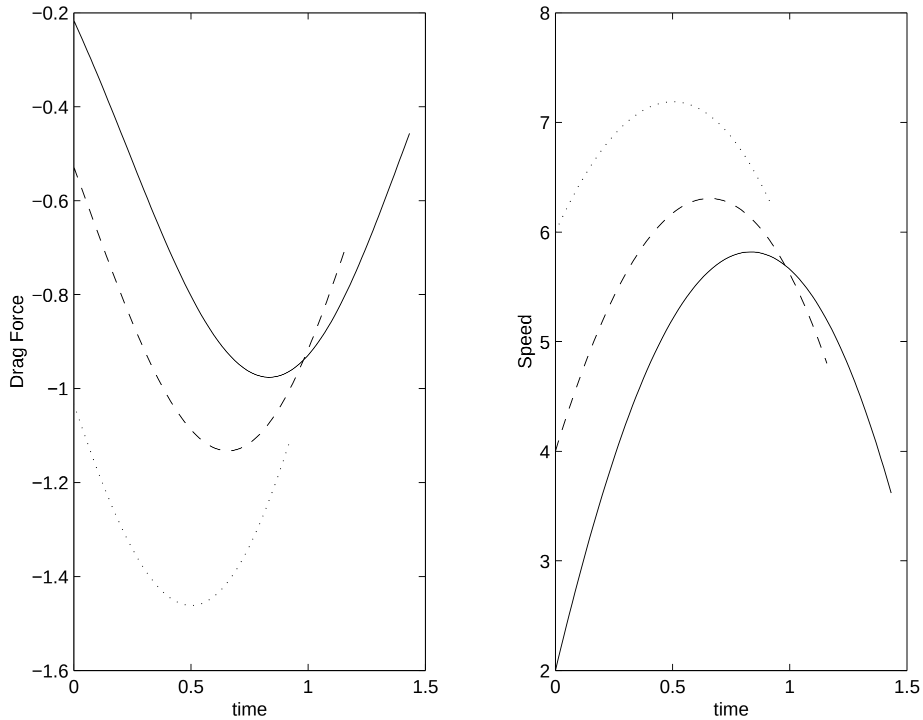

Figure 5: Graphs of the drag force (left) and speed (right) profiles for the curves of minimum descend for $\mu = 0.1$, $\beta = -0.002$ in (58) and initial speeds $v_0 = 2, 4, 6$ (solid, dashed, and dotted respectively).

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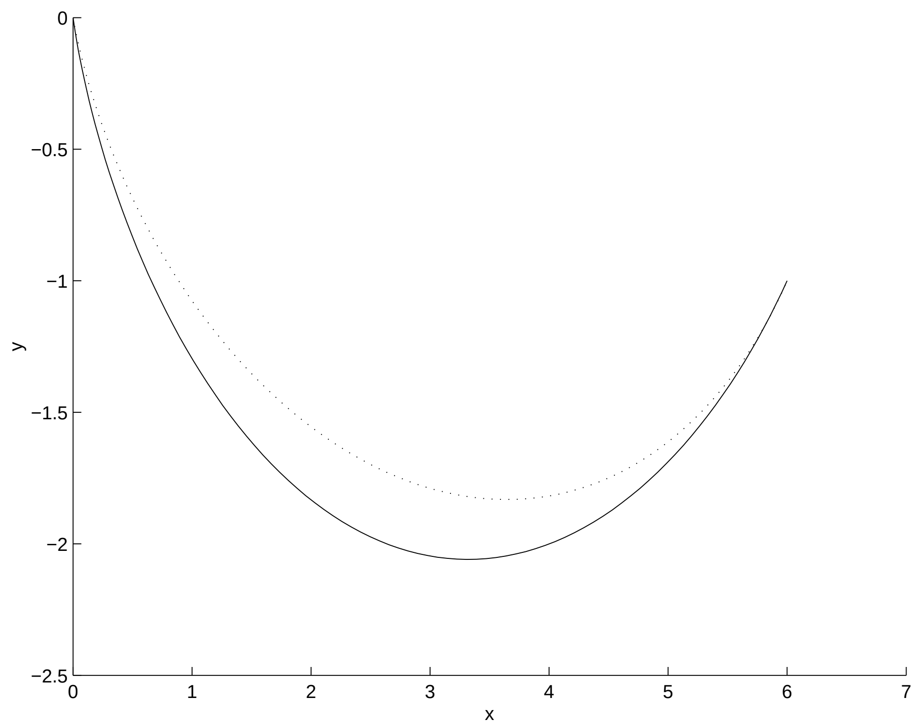

Figure 6: Graphs of the brachistochrone curves for $v_0 = 2$, $\mu = 0.1$, and $\beta = 0.002, -0.002$ (solid and dotted respectively) in (58).

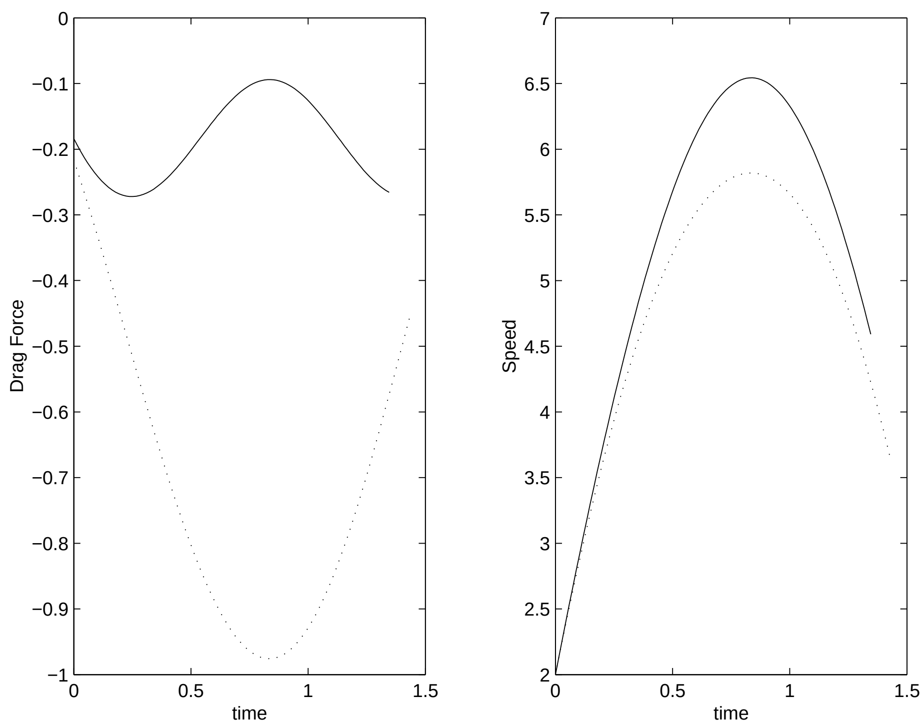

Figure 7: Graphs of the drag force (left) and speed (right) profiles for the curves of minimum descend for $v_0 = 2$, $\mu = 0.1$, and $\beta = 0.002, -0.002$ (solid and dotted respectively) in (58).